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G. FURLAN
F. LANNOY
C. ROSSETTI
AND
G. SEGRÈ

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G. Furlan
Istituto di Fisica dell'Università, Trieste

F. Lannoy
C.E.R.N., Geneva

C. Rossetti **
C.E.R.N., Geneva

G. Segrè
Istituto di Fisica dell'Università, Torino

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** On leave from Istituto di Fisica dell'Università, Torino.

SUMMARY

A systematic analysis of the equal time commutation relations of the generators of an algebra with certain physical operators is made. A method is then introduced whereby considering matrix elements of such commutators between physical one-particle states and using completeness and invariance under space-time translations, corrections to broken symmetry group theoretical formulae are obtained. Several applications to weak, electromagnetic and strong interactions are then made.

THEORY OF CORRECTIONS TO UNITARY SYMMETRY FORMULAE

1. INTRODUCTION

One of the most powerful tools for studying the physics of elementary particles has been the use of symmetry groups. In particular, SU_3 [1] has led to very many well-verified predictions concerning the classification and behaviour of elementary particles and promising results appear also to follow from the application of still higher symmetry groups.

It is nevertheless still not well understood why some of the group theoretical predictions are so good despite the fact that the breaking of the symmetry is large and the group theoretical results correspond to a quasi-perturbation theoretical approach in this breaking.

We shall present a method for studying in some cases the corrections to simple group theoretical formulae and thereby attempt to achieve an understanding of the validity of these results. A first step along this dection has been taken in Refs. (2) and (3) in which the renormalization of the weak interaction current vertex due to symmetry breaking was estimated. This paper will be devoted to a generalization of the method and to its application to a wider class of problems.

The method is based on studying the equal time commutators of the generators of the group algebra, as constructed from the integral over all space of the fourth components of the currents. This method has been repeatedly emphasized by GELL-MANN [4] over the course of the past few years and has the advantage that the commutation relations remain unchanged even when the symmetry is broken and therefore the currents are no longer conserved.

We shall show how to construct a scheme for evaluating corrections to group theoretical formulae by a judicious use of commutators, completeness and invariance under space-time translations.

Sections 2 and 3 contain a general outline of the method which was applied in Refs. (2) and (3) to the renormalization of the weak current and show how it may be generalized to treat a wider class of problems. Section 4 treats the influence of kinematical factors and the choice of frame of reference for evaluating the corrections and Section 5 shows how the corrections may be evaluated. The method is finally applied to mass formulae in Section 6 and to relations between electromagnetic form factors in Section 7. An appendix containing some numerical results on the evaluation of corrections to mass formulae is also included; they are in reasonable agreement with experiment.

2. GENERAL OUTLINE OF THE METHOD

One of the fundamental consequences of the invariance of a theory under a group is the existence of a set of conserved currents $J_\mu^{(\sigma)}$ associated with the group transformations. The fourth components of these currents, integrated over all space, which we shall call generalized "charges" are the generators of the infinitesimal transformations of the group (at time t)

$$Q_\sigma(t) = \int J_o^{(\sigma)}(\bar{x}, t)\, d^3x \qquad (2.1)$$

If we assume the symmetry to correspond to a (semi-simple) Lie group, the generalized charges satisfy the equal time commutation relations

$$[Q_\sigma(t), Q_\varsigma(t')]_{t=t'} = C_{\sigma\varsigma}^{\tau}\, Q_\tau(t)$$

where the $C_{\sigma\varsigma}^{\tau}$'s are the structure constants of the Lie algebra. In the following, we shall always employ the generators in the standard form. Recalling the Racah notation [5], we label as Q_i the mutually-commuting (always at equal times) generators and Q_α those corresponding to the non-null roots α . In a given representation the operator Q_α connects the state $|m\rangle$ belonging to a weight m only with the state belonging to the weight $m+\alpha$.

$$\langle m+\alpha, \vec{p}\,|\, Q_\alpha \,|\, m, \vec{p}\,'\rangle = C(\alpha, m)\, \delta^{(3)}(\vec{p} - \vec{p}\,') \qquad (2.2)$$

where $C(\alpha, m)$ is a constant determined by the group structure. For instance, if we consider the state belonging to the highest weight M of a given representation, we have

$$\langle M, \vec{p}\,|\, Q_\alpha \,|\, M-\alpha, \vec{p}\,'\rangle = \sqrt{M^i\alpha_i}\;\delta^{(3)}(\vec{p} - \vec{p}\,') \qquad (2.2')$$

the α_i's being the components of the root α .

The equal time commutation relations hold even when the symmetry is broken, that is to say when the currents are not all conserved, and the Q_σ are no longer constants in time. A set of one-particle states, however, which formed an irreducible representation of the group in the symmetry limit now contains admixtures of other representations as the states are eigenstates of the total Hamiltonian which contains both a symmetry-preserving and a symmetry-breaking part. The action of this symmetry breaking is then reflected in the matrix element of Q_α, which now equals

$$\langle m, \vec{p} | Q_\alpha | m-\alpha, \vec{p}' \rangle = C(\alpha, m) F^{(\alpha)}(p) \delta(\vec{p}' - \vec{p}) \quad (2.3)$$

and, in the particular case of the highest weight

$$\langle M, \vec{p} | Q_\alpha | M-\alpha, \vec{p}' \rangle = \sqrt{M;\alpha} \, F^{(\alpha)}(p) \delta(\vec{p}' - \vec{p}) \quad (2.3')$$

the deviation of $F^{(\alpha)}$ from unity being a measure of the symmetry breaking.* It is then clear that the above-defined quantity $F^{(\alpha)}$ is simply connected with the quantity $G^{(\alpha)}$ considered in I and II (which in the limit of zero momentum transfer is the renormalized coupling constant), $F^{(\alpha)} = G^{(\alpha)}/G_0^{(\alpha)}$. In addition Q_α now has also non-vanishing matrix elements between one- and many-particle states. The reason for this is that a multiplet of particles transforming (in the symmetry limit) as an irreducible representation of our group no longer has well-defined transformation properties under all group rotations, but only under those which leave unchanged the total Hamiltonian, i.e., which correspond to constants of the motion.

We have already said that the deviation of $F^{(\alpha)}$ from unity, i.e., of $G^{(\alpha)}$ from $G_0^{(\alpha)}$, is a measure of the symmetry breaking; another is given by the matrix element of the commutator of the total Hamiltonian H and a "charge" Q_α between one-particle states and many-particle states; it is, of course, clear that Q_α commutes with the symmetry-preserving part H_S of the

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* We notice a slight change between our present notation and the one used in I and II; in I and II, in fact, we have included the unrenormalized coupling constants $G_0^{(\alpha)}$ in the definition of the currents.

Hamiltonian, but no longer with the breaking part H_B. Then the non-vanishing matrix elements of the Q_α's between one- and many-particle physical states can be connected with those of $[Q_\alpha, H]$ because

$$\langle M | Q_\alpha | m \rangle = \frac{\langle M | [Q_\alpha, H] | m \rangle}{E_m - E_M} \tag{2.4}$$

where $|m\rangle$ stands for a m-particle state and E_x is the total energy of the $|x\rangle$ state. In the limit of exact symmetry the numerator in the r.h.s. of (2.4) vanishes being of order ϱ, where ϱ is a dimensionless coupling constant characterizing the strength of the symmetry-breaking Hamiltonian.

In this paper we will examine how, by an appropriate use of the Lie algebra of the group and of completeness, we can treat a wide class of phenomena in order to obtain, as a first approximation, relations valid in the exact symmetry limit, and then the corrections to these relations due to the approximate validity of the symmetry in nature.

As for the applications of our method in this paper we shall be concerned only with SU_3 implications; in so far as SU_3 is concerned we shall employ the de Swart convention [6] for the generators. We define the generalized charges corresponding to the non-zero roots of SU_3 as

$$Q_A^{(\pm)} = \int (J_0)_A^{(\pm)} \, d^3x \sim A^{(\pm)} \tag{2.5}$$

$$A = I, K, L.$$

where the symbol $\sim$ means: "has the same SU_3 transformation properties as". The I-like operators are translation operators in the I-spin subspace, in the sense that they connect states with $\Delta I = 1$; in the same manner the L-like operators are translation operators in the U-spin subspace and the same is for the K-like operators in the V-spin subspace (see Fig. 1). For the generators corresponding to the null roots we choose $Q_3 \sim I_3$ and $Q_Y \sim Y$ (the hypercharge). In some cases we use also the electric charge $Q = Q_3 + \frac{1}{2} Q_Y$.

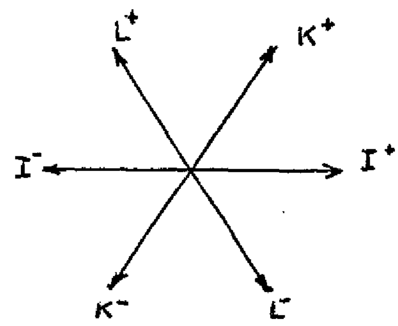

Fig. 1

In what follows we often use the commutators of the total Hamiltonian H with the "charges" $Q_A^{\pm}$. We define:

$$[Q_A^{\pm}, H] = \mp N_A^{\pm} \qquad (A = I, K, L) \tag{2.6}$$

It is clear that, if

$$H = H_S + H_B$$

H_S being the symmetry-preserving part and H_B the symmetry-breaking part of the Hamiltonian, as long as we consider a breaking which transforms under SU_3 like hypercharge, then $N_K^{\pm} \sim K^{\pm}$ and $N_L^{\pm} \sim L^{\pm}$ whereas $N_I^{\pm}$ is zero.

From (2.6) and (2.5) and making use of the dynamical equation

$$[Q_A^{\pm}, H] = i\, \dot{Q}_A^{\pm}$$

we can write

$$N_A^{\pm} = \mp i \int \frac{\partial}{\partial t} (J_0)_A^{\pm}\, d^3x = \mp i \int (\partial_\mu J_\mu)_A^{\pm}\, d^3x$$

and putting

$$(\partial_\mu J_\mu(x))_A^{\pm} = D_A^{\pm}(x)$$

we have

$$[Q_A^{\pm}, H] = \mp N_A^{\pm} = i \int D_A^{\pm}(x)\, d^3x \tag{2.7}$$

where the $D_A^{\pm}$ are Lorentz scalars.

