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THE $\tilde{U}(12)$ SYMMETRY

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1. INTRODUCTION

The search for a relativistic version of the spin-unitary spin symmetry[1] SU(6) led in early 1965 to a consideration of SL(6, C)[2] and U(6, 6)[3] (= $\widetilde{U}(12)$[4] = M(12)[5] = $U_{\mathcal{L}}(12)$[6]) as two possible approximate symmetries for a phenomenological description of known particles and their strong interactions. Even though one had recognised right at the outset that symmetries of the SU(6) class must be dynamical in origin, and that this origin is extremely obscure at present, a certain amount of unpopularity has arisen over the relativistic generalisations SL(6, C) and U(6, 6) on account of the fact that they cannot per se be expected to represent the symmetry of the S-matrix. At best they apply to a local interaction Lagrangian and in a restricted sense to many-particle free states. Without attempting to discover the origin of the postulated symmetry, we gave in reference 4I a description of the symmetry in the context of BARGMANN-WIGNER (B. W) equations[7] applied to finite-dimensional representations of U(6, 6); in paper II this context was extended to include U(6, 6) invariant phenomenological interaction Lagrangian densities ($\mathcal{L}_{int}$) between the corresponding fields. Right from the beginning it was recognized that notwithstanding the invariance of $\mathcal{L}_{int}$ the resulting S-matrix could not be U(6, 6) invariant because the B. W equations are not U(6, 6) covariant[8].

Accepting this heuristic description of the possible symmetry, the crucial question which arises is whether one can systematically identify within the theory expressions for some S-matrix elements which display approximate residual symmetries higher than just SU(3). If such residual symmetries survive - and we know empirically that they do from the very existence of the SU(6) supermultiplet structure itself[1] and from the rather remarkable correlation of experimental data achieved by considering the lowest baryon-meson vertex[10] - the further question may be asked: "What reasons make the approximations so good?" This is then back to the dynamical problem with its intimate link with the dynamical origin of the symmetry in the first place. No systematic study of these problems has yet been made; the problem of <u>residual</u> symmetries has however been considerably clarified[11]. In this (frankly biased) review we restate the U(6, 6) theory incorporating some of this recent work; we show that the theory has a logical basis, that even though it uses a non-compact group it employs perfectly unitary representations, and that it does not violate unitarity[9] nor antiparticle conjugation[12] in the formulation of Ref. 4.

The logical structure of the theory to be developed is the following:

(1) We may take it as an empirical fact that the known particles at-rest correspond to the representations of a compact non-covariant $U(6) \otimes U(6)$ group structure[13]. This structure is wider than SU(6), incorporates more particles in the multiplets (Ref. 4. II addendum), and forms the crucial point of departure of U(6, 6) versus SL(6, C) theory. If one is proceeding postulationally we may assume that the existence of U(6) x U(6) structure is an abstraction which follows from the observation that a free SU(3) invariant quark Lagrangian happens to possess extra invariances (see sections 3 and 4) which for quarks at rest devolve to a $U(6) \otimes U(6)$ symmetry.

(2) The B.W equations applied to the finite-dimensional non-unitary representations of a U(6, 6) group reproduce precisely the relativistic structure of such $U(6) \otimes U(6)$ multiplets, no more and no less. The B.W. equations can thus be looked upon as the <u>relativistic boost</u>, generating for a single particle state what we may call the little group structure $[U(6) \otimes U(6)]_p$. The introduction of momenta through the use of the equations (just as in the case of transition from finite-dimensional representations of the homogeneous Lorentz group to the unitary representations of the inhomogeneous group) allows for the introduction of a <u>unitary</u> norm for these multiplets. This norm is not merely the norm for representations of $I\mathcal{L}_4 \otimes SU(3)$; it corresponds to the group structure $[U(6) \otimes U(6)]_p$ which for the rest system reduces to $U(6) \otimes U(6)$. This was the content of section 4 of Paper I and is recapitulated in Section 3 of this paper.

(3) One can write free Lagrangians which yield field equations with the same content of the B.W equations as was done in Reference II. A local interaction Lagrangian among the multiplets of a higher symmetry scheme must naturally possess at least the symmetry of the free Lagrangian, and the interaction was therefore assumed to possess the U(6,6) symmetry. SCHWINGER[14] has shown that for the meson-meson interaction at least, the demand for relativistic invariance imposes this minimal symmetry. We show in Section 4e that a U(6,6) invariant quark-quark Lagrangian can lead in a strong interaction limit to U(6) $\otimes$ U(6) bound state multiplets. The internal consistency argument then gives us a heuristic reason for postulating U(6,6) invariant phenomenological interaction Lagrangians also for the bound state composite fields.

(4) The lowest order calculation for the baryon-meson vertex function gives results for the ratio of proton electric and magnetic form factors and the proton magnetic moment which are in good agreement with experiment[15]. These results are an essential consequence of the starting assumption that the free-particle multiplet structure is the B.W boosted U(6) $\otimes$ U(6) rather than U(6)[16]. Unlike SL(6,C), U(6,6) does provide a definite value for the proton magnetic moment which agrees fairly with experiment. It would be true to say that the explanation of the hitherto mysterious equality of electric and magnetic form factors is the most striking prediction of $\widetilde{U}(12)$ theory.

(5) To check whether these conclusions survive for higher order calculations of the vertex function, one must investigate the possible residual symmetries of the general many-particle S-matrix elements. For an n-particle system this symmetry can at best be the intersection of every $[U(6) \otimes U(6)]_{p_n}$. It is found that for the vertex function and all collinear processes this symmetry is the so-called $U(6)_W$ symmetry[17]. It happens that $U(6)_W$ and U(6,6) effectively coincide on the mass shell of the vertex function; however the reasons for the