As far as we are concerned, we shall always consider matrix elements of operators between physical states, i.e. eigenstates of the total Hamiltonian; we have then, using (2.7), the following relation between the matrix element of a N_A and the corresponding Q_A:

$$\langle a | N_A^{\pm} | b \rangle = \pm (E_a - E_b) \langle a | Q_A^{\pm} | b \rangle \tag{2.8}$$

E_x being the total energy of the state $|x\rangle$. If $|a\rangle$ and $|b\rangle$ are one-particle states belonging, in the symmetry limit, to the same irreducible representation, then $E_a = E_b$ as long as the Hamiltonian preserves the symmetry, and we find obviously that N_A is zero; but if the symmetry is broken by a part of the Hamiltonian of strength f, we find that $(E_a - E_b)$ is a quantity $O(f)$, i.e. a measure of the broken symmetry. On the contrary if $|a\rangle$ and $|b\rangle$ do not belong to the same representation, then, as already stressed, the matrix elements of Q_A * can be different from zero only if the symmetry is broken and, in this case, they are of the first order in the breaking as one easily sees by reading (2.8) in reverse order:

$$\langle a | Q_A^{\pm} | b \rangle = \pm \frac{\langle a | N_A^{\pm} | b \rangle}{E_a - E_b} \tag{2.9}$$

and noting that N_A is of order f by its definition, whereas $E_a - E_b$ has, in this case, nothing to do with the breaking. By using (2.7) we can also write the matrix elements of the Q_A's as

$$\langle a | Q_A^{\pm} | b \rangle = -i (2\pi)^3 \frac{\langle a | D_A^{\pm}(0) | b \rangle}{E_a - E_b} \delta(\vec{p}_a - \vec{p}_b) \tag{2.10}$$

a form which will prove to be very useful and often employed.

Consider now a physical operator M, whose matrix elements are measurable, and assume that it has well-defined transformation properties under group rotations, say

$$[Q_A , M] = M_A . \tag{2.11}$$

M_A being determined from the group algebra. One can then obtain

* In the following, we shall call such matrix elements the "off-symmetry" matrix elements.

relations between the matrix elements of M and those of M_A. Taking the matrix element of (2.11) between two suitable one-particle states $|a\rangle$ and $|a'\rangle$, which, of course, are taken to belong to the same irreducible representation in the symmetry limit, and using completeness we have

$$\langle a|M_A|a'\rangle = \\ = \Sigma_\alpha \left\{ \langle a|Q_A|\alpha\rangle\langle\alpha|M|a'\rangle - \langle a|M|\alpha\rangle\langle\alpha|Q_A|a'\rangle \right\} \tag{2.12}$$

Of course, in the symmetry limit only the one-particle intermediate states $|a_i\rangle$ belonging to the same representation as $|a\rangle$ and $|a'\rangle$ contribute to the sum; we can then write (2.12) in the form

$$\langle a|M_A|a'\rangle = \\ = \Sigma_i \left\{ \langle a|Q_A|a_i\rangle\langle a_i|M|a'\rangle - \langle a|M|a_i\rangle\langle a_i|Q_A|a'\rangle \right\} + C \tag{2.13}$$

where, calling $\not\Sigma_\alpha$ the sum over all physical states which do not belong to the same irreducible representation as a and a',

$$C = \not\Sigma_\alpha \left\{ \langle a|Q_A|\alpha\rangle\langle\alpha|M|a'\rangle - \right. \\ \left. - \langle a|M|\alpha\rangle\langle\alpha|Q_A|a'\rangle \right\} \tag{2.14}$$

is zero in the symmetry limit and should be regarded as a (small) correction term to the relation

$$\langle a|M_A|a'\rangle = \\ = \Sigma_i \left\{ \langle a|Q_A|a_i\rangle\langle a_i|M|a'\rangle - \langle a|M|a_i\rangle\langle a_i|Q_A|a'\rangle \right\} \tag{2.15}$$

valid as a first approximation.

With the aid of (2.10) the correction term can be written as

$$C = i(2\pi)^3 \not\Sigma_\alpha \left\{ \frac{\langle a|D_A(0)|\alpha\rangle}{E_\alpha - E_a}\langle\alpha|M|a'\rangle\, \delta(\vec{p}_\alpha - \vec{p}_a) \right. \\ \left. - \langle a|M|\alpha\rangle \frac{\langle\alpha|D_A(0)|a'\rangle}{E_{a'} - E_\alpha}\, \delta(\vec{p}_\alpha - \vec{p}_{a'}) \right\} \tag{2.16}$$

The breaking of the symmetry is explicitly taken into account by $D_A(\omega)$ and therefore if we are not interested in still higher order corrections in the symmetry breaking we may take the symmetry limit values of all other quantities. This means, for instance, that the mass of the particle a can be considered as equal to that of particle a' and that the matrix elements of M in (2.16) may be calculated in the symmetry limit. This is a consistent procedure when the symmetry breaking is not too large. We would like to emphasize, however, that our method is not equivalent to a perturbation theoretical one in that we make use of the fact that the states $|a\rangle$ are physical eigenstates to take the physical values for the matrix elements of M in (2.15) and not just a value up to a given order in perturbation theory. The correction (2.14) is then, as we have stated before, caused by the fact that the states $|a\rangle$ do not transform like an irreducible representation of the symmetry group, but contain admixtures to all orders in f of other representations.

3. SUM RULES

From the formula (2.13) we can, by specifying the nature of the M operator, obtain a large number of sum rules connecting the various matrix elements of M with those of M_A.

We shall in this section examine the most interesting results we can obtain if we choose for M some particular operators.

(i) First of all, we can identify M with another generator or, better, generalized charge $Q_{A'}$; in such a case, the commutation rule

$$[Q_A , Q_{A'}] = C_{AA'}^{A''} Q_{A''} \tag{3.1}$$

(the $C_{AA'}^{A''}$ being the structure constants of the algebra) allows us to obtain relations between coupling constants. In particular, in I and II it was shown how to obtain relations between the bare coupling constant of a current and the renormalized one by considering the commutator of opposite charges. Taking the commutator

$$[Q_\alpha , Q_{-\alpha}] = \alpha^i Q_i \qquad (3.2)$$

between physical states corresponding to the highest weight of a given irreducible representation one has

$$\langle M(p)|[Q_\alpha , Q_{-\alpha}]|M(p')\rangle = \alpha^i M_i \, \delta(\vec{p}-\vec{p}') \qquad (3.3)$$

and then, inserting in the commutator a complete set of intermediate physical states, if follows that

$$\{F^{(\alpha)}(p)\}^2 \, \delta(\vec{p}-\vec{p}') + \delta\{F^{(\alpha)}(p)\}^2 = \delta(\vec{p}-\vec{p}') \qquad (3.4)$$

where the term in $\{F^{(\alpha)}\}^2$ is the contribution of the one-particle intermediate state corresponding to the weight $M-\alpha$ (see formula (2.3')) and $\delta\{F^{(\alpha)}\}^2$ is given by the contribution of all other states in the completeness relation. As it has been discussed in I and II, the fact that the matrix elements of Q_α between an one-particle and a many-particle state is $O(\ell)$ implies that the deviation of $\{F^{(\alpha)}\}^2 = \{G^{(\alpha)}/G_0^{(\alpha)}\}^2$ from unity is $O(\ell^2)$ reproducing the result of ADEMOLLO and GATTO [7]. An examination of the correction term δF^2 then allows us to determine the magnitude of this deviation. In II we have done, as an example, the explicit evaluation of the renormalization of the strangeness changing vector current due to the breaking of SU_3 symmetry under some simplifying assumptions (as the one of taking into account only the lowest mass intermediate states and so on) obtaining for δF^2 the value of 0.067 which leads us to the conclusion that the renormalization effect due to the breaking of SU_3 does not change the universality relation in any remarkable way.

(ii) The second case we shall consider is that in which M is an N_A-like operator, i.e. the case in which M itself is a commutator of a charge Q_A and the total Hamiltonian; this allows us to obtain relations, valid at the first order in the breaking, among the energies of the various particles belonging to a given supermultiplet in the symmetry limit. Taking suitable limits one obtains then "mass formulae". Interesting results can also be obtained taking, instead of N_A, its "density" D_A. Considering the commutator between a Q_A and a D_A, we shall do sum rules which interconnect directly the masses of the particles. Again in the symmetry limit we are led to the SU_3 mass formulae. The main difference between these relations and the ones obtained by considering N_A-like operators is that in this case we have no "a priori" choice between linear and quadratic mass formulae. Using the commutator between Q_A and N_A we obtain linear or quadratic mass formulae, for both bosons and fermions, as different limits of our energy relations. This is a consequence of working with no invariant operators.

On the contrary, if we take the commutator between a Q_A and a D_A we are led to a covariant expression which now involves masses for fermions and squared masses for bosons.

These two cases will be discussed in Section 6 where we show how, as a simple application of our method, one can derive the classical SU_3 mass formulae and discuss the possibility of evaluating the $O(f^2)$ corrections. An explicit evaluation of the corrections is also done for the case of pseudoscalar mesons.

(iii) Fruitful information can also be obtained by considering the case in which M represents a current. A previous discussion of such a type of commutator has been done in I. In Section 7 of the present paper we shall discuss the particular case the electromagnetic current and we shall see how our method enables us to obtain relations among the form factors of different particles, valid in the symmetry limit and their corrections due to the breaking.

As a particular case, we easily obtain the classical SU_3 relations among magnetic moments. We notice that the corrections are in this case of the first order in the strength of the breaking due to the fact that while the off-symmetry matrix elements of the Q's are small, the same does not happen for the corresponding currents which could have off-symmetry matrix elements different from zero also in the symmetry limit.

The sum rules we obtain in the various cases have a common structure as we have shown in the general treatment of Section 2. All our sum rules are of the type (2.13), i.e., we have a relation valid as a first approximation in the symmetry limit and a correction term which takes into account the breaking of the symmetry. Clearly, any relation of (2.13) type actually constitutes a continuous set of sum rules depending on which value we take for the momenta of the considered external particles. A complete discussion of our sum rules can thus not be done without an examination of the various frames of reference.

The problem of the dependence of our relations on the common momentum $\vec{p}$ (and consequently of the best sum rule) is a display of the fact that the method based on the introduction of the energy denominators gives a non-covariant separation between the single- and many-particle contributions. In other words, though the choice of the frame of reference does not change the physical content of the sum rule, it gives a different splitting between the zero order terms and the corrections. In particular, starting from the same relation, one can obtain sum rules which look formally different by taking different values of $\vec{p}$ *. In the following section we shall then be concerned with the problem of the choice of the frame of reference and we shall see that in some cases, there exists an "a priori" frame in which one can define the best sum rule, i.e. the one for which the correction is smallest.

4. THE CHOICE OF THE FRAME OF REFERENCE

Clearly, a detailed discussion of the correction (2.14) and its explicit evaluation depends first of all on the nature of the operator M. We shall then distinguish between the various cases treated in Section 3.

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* This occurs for instance in the case of mass formulae obtained from the commutator $[Q_A, N_A]$ (see Section 6) where the linear one corresponds to $\vec{p} = 0$ and the quadratic one to $|\vec{p}| \to \infty$.

(a) We shall first refer to the case in which M is itself a "charge". If M is a charge, $M = Q_A$, so is M_A, as the "charges" satisfy the commutation relation of the group algebra. A particular case of that type has been discussed in detail in I, where it was shown that the correction was smallest in the frame $|\vec{p}| \to \infty$ ($\vec{p}$ being the three-momentum of the external particle). We shall merely sketch the argument of I : if $M = Q_A$, the correction term (2.14), taking into account (2.10), can be written as

$$C = \not{\Sigma}_\alpha \left(C'_\alpha - C''_\alpha \right) \delta(\vec{p}_a - \vec{p}_{a'}) \tag{4.1}$$

where

$$C' = (2\pi)^6 \frac{\langle a | D_A(o) | \alpha \rangle \langle \alpha | D_{A'}(o) | a' \rangle}{(E_\alpha - E_a)^2} \delta(\vec{p}_\alpha - \vec{p}_a) \tag{4.2}$$

and analogously for C''_α, where, as previously said, we have put $E_{a'} = E_a$. Now, for kinematical reasons

$$\langle a | D_a(o) | \alpha \rangle = \frac{d_A^{a\alpha}(\Delta^2)}{\sqrt{4 E_a E_\alpha}} \frac{1}{(2\pi)^3} \tag{4.3}$$

where $\Delta^2 = (p_a - p_\alpha)^2$ and $d_A^{a\alpha}(\Delta^2)$ is a Lorentz invariant function; then, dropping the δ functions, C' can be written in the form

$$C' = \frac{d_A^{a\alpha}(\Delta^2)\, d_{A'}^{\alpha a'}(\Delta^2)}{m_\alpha^2 - m_a^2} \left(\frac{E_\alpha + E_a}{\sqrt{4 E_\alpha E_a}} \right)^2 \tag{4.4}$$

where we have taken into account that $E_a = (m_a^2 + \vec{p}^{\,2})^{1/2}$ and, as a consequence of the δ function $E_\alpha = (m_\alpha^2 + \vec{p}^{\,2})^{1/2}$. The kinematical factor in the brackets reaches its minimum value one for $|\vec{p}| \to \infty$ and its maximum of $(m_\alpha + m_a)^2 / 4 m_\alpha m_a$ for $\vec{p} \to o$ and $d_A^{a\alpha}(\Delta^2)$ is expected to be an increasing function of the time-like variable $\Delta^2 = (p - p_\alpha)^2 = (E_\alpha - E_a)^2$, so that the minimum of $d_A^{a\alpha}(\Delta^2)$ should also be reached at $|\vec{p}| \to \infty$, i.e. when $\Delta^2 \to o$.

Unfortunately the effect of the kinematical factors is not always so unambiguous, as we shall see explicitly when we treat the case of the mass formulae. The above discussion was presented only as an example of the type of analysis which should be performed prior to making an explicit evaluation of the corrections due to intermediate many-particle states.

This particular case, in which the frame $|\vec{p}| \to \infty$ is a privileged one, has been extensively studied in Ref. (3). Nevertheless, we will treat it also here as a good example of the method for calculating the corrections of the form (2.14) to a sum rule in the frame $|\vec{p}| \to \infty$.

If we limit ourselves to the two-particle intermediate state contributions, then Eq. (4.2) becomes

$$C'_\alpha = (2\pi)^6 \int \frac{\langle a(p)|\, D_A \,|\alpha_1(p_1)\,\alpha_2(p_2)\rangle \langle \alpha_1(p_1)\,\alpha_2(p_2)|\, D_{A'} \,|a'(p)\rangle}{(E_1 + E_2 - E_a)^2}\, d^3p_1\, d^3p_2\, \delta(\vec{p}_1 + \vec{p}_2 - \vec{p}) \tag{4.5}$$

$$= \frac{1}{(2\pi)^3}\frac{1}{2E_a}\int \frac{d^3p_1\, d^3p_2}{4\, E_1\, E_2}\, \frac{\delta(\vec{p}_1 + \vec{p}_2 - \vec{p})}{(E_1 + E_2 - E_a)^2}\, \phi^\alpha_{inv}$$

ϕ^α_{inv} being a Lorentz invariant function depending on the invariants of the problem. To take the limit $|\vec{p}| \to \infty$ we transform (4.5) to a more useful form by means of the substitution

$$P = p_1 + p_2 \;; \qquad q = p_1 - p_2$$

and we choose as invariants

$$s = P^2; \qquad \Delta^2 = (P - p)^2; \qquad p\cdot q\,.$$

By integrating over d^3P, we rewrite (4.5) as

$$C'_\alpha = \frac{1}{2}\frac{1}{(2\pi)^3}\frac{1}{2E_a}\int \frac{ds}{\sqrt{\vec{p}^2 + s}}\, \frac{1}{(\sqrt{\vec{p}^2 + s} - \sqrt{\vec{p}^2 + m_a^2})^2}\, \mathcal{J}^\alpha(s) \tag{4.6}$$

where

$$\mathcal{J}^\alpha(s) = \int d^4q\; \delta\{(P+q)^2 - 4m_1^2\}\, \delta\{(P-q)^2 - 4m_2^2\}\, \theta(P_0 + q_0)\, \theta(P_0 - q_0)\, \phi^\alpha(s, \Delta^2, pq)$$

In the limit $|\vec{p}| \to \infty$ we have $\Delta^2 = 0$; we can then evaluate the invariant integral $\mathcal{J}^\alpha$ in any frame where $\Delta^2 = 0$. In particular, we can choose the frame $\vec{p} = 0$, $p_0 = m_a$; in that frame, from $0 = \Delta^2 = (P-p)^2$, we have $P_0 = (s+m_a^2)/2m_a$ and from $s = P_0^2 - \vec{P}^2$ follows $|\vec{P}| = (s-m_a^2)/2m_a$. In this way we obtain in the limit $|\vec{p}| \to \infty$

$$\mathcal{J}^\alpha(s) = \frac{\pi}{2}\,\frac{1}{s-m_a^2}\int_{\eta_1}^{\eta_2} d\eta\;\phi^\alpha(s,\Delta^2=0,\,p\cdot q=\eta)$$

$$\eta_{1,2} = \frac{s+m_a^2}{2s}(m_1^2-m_2^2) \mp \frac{s-m_a^2}{2s}\sqrt{\{s-(m_1-m_2)^2\}\{s-(m_1+m_2)^2\}}$$

and taking the limit in (4.6), we finally obtain

$$\lim_{|\vec{p}|\to\infty} C'_\alpha = \frac{\pi}{2}\,\frac{1}{(2\pi)^3}\int_{(m_1+m_2)^2}^{\infty}\frac{ds}{(s-m_a^2)^3}\int_{\eta_1}^{\eta_2} d\eta\;\phi^\alpha(s,\Delta^2=0,\,p\cdot q=\eta) \qquad (4.7)$$

The above formula is useful for numerical computations, but it is surely not the most elegant one. Writing down the limit of (4.6) as

$$\lim_{|\vec{p}|\to\infty} C'_\alpha = \frac{1}{(2\pi)^3}\int\frac{ds}{(s-m_a^2)^2}\,\lim_{|\vec{p}|\to\infty}\mathcal{J}^\alpha(s)$$

using the formal equality

$$\int ds = \frac{2m_a^2}{\pi}\int\frac{d^4P}{P^2-m_a^2}\;\delta\{(P-p)^2\}$$

and re-introducing our original variable p_1 and p_2 one has finally

$$\lim_{|\vec{p}|\to\infty} C'_\alpha = \frac{4m_a^2}{(2\pi)^4}\int\frac{d^4p_1\,d^4p_2}{\{(p_1+p_2)^2-m_a^2\}^3}\;\delta\{(p_1+p_2-p)^2\}\;\delta(p_1^2-m_1^2)\;\delta(p_2^2-m_2^2)\cdot$$

$$\cdot\,\theta(p_{10})\;\theta(p_{20})\;\phi^\alpha\{(p_1+p_2)^2,(p_1+p_2-p)^2,(p_1-p_2)\cdot p\}\,. \qquad (4.8)$$

This is a covariant expression for the two-particle contribution to the correction at $|\vec{p}| = \infty$ and this form can be immediately generalized to many-particle intermediate state contributions.

(b) The second case we are interested in is that in which M is itself a commutator between a "charge" $Q_{A'}$, and the total Hamiltonian: $\quad M = [H, Q_{A'}] = N_{A'}$

The expression for the correction in this case is analogous to the one of the previous case; the only difference lies in the fact that in the formula corresponding to (4.2) only one energy denominator appears

$$C'_\alpha = (2\pi)^6 \frac{\langle a(p)| D_A | \alpha \rangle \langle \alpha | D_{A'} | a'(p) \rangle}{E_\alpha - E_a} \delta(\vec{p} - \vec{p}_\alpha) \tag{4.9}$$

so that the correction behaves like $1/p$ as $|\vec{p}| \to \infty$. Of course, also the fundamental term which constitutes the particular case of (2.15), behaves like $1/p$ as $|\vec{p}| \to \infty$. We are thus interested in the evaluation of the $\lim_{|\vec{p}| \to \infty} p\,C$ and it is easily shown that the formula

$$\lim_{|\vec{p}| \to \infty} \frac{2 m_a^2}{(2\pi)^4} \int \frac{d^4 p_1\, d^4 p_2}{\{(p_1+p_2)^2 - m_a^2\}^2} \delta\{(p_1+p_2-p)^2\}\, \delta(p_1^2 - m_1^2)\, \delta(p_2^2 - m_2^2) \cdot$$

$$\cdot\, \theta(p_{1_0})\, \theta(p_{2_0})\, \phi^\alpha\{(p_1+p_2)^2, (p_1+p_2-p)^2, (p_1-p_2)\cdot p\} \tag{4.10}$$

analogous to (4.8) holds. However, the frame $|\vec{p}| \to \infty$ does not have such a preferred character as in the preceding case and one should also be interested in the frame $\vec{p} = 0$; taking the $\vec{p} = 0$ limit at the stage analogous to the (4.6) one obtains immediately

$$\lim_{|\vec{p}| \to 0} C'_\alpha = \frac{\pi}{8 m_a} \frac{1}{(2\pi)^3} \int \frac{ds}{s^{3/2}} \frac{\sqrt{\{s-(m_1-m_2)^2\}\{s-(m_1+m_2)^2\}}}{\sqrt{s} - m_a} \phi^\alpha_0(s) \tag{4.11}$$

the invariant function ϕ^α in the $|\vec{p}| = 0$ limit becoming a function of s only.

(b') If, instead of N_A, we consider its density $D_A(o)$, the expression we obtain for the corrections is slightly different because we have no momentum conservation between $a(p)$ and $a'(p')$. We obtain in this case

$$C = \Sigma_\alpha (C'_\alpha - C''_\alpha) \tag{4.12}$$

with

$$C'_{\alpha} = i(2\pi)^3 \frac{\langle a(p)|\, D_A(0)|\, \alpha\rangle\langle\alpha|\, D_{A'}(0)\,|\, a'(p')\rangle}{E_{\alpha} - E_a}\, \delta(\vec{p} - \vec{p}_{\alpha}) \tag{4.13}$$

$$C''_{\alpha} = -i(2\pi)^3 \frac{\langle a(p)|\, D_{A'}(0)|\, \alpha\rangle\langle\alpha|\, D_A(0)\,|\, a'(p')\rangle}{E_{\alpha} - E_a}\, \delta(\vec{p}' - \vec{p}_{\alpha})$$

If we choose the system $\vec{p}' = \vec{p}$ the correction reduces (apart from a $-i(2\pi)^3$ factor) to the above-discussed form (2.9) and then we can use (4.10) for the explicit evaluation of the corrections in the frame $|\vec{p}| \to \infty$. A further discussion on the choice of the frame in this case will be carried out in Section 6.

(c) Finally we shall examine the case in which the operator M is not directly connected to the breaking of the symmetry, i.e. the case in which the matrix elements of M between one- and many-particle states are not of order ℓ , but are different from zero also in the exact symmetry limit. This is the case in which M represents, for instance, the electromagnetic current. A two-particle intermediate state correction term is given by (see (2.16)):

$$C'_{\alpha} = i(2\pi)^3 \frac{\langle a(p)|\, D_A(0)|\, \alpha_1(p_1)\, \alpha_2(p_2)\rangle}{E_1 + E_2 - E_a}\langle\alpha_1(p_1)\,\alpha_2(p_2)|\, M\,|\, a'(p')\rangle\;\; \delta(\vec{p}_1 + \vec{p}_2 - \vec{p})$$

The answer, if any, to the question in which frame the correction is the smallest, clearly depends in the asymptotic behaviours of M the matrix elements of M , and nothing can be said until we specify the nature of M .

5. DISPERSIVE EVALUATION OF THE CORRECTIONS

For an explicit evaluation of the corrections to our formulae, we need to be able to calculate the ϕ^{α} functions, which in turn requires a knowledge of the matrix elements of the function $D_A(0)$ between one- and many-particle states. We shall limit ourselves to the contribution of two-particle states, though our treatment is, in principle, generalizable.

We start by considering the Lorentz invariant quantity*

* $\phi^{\alpha} \equiv \Phi^{\beta_1 \beta_2}$ can be ~~written~~ as $\phi^{\beta_1 \beta_2} = R\, \tilde{R}$

$$R = \sqrt{(2\pi)^9\, 8\, E(p)\, E(p_1)\, E(p_2)}\ \langle a(p) |\ D_A(0) \,|\, \beta_1(p_1)\, \beta_2(p_2) \rangle \qquad (5.1)$$

In this expression it is understood (see Eq. (4.9)) that R is multiplied by a δ^3 function which guarantees three-momentum conservation $\vec{p} = \vec{p}_1 + \vec{p}_2$. To visualize the fact that there is no four-momentum conservation we introduce a time-like vector $\Delta = (p_1 + p_2 - p)$; $\Delta^2 = (E_1 + E_2 - E)^2$. Then we consider Δ^2 as the $(\text{mass})^2$ of an effective spurion which carries off the energy, and which is described by the field $D_A(0)$ *, so that it has the transformation properties of D_A under internal and spatial symmetries. The fact that it is coupled to our system is a display of the breaking of the symmetry and, of course, the "coupling constant" is of the order of ℓ . In other words, R can be considered as describing the scattering process $a +$ spurion $\rightarrow \beta_1 + \beta_2$

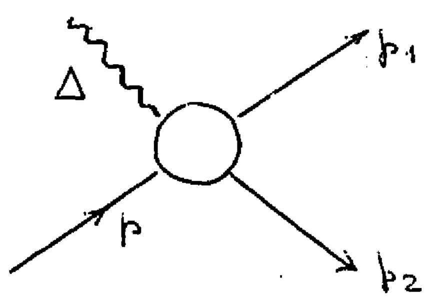

Fig. 2

R will depend on the invariant variables

$$s = (p_1 + p_2)^2 , \quad t = (p_1 - p)^2 , \quad u = (p_2 - p)^2$$

$$p_1 + p_2 = p + \Delta , \quad s + t + u = m_a^2 + m_1^2 + m_2^2 + \Delta^2 \qquad (5.2)$$

and to evaluate it we shall use a dispersion-like approach. This means that we assume for R analyticity properties in s , t , u with the poles and cuts required by unitarity. Then we shall do a "pole

*For a model in which, for instance, the divergence of the strangeness changing vector current is proportional to the field of the κ-meson, see Ref. (8).

approximation" by retaining only the pole contributions. In this way, the final result will depend on physical parameters only (i.e., physical matrix elements evaluated for special values of the kinematical variables) and, in particular, it will be shown that R in this approximation can be expressed in terms of the physical mass differences, without any hypothesis on the transformation properties of the symmetry-breaking Hamiltonian.

To make things clearer it is convenient to work out an explicit example. Let us consider the case where $D_A = D_K^+$ and a is "π" β_1 a "ρ" and β_2 a "K" meson dropping charge indices. As to the analyticity properties of R in this case, we can say that in the variable s there are a pole at $s = m_K^2$ and a cut starting from $(m_K + 2m_\pi)^2$, in the variable u a pole at $u = m_\pi^2$ and a cut for $u \geq (3m_\pi)^2$, in the variable t a pole at $t = m_{K^*}^2$ and a cut for $t \geq (m_{K^*} + 2m_\pi)^2$. Graphically, the situation is pictured in Fig. 3

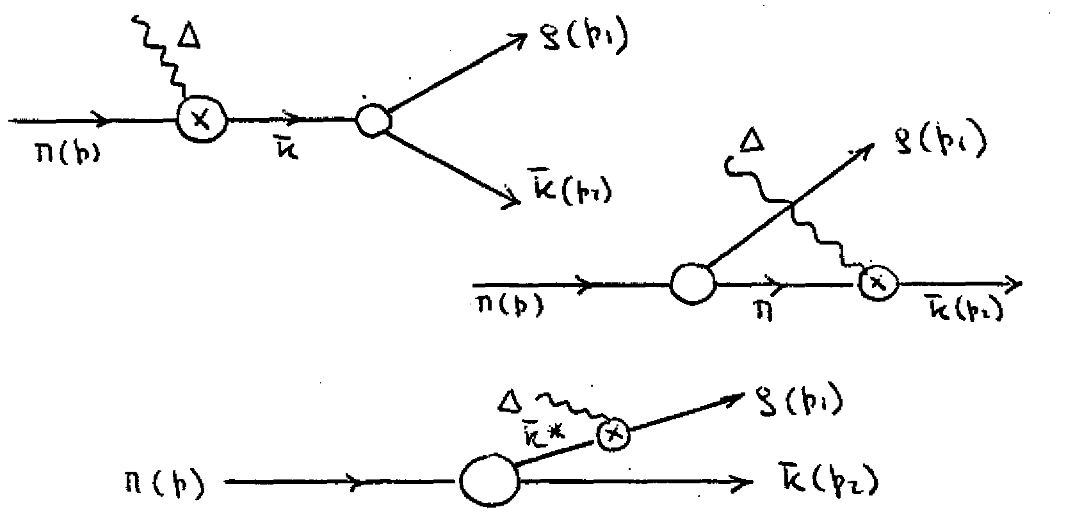

Fig. 3

As we mentioned at the beginning, we shall limit ourselves to the pole approximation in all applications. This approximation is in agreement with the whole spirit of our calculation and we believe it can give a reasonable indication for the total correction.

Now, on invariance grounds, we may write

$$R(s,t,u) = (p \cdot \varepsilon^{(\rho)})\, R_1 + (p_2 \cdot \varepsilon^{(\rho)})\, R_2 \tag{5.3}$$

(where $\varepsilon_\mu^{(\rho)}$ is the ρ -meson polarization vector)

and for each R_i we shall take an expression of the form

$$R_i \simeq \frac{A_i}{s - m_K^2} + \frac{B_i}{t - m_{K^*}^2} + \frac{C_i}{u - m_\pi^2} \qquad i = 1, 2 \tag{5.4}$$

where A_i, B_i, and C_i are the residua at the poles and they can be expressed in terms of matrix elements of physical operators between physical states. In particular, they are given by matrix elements of D_K and of strong currents between one-particle states.

To be more definite let us study in detail the contribution at the $s = m_K^2$ pole. Using standard reduction techniques, we find

$$\langle \pi | D_K^+(0) | \bar{K} \rho \rangle = i \int d^4x \, \langle \pi | [D_K^+(0), J_\mu^{(\rho)}(x)] | \bar{K} \rangle \, \frac{\theta(-x_0)\varepsilon_\mu e^{-ip_1 x}}{\sqrt{(2\pi)^3\, 2E_1}} \tag{5.5}$$

and the corresponding discontinuity around the $\bar{K}$ -pole at $s = m_K^2$ is

$$\frac{\pi (2\pi)^{3/2}}{\sqrt{2E_1}} \sum_{\alpha = \bar{K}} \delta^4(p_1 + p_2 - p_\alpha) \langle \pi | D_K^+ | \bar{K}_\alpha \rangle \langle \bar{K}_\alpha | J \cdot \varepsilon | \bar{K} \rangle \tag{5.6}$$

Next we introduce the definitions

$$\langle \pi(p) | D(0) | \bar{K}(p_\alpha) \rangle = \frac{1}{(2\pi)^3} \frac{1}{\sqrt{4EE_\alpha}} F\{p_\alpha^2, m_\pi^2, (p_\alpha - p)^2\}$$

$$\langle \bar{K}(p_\alpha) | J \cdot \varepsilon | \bar{K}(p_2) \rangle = \frac{1}{(2\pi)^3} \frac{1}{\sqrt{4E_2 E_\alpha}} (\varepsilon \cdot p_2)\, G\{m_K^2, p_\alpha^2, (p_2 - p_\alpha)^2\} \tag{5.7}$$

where F and G are the form factors describing the corresponding ($K\pi$ -spurion) and $(KK\rho)$ vertices. We find for Eq. (5.6)

$$\frac{\pi (2\pi)^{3/2}}{\sqrt{2E_1}} \delta(s - m_K^2)\, F(s = m_K^2, m_\pi^2, \Delta^2)\, G_K(m_K^2, s = m_K^2, m_\rho^2) \tag{5.8}$$

and the coefficient of the δ -function is just the residuum at the pole. The quantity $G_K(m_K^2, m_K^2, m_\rho^2) = g_{\rho KK}$ is the ρKK coupling constant while $F(m_K^2, m_\pi^2, \Delta^2)$ will be discussed later. Using analogous considerations the following can be derived in a straightforward way:

$$R_1 = \frac{F(m_K^2, m_\pi^2, \Delta^2)}{m_K^2 - s} g_{\rho KK} + \frac{F(m_\rho^2, m_{K^*}^2, \Delta^2)}{m_{K^*}^2 - t} \left\{1 + \frac{m_\pi^2 - m_K^2}{m_{K^*}^2}\right\} g_{K^* K\pi}$$
$$(5.9)$$
$$R_2 = \frac{F(m_\pi^2, m_K^2, \Delta^2)}{m_\pi^2 - u} g_{\rho\pi\pi} + \frac{F(m_\rho^2, m_{K^*}^2, \Delta^2)}{m_{K^*}^2 - t} \left\{1 + \frac{m_\pi^2 - m_K^2}{m_{K^*}^2}\right\} g_{K^* K\pi}$$

Our final step is the evaluation of F. To this end, let us consider

$$\langle \pi(p) | J_\mu^{(K^+)} | \bar{K}(p_d) \rangle = \frac{1}{(2\pi)^3} \frac{1}{\sqrt{4 E E_d}} \left\{ (p_d + p)_\mu F_1(\Delta^2) + (p_d - p)_\mu F_2(\Delta^2) \right\} \quad (5.10)$$

where, as a manifestation of the non-conservation of the current $J_\mu^{(K^+)}(x)$, we have two form factors, $F_1(\Delta^2)$ and $F_2(\Delta^2)$. The matrix element of D_K^+ then equals

$$\langle \pi(p) | D_K^+ | \bar{K}(p_d) \rangle = \frac{i}{(2\pi)^3} \frac{1}{\sqrt{4 E E_d}} \left\{ (m_\pi^2 - m_K^2) F_1(\Delta^2) - \Delta^2 F_2(\Delta^2) \right\} \quad (5.11)$$

By comparison, we obtain

$$i F(m_K^2, m_\pi^2, \Delta^2) = (m_K^2 - m_\pi^2) F_1(\Delta^2) + \Delta^2 F_2(\Delta^2) \quad (5.12)$$

In particular, in the limit $|\vec{p}| \to \infty$, $\Delta^2 \to 0$ we obtain

$$i F(m_K^2, m_\pi^2, 0) = (m_K^2 - m_\pi^2) F_1(0) \quad (5.13)$$

where $F_1(0) = C_{K\pi}^{K^+} \tau$ (τ being the renormalization ratio G/G_0) and then, in the symmetry limit, reduces to the simple $C_{K\pi}^{K^+}$ Clebsch-Gordan coefficient*.

---

* In this respect, it is worthwhile mentioning the analogous result which holds for spin-$\frac{1}{2}$ particles. If we define

$$\langle p_2 | D_A | p_1 \rangle = \frac{1}{(2\pi)^3} \sqrt{\frac{m_1 m_2}{E_1 E_2}}\, \bar{u}_2 u_1 F^A[m_1^2, m_2^2, (p_1 - p_2)^2] \quad (5.11')$$

it is easy to derive the relation

$$i F^A(m_1^2, m_2^2, 0) = (m_1 - m_2) C_{21}^A \tau^A \quad (5.13')$$

With Eq. (5.13) our final goal is achieved and, in the limit $|\vec{p}| \to \infty$ we obtain for R_i (neglecting higher-order terms in the mass difference)

$$R_1 = i\, C^{\kappa}_{\kappa\pi}\; g_{\rho\kappa\kappa} \frac{(m_\kappa^2 - m_\pi^2)}{s - m_\kappa^2} + i\, C^{\kappa}_{\kappa^*\rho}\; g_{\kappa^*\kappa\pi} \frac{(m_\rho^2 - m_{\kappa^*}^2)}{t - m_{\kappa^*}^2}$$

$$R_2 = i\, C^{\kappa}_{\kappa\pi}\; g_{\rho\pi\pi} \frac{(m_\pi^2 - m_\kappa^2)}{u - m_\pi^2} + i\, C^{\kappa}_{\kappa^*\rho}\; g_{\kappa^*\kappa\pi} \frac{(m_\rho^2 - m_{\kappa^*}^2)}{t - m_{\kappa^*}^2} \qquad (5.14)$$

The fact that D_κ is an operator which is proportional to the symmetry breaking is reflected by the R_i's being proportional to the difference of masses of particles belonging to the same representation, namely π and κ and ρ and κ^*. It is rather remarkable, however, that no assumption has been made on how the symmetry is broken in calculating R (at least in the pole approximation). All that has been used is the fact that our states are eigenstates of the total Hamiltonian. Regarding the possibility of improving our calculations, we notice that a simple way of taking into account the higher-lying states would be to ascribe a s-dependence to the form factors of Eq. (5.7). In the same way we could "dress" every vertex introducing the final state interactions.

6. ON THE MASS FORMULAE AND THEIR CORRECTIONS

6.1 It is well known that in the symmetry limit the masses of all the particles in a given supermultiplet should be exactly the same, but if in the Hamiltonian a breaking effect of strength f is present, then the masses of the components will differ by a quantity $O(f)$. There exist, however, particular linear combinations of such mass differences which are valid up to a higher order in f; such combinations are the so-called mass formulae and their agreement with experimental data is expected to be particularly good (as long as f is not too large) as the corrections are expected to be $O(f^2)$. In particular, believing in SU_3, if one makes some assumptions on the breaking Hamiltonian and treats the breaking as a small perturbation, one gets the well-known relations among the masses of particles of a given supermultiplet [9]:

$$
\begin{aligned}
&4K - 3\eta - \pi = 0 \\
&4K^* - 3\omega_0 - \rho = 0 \\
&2N + 2\Xi - 3\Lambda - \Sigma = 0 \\
&N^* - Y^* = Y^* - \Xi^* = \Xi^* - \Omega^- \, ,
\end{aligned}
\tag{6.1}
$$

where it is common convention to consider the particle symbols as their masses for fermions and as their squared masses for bosons; in the vector meson formula, moreover, one introduces the eighth component of the octet $\omega_0 = \omega \sin\theta + \varphi \cos\theta$, i.e., as a mixture of the physical particles ω and φ [10]. It should, however, be emphasized that these conventions are introduced into the theory from the outside and they are not actually supported by any firm theoretical arguments. In particular, the choice between linear and quadratic mass formulae seems to be rather arbitrary, being really supported only by the agreement with experimental data. We will show in this section how, using suitable commutation relations and completeness, one can obtain the SU_3 mass formulae as a limit of more general formulae.

As mentioned in Section 3, we can obtain mass formulae in two different ways. The first one, based on the consideration of N_A-like operators, provides actually relations which connect the energies [11] of the various constituents of a supermultiplet. Clearly, from these one obtains immediately relations among masses or squared masses (both for fermions amd bosons) by taking suitable limits for the external momenta. It should be emphasized that both formulae, the linear and the quadratic one, have actually the same validity in so far as SU_3 and its breaking are concerned, the difference between the two consisting only in the role plaid by kinematical factors, as remarked in Section 2.

The second method, based on the use of the commutator between a "charge" Q_A and a "divergence" D_A requires the further assumption that the D_A's themselves belong to an octet. The relations we obtain in this way are linear combinations of masses in the fermion case and of squared masses in the boson case; the coefficients, depending on some kinematical factors and "form factors", reduce to the well-known coefficients of the SU_3 mass formulae if we neglect corrections of order higher than two in the symmetry breaking. Thus, we obtain in this case linear mass formulae for fermions and quadratic for bosons, no matter which frame of reference we choose.

Moreover, our method enables us to evaluate explicitly the second order corrections to the mass formulae; however, as we shall see later, it is not easy to establish in which frame of reference the corrections should be expected to be minimal, that is, there are no completely general arguments in favour of a particular frame of reference. Nevertheless, on the basis of some heuristic model we can believe that, at least in the second case, the frame $p \to \infty$ should be preferred.

6.2 We shall derive here the energy sum rules using the above-mentioned method. To reach our goal, we remember now the definition (2.6) of the N_A operators. Then we assume, as usual, that the SU_3 breaking part of the Hamiltonian (as far as the so-called semi-strong interactions are concerned) $\sim$ the hypercharge Y . It is then clear from the group algebra that $N_A \sim Q_A$ and thus

$$[Q_A^{\pm}, N_A^{\pm}]_- = 0 \qquad A = I, K, L. \tag{6.2}$$

If we keep in mind that the operators Q_A and thus also N_A, are translation operators in the A-spin subspace and we work in the V-spin subspace

$$[Q_K^{\pm}, N_K^{\pm}]_- = 0 \tag{6.3}$$

or in the -spin subspace

$$[Q_L^{\pm}, N_L^{\pm}]_- = 0 \tag{6.4}$$

it is clear that one can obtain relations among the energies of the constituents of different I-spin multiplets in a given SU_3 representation. It is simply a matter of taking matrix elements of (6.3) or (6.4) (between suitable one-particle states belonging in the symmetry limit to a given SU_3 representation) using completeness and taking into account, as a first approximation, only the one-particle intermediate states (belonging to the same irreducible representation).

Each single term one obtains in the development commutator, taking into account only the abovementioned states, is clearly $O(\varepsilon)$, as N_A itself is $O(\varepsilon)$. The contribution of the remaining states is $O(\varepsilon^2)$ and it constitutes the correction to the relation obtained which in first approximation (order ε) equals zero.

If now, in the evaluation of the corrections, we limit ourselves to the ε^2 order it is clear that the matrix elements between one-particle states of the "charges" Q_A can be simply taken as given by their symmetry limit since the introduction of $F^{(A)}$ contributes to the corrections only with terms $O(\varepsilon^3)$, the difference between $F^{(A)}$ and 1 being itself $O(\varepsilon^2)$. In other words, we can use (2.2) instead of (2.3) for the matrix elements of Q_A.

As a practical example, we shall consider the case of the pseudoscalar mesons; we consider then, for instance, the matrix element of (6.3) between K^+ and K^- states. Introducing a complete set of physical intermediate states, expliciting the one-particle intermediate state contribution and using (2.8) and (2.14), we have

$$\begin{aligned} 0 = \langle K^+(p)|[Q^+_K, H^+_K]_-|K^-(p')\rangle &= \langle K^+(p)|Q^+_K|\pi^0\rangle \\ \langle\pi^0|H^+_K|K^-(p')\rangle &+ \langle K^+(p)|Q^+_K|\eta\rangle\langle\eta|H^+_K|K^-(p')\rangle \\ &- \langle K^+(p)|H^+_K|\pi^0\rangle\langle\pi^0|Q^+_K|K^-(p')\rangle - \\ &- \langle K^+(p)|H^+_K|\eta\rangle\langle\eta|Q^+_K|K^-(p')\rangle + C \end{aligned} \tag{6.5}$$

so that one finally has

$$4E_K(p) - 3E_\eta(p) - E_\pi(p) = C \tag{6.6}$$

where we have dropped the overall δ function which states that $\vec{p}' = \vec{p}$ and all the energies of the particles should be evaluated for the same value p of the three-momentum; the correction term C is given by

$$C = 2(2\pi)^6 \sum_\alpha \frac{\langle K^+ | D^+_K | \alpha \rangle \langle \alpha | D^+_K | K^- \rangle}{E_\alpha - E_K} \delta(\vec{p}_\alpha - \vec{p}) \qquad (6.7)$$

which clearly shows the $O(f^2)$ character of C and is of the form of (4.9), as discussed in Section 4.

Eq. (6.6) represents a continuous set of mass sum rules, one for each value of the momentum , which contains the linear mass formula as the limit value for $p \to 0$:

$$4 m_K - 3 m_\eta - m_\pi = C_0 \qquad (6.8)$$

$$C_0 = \lim_{p \to 0} C \qquad (6.9)$$

and the quadratic one as the $p \to \infty$ limit

$$4 m^2_K - 3 m^2_\eta - m^2_\pi = C_\infty \qquad (6.10)$$

$$C_\infty = \lim_{p \to \infty} 2pC \qquad (6.11)$$

In order to determine whether (6.8) or (6.10) is the a priori better relation, we shall compare the two corrections. For this purpose, we note that, if we write the physical masses as

$$m_a = m_0 + \delta_a$$

m_0 being the octet bare mass and $\delta_a = O(f)$ the renormalization effect due to the symmetry breaking. (6.8) gives

$$4\delta_K - 3\delta_\eta - \delta_\pi = C_0$$

and (6.10)

$$4\delta_K - 3\delta_\eta - \delta_\pi = \frac{C_\infty}{2m_0} - \frac{1}{2m_0}(4\delta^2_K - 3\delta^2_\eta - \delta^2_\pi)$$

so that we should have

$$C_o = \frac{C_\infty}{2\omega_o} - \frac{1}{2\omega_o}\left(4\delta^2_K - 3\delta^2_\eta - \delta^2_\pi\right)$$

In order to establish if there exists a better a priori mass formula, we shall try to see if the further correction in the $\bar{\delta}^2_\alpha$'s which appear in the expression of the quadratic formula, improves or worsens the value of the correction. We are thus led to compare C_o to $C_\infty/2\omega_o$. Remembering now (4.3) and taking into account the fact that, as a consequence of the Wigner-Eckart theorem, one has

$$d_A^{ab}(\Delta^2) = C_A^{ab}\, d_A(\Delta^2) \tag{6.12}$$

the C_A^{ab} being the SU_3 Clebsch-Gordan coefficients. From (3.7) we can say that C is a sum of terms of the type

$$C' = C_{K^+}^{K^+\alpha}\, C_{K^+}^{\alpha K^-}\, \frac{\{d_{K^+}(\Delta^2)\}^2}{\omega^2_\alpha - \omega^2_K}\, \frac{E_\alpha + E_K}{4E_\alpha E_K} \tag{6.13}$$

where $E_K = (p^2+\omega^2_K)^{1/2}$, $E_\alpha = (p^2+\omega^2_\alpha)^{1/2}$, ω_α being the invariant mass of the intermediate state; and $\Delta^2 = (p_\alpha - p)^2 = (E_\alpha - E_K)^2$ is the squared momentum transfer. From (6.13) it follows that

$$\frac{C_o'}{C'_\infty/2\omega_o} = \frac{\lim\limits_{p\to 0} C}{\lim\limits_{p\to\infty} \frac{2pC}{2\omega_o}} = \left\{\frac{d_{K^+}[(\omega_\alpha - \omega_K)^2]}{d_{K^+}(0)}\right\}^2 \frac{\omega_\alpha + \omega_K}{2\omega_\alpha}\, \frac{\omega_o}{\omega_K} \tag{6.14}$$

As we said in Section 2, it is now reasonable to suppose the "form factors" $d_A(\Delta^2)$ to be increasing functions of Δ^2 when Δ^2 becomes larger and time-like (i.e., when Δ^2 approaches, and subsequently runs the singularity region). In this wasy, $d_A(\Delta^2)$ will reach its minimum value for $\Delta^2 = 0$, i.e. $p \to \infty$. The factor $(d_K[(\omega_\alpha - \omega_K)^2])^2 \cdot (d_{K^+}(0))^{-2}$ in (3.14) will then be greater than one; the factor $(\omega_\alpha + \omega_K)/2\omega_\alpha$ is smaller than one (ω_α being greater than ω_K) whereas $\omega_o/\omega_K \simeq 1$. Thus it is not easy to ascertain a priori when the correction is smallest and consequently conclude whether the linear or the quadratic mass formula is better (at least without making specific hypotheses on the behaviour of the form factor).

An explicit evaluation of the corrections has been made along the lines suggested in Sections 4 and 5, for the squared mass formula as well as for the linear one. In both cases, the numerical value of the calculated correction is in satisfactory agreement with the experimental values. The details of calculation (assumptions, approximation, numerical values) are given in the appendix.

We have discussed in some details the mass formulae for the pseudoscalar meson case. It is, however, clear that exactly the same argument can be given for the other SU_3 mass formulae. Formulae of the (6.3) and (6.4) type taken between physical states belonging in the symmetry limit to other irreducible representations, give relations of the (6.6) type corresponding to the various SU_3 mass formulae.

For instance, taking the matrix element of (6.3) between a proton and a Ξ^- state, one has

$$2E_N(p)+2E_\Xi(p)-3E_\Lambda(p)-E_\Sigma(p)=C_B \qquad (6.15)$$

with

$$C_B=2(2\pi)^6\sum_\alpha \frac{\langle P|D_K^+|\alpha\rangle\langle\alpha|D_K^+|\Xi^-\rangle}{E_\alpha-E}\,\delta(\vec{p}_\alpha-\vec{p}) \qquad (6.16)$$

where we have made in C_B the approximation $E_P=E_\Xi=E=(\omega_0^2+p^2)^{1/2}$ and dropped the δ function stating $\vec{p}_P=\vec{p}_\Xi=\vec{p}$.

For the 3/2 resonances, Eq. (6.3) taken between N^{*++} and Ξ^{*0} and subsequently between Y^{*+} and Ω^-, gives

$$(E_{N^*}-E_{Y^*})-(E_{Y^*}-E_{\Xi^*})=\frac{1}{2\sqrt{3}}C_1$$

$$(E_{Y^*}-E_{\Xi^*})-(E_{\Xi^*}-E_{\Omega^-})=\frac{\lambda}{2\sqrt{3}}C_2$$

where, with the usual conventions

$$C_1=2(2\pi)^6\sum_\alpha \frac{\langle N^*|D_K^+|\alpha\rangle\langle\alpha|D_K^+|\Xi^*\rangle}{E_\alpha-E}\,\delta(\vec{p}_\alpha-\vec{p})$$

$$C_2=2(2\pi)^6\sum_\alpha \frac{\langle Y^*|D_K^+|\alpha\rangle\langle\alpha|D_K^+|\Omega\rangle}{E_\alpha-E}\,\delta(\vec{p}_\alpha-\vec{p})$$

For the vector meson it is then clear that one has a formula like (6.6) substituting the corresponding vector mesons in place of the pseudoscalar mesons; obviously, in order to obtain a good agreement with experimental data one should replace the η with a mixture of ω and φ and the ratio of the mixture determined, as usual, from experimental data, the ω-φ mixing angle not being predicted by SU_3.

To conclude this section, we would like to emphasize that our rules, obtained as a consequence of formulae like (6.3) and (6.4) and completeness, should be valid also if the breaking Hamiltonian does not simply $\sim$ the hypercharge. In fact, the same mass formulae (and the same method for evaluating the corrections) hold for every breaking Hamiltonian such that at least one of the following relations is verified

$$[Q^+_K, [Q^+_K, H]] = 0 \tag{6.17}$$

$$[Q^+_L, [Q^+_L, H]] = 0 \tag{6.18}$$

$$[Q^+_K, [Q^+_L, H]] = 0 \tag{6.19}$$

In particular, when working in U-spin subspace, it is easily recognized that (6.18) is satisfied for every breaking Hamiltonian of the type $H_B \sim \Sigma_\mu (a_\mu + b_\mu Y) Q^\mu$ because $Q \sim Q_3 + \frac{1}{2} Q_Y$ is a U-spin scalar; this fact suggests that the mass formulae written down for particles of the same charge should be valid also if one takes into account the simultaneous breaking of SU_3 (supposed $\sim Y$) and the electromagnetic interaction.

6.3 The second method which allows us to obtain mass formulae is based on the hypothesis that the divergences D_A belong to an octet; we admit in particular the validity of the equal time commutation relations

$$[Q_A^\pm(t), D_A^\pm(\bar{x}, t)] = 0 \tag{6.20}$$

$$A = I, K, L .$$

for every value of $\bar{x}$. We choose for simplicity $\bar{x} = 0$ and $t = 0$ and we work, as in the previous case, in the V-spin subspace taking the matrix element of

$$[Q^{+}_{K}, D^{+}_{K}] = 0 \tag{6.21}$$

between suitable physical states and using completeness.

We start by considering the pseudoscalar meson case. We obtain

$$0 = \langle K^+|[Q^+_K, D^+_K]|K^-\rangle = \langle K^+|Q^+_K|\pi^0\rangle\langle\pi^0|D^+_K|K^-\rangle \tag{6.22}$$

$$- \langle K^+|D^+_K|\pi^0\rangle\langle\pi^0|Q^+_K|K^-\rangle + (\eta \leftrightarrow \pi_0) + C$$

The matrix elements we need are of the type $\langle p_2|Q^+_K|p_1\rangle$ and $\langle p_2|D^+_K|p_1\rangle$. As far as the first one is concerned, we remark that, as done before, we can take for it its symmetric value, the deviations being of the order φ^2. For the second term, we can use Eq. (5.11)

$$\langle Q_2(p_2)|D^+_K|p_1\rangle = \frac{1}{(2\pi)^3}\frac{i}{\sqrt{4E_2E_1}}(m_2^2 - m_1^2)\, C^{K^+}_{21}\, G[(p_2-p_1)^2] \tag{6.23}$$

where $C^{K^+}_{21}$ is the appropriate Clebsch-Gordan coefficient and the form factor $G(0) = \kappa$ (the renormalization ratio). Moreover, for the purpose of simplifying the derivation, we assume $\vec{p}_2 = \vec{p}_1 = \vec{p}$. In so doing, Eq. (6.22) becomes

$$(m^2_\pi - m^2_K)\frac{G_{\pi K}(\Delta^2_\pi)}{\sqrt{4EE_\pi}} + 3(m^2_\eta - m^2_K)\frac{G_{\eta K}(\Delta^2_\eta)}{\sqrt{4EE_\eta}} = O(\varphi^2) \tag{6.24}$$

where

$$\Delta^2_{\pi,\eta} = (E_K - E_{\pi,\eta})^2 = \left(\sqrt{\vec{p}^2 + m^2_K} - \sqrt{\vec{p}^2 + m^2_{\pi,\eta}}\right)^2 \tag{6.25}$$

We note now that the coefficients of the two squared mass differences differ by terms which are of the order φ and which can be collected in the corrections on the r.h.s. Thus Eq. (6.24) gives the well-known mass formula

$$4m^2_K - 3m^2_\eta - m^2_\pi = O(\varphi^2) \tag{6.26}$$

In order to obtain the mass formula for the baryon octet, it is now sufficient to take the commutator (6.21) between a proton and a Ξ^-, apply the standard rules of our game and use Eq. (5.11') for the matrix element of D^+_K between spin-$\frac{1}{2}$ states

$$\langle p_2 | D^+_K | p_1 \rangle = \frac{i}{(2\pi)^3} \sqrt{\frac{m_2 m_1}{E_2 E_1}}\; \bar{u}(p_2) u(p_1) (m_2 - m_1)\, G[(p_2 - p_1)^2] \tag{6.27}$$

which involves linear mass differences. Thus, we get the mass formula

$$2m_{\Xi} + 2m_N - 3m_{\Lambda} - m_{\Sigma} = O(p^2). \tag{6.28}$$

It is important to realize the different role that the p-dependence has in this case and in Subsection (6.2). Here the classical SU_3 mass formulae can be obtained independently of the value of p which comes in only when we discuss the corrections. On the contrary, in the previous section we have actually energy relations and different choices for p can give different "mass formulae".

Finally, we would like to point out that it is possible to do the whole derivation of Eqs. (6.26) and (6.28) taking into account the complete form of $\langle p_2 | Q^+_K | p_1 \rangle$ (i.e., including form factors and kinematical factors) In this case, no kinematical factors appear in the l.h.s. of Eq. (6.28) and the only approximation we make is to take the renormalization ratios $Z = 1$ (which actually contribute to the corrections with $O(p^3)$ terms). In this way, we would get for (6.24) a more complicated expression involving form factors evaluated in different points. However, if we perform the limit $|\vec{p}_1| = |\vec{p}_2| = |\vec{p}| \to \infty$ all the arguments of the form factors tend to zero and we get again Eqs. (6.26) and (6.28)* (after multiplication by $|\vec{p}|$). Thus, the choice of the $|\vec{p}| = \infty$ reference frame presents some definite advantages. It allows a clear-cut separation of the corrections (in the sense pointed out above) and, as shown in

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*Here we did not play all the game but a detailed calculation of this sort is given in the example of Section 7.

Section 4, the many-particle contribution can be put in the covariant form (4.10). Moreover, we can give here the same discussion of Subsection (6.2) about the magnitude of the corrections: in fact, the r.h.s. corrections of Eq. (6.26) and (6.28) are exactly given by Eqs. (6.7) and (6.15). Applying the same considerations we can presumably believe that the corrections assume their minimum value as $\flat \to \infty$.

7. RELATIONS FOR FORM FACTORS AND MAGNETIC MOMENTS

In this section we would like to discuss the case in which the operator M of Eq. (2.13) is a current density [12]. As a particular example, we shall choose the electromagnetic current, though our argument will be quite general and, in principle, applicable also to other currents*. The electromagnetic current transforms under SU_3 rotations as the charge, i.e., it is a scalar in the U-spin space. As a consequence, we have

$$\left[Q_L^{\mp}(t),\ J_\mu(\vec{x}, t) \right] = 0 \qquad (7.1)$$

The operators are taken at equal times and from now on we shall consider $t = 0,\ \vec{x} = 0$. Following our usual procedure, we consider the matrix element of the commutator (4.1) between (physical) proton and Σ^+ states, we insert a complete system of intermediate states and keeping the lowest contributing states we get

$$\langle P(p_2)|\, Q_L^+\, |\Sigma^+(p_1')\rangle\langle \Sigma^+(p_1')|\, J_\mu(0)\, |\Sigma^+(p_1)\rangle -$$

$$-\langle P(p_2)|\, J_\mu(0)\, |P(p_2')\rangle\langle P(p_2')|\, Q_L^+ |\Sigma^+(p_1)\rangle + C = 0 \qquad (7.2)$$

where

$$C = \sum_\alpha \left\{ \langle P|\, Q_L^+\, |\alpha\rangle\langle\alpha|\, J_\mu(0)\, |\Sigma^+\rangle - \langle P|\, J_\mu(0)\, |\alpha\rangle\langle\alpha|\, Q_L^+\, |\Sigma^+\rangle \right\} \qquad (7.2')$$

We can remark that the correction C is of the first order in the symmetry-breaking interaction. We introduce now the following relation

$$\langle P(p)|\, Q_L^+\, |\Sigma^+(p')\rangle =$$

$$= -\sqrt{\frac{m_P\, m_\Sigma}{E_P\, E_\Sigma}}\ \delta(\vec{p} - \vec{p}\,')\ \bar{u}_P(p) \left\{ \gamma_0\, G(t) + q_0\, H(t) \right\} u_\Sigma(p') \qquad (7.3)$$

where $q_0^2 = t = (E_P - E_\Sigma)^2 = \left(\sqrt{\vec{p}^{\,2} + m_P^2} - \sqrt{p^2 + m_\Sigma^2}\right)^2$ and we adopt the normalization $G(0) = \kappa$ (1 in the symmetry limit). The presence of the

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* For a discussion of the weak current case, see Ref. (2).

additional term $q_0 H(\bar{t})$ is another consequence of the breaking of the symmetry (it disappears in fact as $m_p \to m_\Sigma$, $q_0 \to 0$). It can be verified that its presence does not alter our final conclusions, so that we shall omit it in order to make the formalism simpler. Moreover

$$\langle P(p) | J_\mu(0) | P(p') \rangle =$$

$$= \frac{1}{(2\pi)^3} \left(\frac{m_p^2}{E E'} \right)^{1/2} \bar{u}(p)\, e \left\{ \gamma_\mu F_1^p(t) + \frac{k_p}{2m_p} \sigma_{\mu\nu} q_\nu F_2^p(t) \right\} u(p') \tag{7.4}$$

In Eq. (7.4) $q = p - p'$, $t = q^2$ and $F_{1,2}^p(t)$ are the usual electromagnetic form factors normalized to 1 at $t = 0$ ($F_1^p(0) = F_2^p(0) = 1$). k_p is the anomalous part of the magnetic moment (in $e/2m_p$ units). An analogous relation can be written for the electromagnetic vertex of the Σ^+ and in so doing we introduce the quantities $F_1^\Sigma(t)$, $F_2^\Sigma(t)$ and k_Σ (anomalous magnetic moment in units $e/2m_\Sigma$). It is important to notice that in Eqs. (7.3) and (7.4) we are using the physical masses for the involved particles. This is due to the fact that the states we are considering are physical states, eigenstates of the total Hamiltonian (not completely invariant). In this way, we already introduce in the kinematical factors a display of the violation of the SU_3 symmetry.

If we insert these definitions of Eq. (7.2), we find

$$e\, \bar{u}_p(p_2) \left\{ \left[\gamma_\mu F_1^p(t_2) + \sigma_{\mu\nu} q_{2\nu} F_2^p(t_2) \right] \frac{\gamma \cdot p_2' + m_2}{2 E_2'} \gamma_0\, G(\bar{t}_2) - \right.$$

$$\left. - \gamma_0 \frac{\gamma \cdot p_1' + m_\Sigma}{2 E_1'} G(\bar{t}_1) \left[\gamma_\mu F_1^\Sigma(t_1) + \sigma_{\mu\nu} q_{1\nu} F_2^\Sigma(t_1) \right] \right\} u_\Sigma(p_1) =$$

$$= (2\pi)^3 \sqrt{\frac{E_1 E_2}{m_p m_\Sigma}} \cdot C \tag{7.5}$$

where

$$q_1 = (p_1' - p_1)\,; \quad q_1^2 = t_1\,; \quad \bar{t}_1 = (p_2 - p_1')^2\,; \quad \vec{p}_1' = \vec{p}_2$$

$$q_2 = (p_2 - p_2')\,; \quad q_2^2 = t_2\,; \quad \bar{t}_2 = (p_2' - p_1)^2\,; \quad \vec{p}_2' = \vec{p}_1$$

From this expression we see at once that, even neglecting the correction $C = O(f)$, nothing very definite can be said. In fact, owing to the presence of the two arbitrary momenta p_1, p_2 Eq. (7.5) allows a comparison between form factors evaluated in the different points t_1 and t_2. To avoid this difficulty, it is again convenient to choose the best sum rule, i.e., to consider the configuration of $|\vec{p}_1|$, $|\vec{p}_2|$ which minimizes C. With the same arguments as before, this is achieved by choosing $\vec{p}_1 \to \infty$, $\vec{p}_2 \to \infty$ but $\vec{p}_2 - \vec{p}_1 = \vec{h}$ fixed. In this limit

$$q_1 = (\vec{h}, 0) \; ; \quad q_2 = (\vec{h}, 0) \; ; \quad p_1' = p_2 \quad ; \quad p_2' = p_1$$
$$t_1 = t_2 = -h^2 < 0 \; ; \quad \bar{t}_1 = \bar{t}_2 = 0 \tag{7.6}$$

Using the free Dirac equations for the external spinors, Eq. (7.5) becomes ($\tau \simeq 1$)

$$e\, \bar{u}_p(p_2) \left\{ \left[\gamma_\mu F_1^p(t) + \sigma_{\mu\nu} q_\nu F_2^p(t) \frac{k_p}{2m_p} \right] - \right.$$
$$\left. - \left[\gamma_\mu F_1^\Sigma(t) + \sigma_{\mu\nu} q_\nu F_2^\Sigma(t) \frac{k_\Sigma}{2m_\Sigma} \right] \right\} u(p_1)$$
$$= (2\pi)^3 \lim_{\substack{|\vec{p}_1|, |\vec{p}_2| \to \infty \\ \vec{p}_2 - \vec{p}_1 = \vec{h}}} \sqrt{\frac{E_1 E_2}{m_p m_\Sigma}}\; C \tag{7.7}$$

$$q = p_2 - p_1 = (\vec{h}, 0) \; ; \quad t = q^2$$

Thus we get the result

$$F_1^p(t) = F_1^\Sigma(t) + \delta F_1$$
$$\frac{k_p}{m_p} F_2^p(t) = \frac{k_\Sigma}{m_\Sigma} F_2^\Sigma(t) + \delta F_2 \tag{7.8}$$

In particular, at $t = 0$ the second relation gives

$$k_p = k_\Sigma \frac{m_p}{m_\Sigma} + \delta k = k_\Sigma \left(1 + \frac{m_p - m_\Sigma}{m_\Sigma} \right) + \delta k \tag{7.9}$$

and going over to the total magnetic moments

$$\mu_p = (1 + k_p) \frac{e}{2m_p} \quad ; \quad \mu_\Sigma = (1 + k_\Sigma) \frac{e}{2m_\Sigma}$$

$$\frac{\mu_\Sigma}{\mu_p} = 1 - \frac{1}{1 + k_p} \frac{m_\Sigma - m_p}{m_\Sigma} - \frac{\delta k}{\mu_p} \tag{7.10}$$

In this way, we recognize two different types of corrections, both of order f to the symmetric limits $k_p = k_\Sigma$, $\mu_p = \mu_\Sigma$. The first one which is proportional to the mass difference, is of a kinematical origin in the sense that it is due to the fact of taking correctly into account the physical masses of the particles. In Eq. (7.10) for instance, it produces a correction $\approx - 9\%$. The second term δk is related to the existence of non-diagonal matrix elements for the generator Q_L^+ and it can be treated using the formalism discussed in Section 2. The simplest set of states to be introduced in C of Eq. (7.2) would be those contamining one nucleon (Σ^+) and one pseudoscalar meson, the matrix elements $\langle P | J_\mu | P\pi \rangle$ and $\langle \Sigma^+ | J_\mu | \Sigma^+\pi \rangle$ could then be evaluated using data for photoproduction when known or even calculated in a simple model.

Unfortunately, one usually cannot obtain all the relations between electromagnetic form factors of particles in a given representation by taking matrix elements of one commutator. For the baryon octet there are nine magnetic form factors, including the $\Sigma^\circ\Lambda$ transition, and only two of them are linearly independent corresponding to the F and D coupling of the current.

In order to obtain a general formula whose matrix elements give all the required relations between magnetic moments, we observe that if we put briefly

$$J_\mu^{e.m} = M$$

$M \sim Q$, the electromagnetic charge,

$$M = M_3 + \frac{1}{2} M_Y \;;\quad M_3 \sim Q_3 \;,\quad M_Y \sim Q_Y \tag{7.11}$$

Then from the commutation rules of the algebra one has

$$[Q_A^{\pm}, M_i] \sim M_A^{\pm}$$

and $\qquad (i = 3, Y) \qquad$ (7.12)

$$[Q_A^{-}, M_A^{+}] \sim \Sigma_i\, c_i M_i$$

It is then clear that a suitable combination of the commutators $[Q_A^-, [Q_A^+, M]]$ should reproduce M itself.

We find that

$$[[M, Q_K^+], Q_K^-] = M_3 + \tfrac{3}{2} M_Y$$

$$[[M, Q_L^+], Q_L^-] = 0 \qquad (7.13)$$

$$[[M, Q_I^+], Q_I^-] = 2 M_3$$

(the second relation is a trivial one because $[M, Q_L^{\pm}]$ is already zero) and the required general formula is

$$3M = [[M, Q_K^+], Q_K^-] + [[M, Q_I^+], Q_I^-] \qquad (7.14)$$

The simplest single formula which enables us to obtain all the desired relations between form factors and their corrections of order f is

$$[[J_\mu^{e.m}(x), Q_K^+], Q_K^-] + [[J_\mu^{e.m}(x), Q_I^+], Q_I^-] = 3\, J_\mu^{e.m}(x)$$

In the frame in which the three momenta of the initial and final particle are equal, we find as zero order approximation (i.e., neglecting all the mass differences) the following nine relations between the nine magnetic moments

1. $\sqrt{3}\,\mu_{\Sigma\Lambda} + \frac{3}{2}\mu_\Lambda + \frac{1}{2}\mu_{\Sigma^0} + \mu_N = 0$
2. $\mu_P + \mu_N + \mu_{\Sigma^-} = 0$
3. $\mu_N + 2\mu_{\Sigma^0} = 0$
4. $\sqrt{3}\,\mu_{\Sigma\Lambda} + \frac{1}{2}\mu_{\Sigma^0} + \frac{3}{2}\mu_\Lambda + \mu_{\Xi^0} = 0$
5. $\mu_{\Xi^0} + \mu_{\Xi^-} + \mu_{\Sigma^+} = 0$

6. $\mu_{\Xi^0} + 2\mu_{\Sigma^0} = 0$

7. $-\sqrt{3}\,\mu_{\Sigma\Lambda} + 2(\mu_{\Sigma^-} + \mu_{\Sigma^+} - \mu_{\Sigma^0}) + \frac{1}{2}(\mu_{\Xi^-} + \mu_p) = 0$

8. $\frac{3}{2}(\mu_p + \mu_{\Xi^-}) - \sqrt{3}\,\mu_{\Sigma\Lambda} = 0$

9. $\mu_p + \mu_{\Xi^-} - \mu_\Lambda - \mu_{\Sigma^0} - \frac{2}{\sqrt{3}}\mu_{\Sigma\Lambda} = 0$

the first eight of which are obtained by taking matrix elements between two p, N, Σ^-, Ξ^-, Ξ^0, Σ^+, Σ_0, and Λ states respectively and the nineth between a Σ_0 and a Λ. Of course, only seven of these are linearly independent and the calculation was made in the lowest approximation, keeping only one-baryon intermediate states. Corrections to these relations can naturally be calculated, as discussed in the previous example. We wish to emphasize that if we are interested in the first order correction, we should introduce many-particle intermediate states between J_μ and only one Q, i.e., consider only terms of the type

$$\langle a | J_\mu | \alpha_1 \alpha_2 \rangle \langle \alpha_1 \alpha_2 | Q | a'' \rangle \langle a'' | Q | a' \rangle$$

and analogous ones, but no terms like

$$\langle a | Q | \alpha_1 \alpha_2 \rangle \langle \alpha_1 \alpha_2 | J_\mu | \alpha_3 \alpha_4 \rangle \langle \alpha_3 \alpha_4 | Q | a' \rangle$$

which are of order f^2. Thus, since the matrix element of a Q between two one-particle states reduces in this approximation to a simple coefficient, we can always apply, in evaluating the corrections, the methods previously discussed.

Analogous considerations could be developed also for strong "charges". Assuming a Yukawa-like coupling between baryons and mesons sum rules could be derived for the different coupling constants.

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APPENDIX

We wish to calculate the corrections as displayed in (6.7) to the pseudoscalar octet mass formula. Like in Ref. (3), we simplify the problem by considering intermediate states containing one pseudoscalar and one vector meson. We further simplify our calculation by breaking the symmetry only on the lines and, as suggested in Section 5, making the pole approximation in the dispersive evaluation of the matrix elements of D (see Fig. 1).

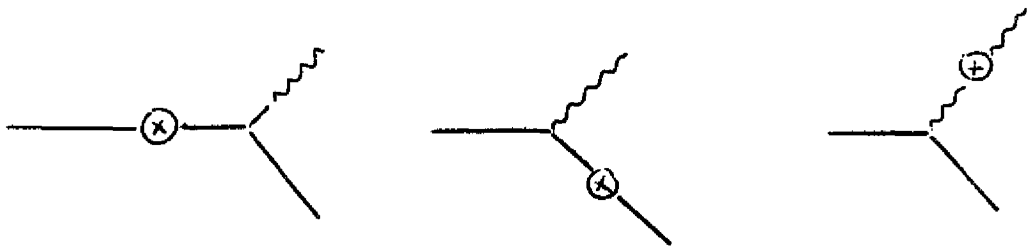

Fig. 1

— denotes a pseudoscalar meson

$\sim$ denotes a vector meson

$\otimes$ denotes the breaking of SU_3

We shall now give some of the details of the calculation, focussing on the $|\vec{p}| \to \infty$ case

$$4\, m_K^2 - 3\, m_\eta^2 - m_\pi^2 = C_\infty$$

where C_∞, remembering Section 4, can be written in the form

$$C_\infty = \frac{1}{2}\, \frac{1}{(2\pi)^2} \sum_\alpha \int \frac{ds}{(s-m_K)^2} \int_{\eta_1}^{\eta_2} d\eta\; \phi^\alpha(s, \Delta^2 = 0, \mathrm{th}\,\eta = \tau)$$

and the invariant function ϕ^α is studied dispersively, as in Section 5. Unfortunately, because of our simplified hypothesis, C_∞ contains a divergent integral over s. A natural way to get rid of this difficulty would be to introduce a strong interaction form factor instead of a point-like coupling between the vector and the pseudoscalar mesons. However, to avoid additional complications, we shall introduce an s-dependent vertex of the form

$$g(s) = g_{SU_3}\frac{\Lambda}{s+\Lambda}$$

where g_{SU_3} is determined to be

$$g^2_{SU_3}/4\pi = 0.7$$

by taking an average to the fit of the width of the K^{*+} (51 Mev) and the ρ^+ (115 Mev) [13], using the Hamiltonian

$$i\, g_{SU_3}\, K^{*+}_\mu \left(K^- \partial_\mu \pi^0 - \pi^0 \partial_\mu K^-\right)$$
$$+\, 2 i\, g_{SU_3}\, \rho^+_\mu \left(\pi^- \partial_\mu \pi^0 - \pi^0 \partial_\mu \pi^-\right) + \cdots$$

We have moreover, again for the purpose of simplifying calculations, taken the (mass differences)^2 found in the evaluation of the matrix elements of D in the pole approximation, as given by their first-order broken SU_3 limits, that is to say

$$m_\pi^2 = m_0^2 - \frac{2}{\sqrt{3}}\,\delta m^2$$
$$m_\eta^2 = m_0^2 + \frac{2}{\sqrt{3}}\,\delta m^2$$
$$m_K^2 = m_0^2 + \frac{1}{\sqrt{3}}\,\delta m^2$$

which implies, using the known values of the masses [13]

$$m_0^2 = (410\ \mathrm{MeV})^2$$
$$\delta m^2 = 12.75\cdot 10^4\ \mathrm{MeV}^2$$

Similarly, for the vector meson masses, we find

$$M_0^2 = (848\ \mathrm{MeV})^2$$
$$\delta M^2 = 12.8\cdot 10^4\ \mathrm{MeV}^2$$

Finally, we present the value of C_∞ calculated for two different values of the cut-off Λ: $\Lambda_1 = (2M_B)^2$, M_B being the mean baryon mass and $\Lambda_2 = \frac{1}{2}\Lambda_1$

$$C_\infty(\Lambda_1) = 0.55\ m_0^2$$
$$C_\infty(\Lambda_2) = 0.19\ m_0^2$$

experimental value $\qquad C_\infty^{exp} = 0.36\ m_0^2$

As an indication, we have also evaluated the correction in the $\vec{p}=0$ limit, using a formula of the (4.11) type. Now, taking linear mass formulae, e.g.,

$$m_\pi = m_0 - \frac{2}{\sqrt{3}}\,\delta m$$
$$m_\eta = m_0 + \frac{2}{\sqrt{3}}\,\delta m$$
$$m_K = m_0 + \frac{1}{\sqrt{3}}\,\delta m$$

we find

$$m_0 = 368 \text{ MeV}$$
$$\delta m = 195 \text{ MeV}$$
$$M_0 = 846 \text{ MeV}$$
$$\delta M = 78 \text{ MeV}$$

and for the two values of the cut-off C_0 equals

$$C_0(\Lambda_1) = 0.79\, m_0$$
$$C_0(\Lambda_2) = 0.27\, m_0$$

experimental value $\quad C_0^{exp} = 0.54\, m_0$.

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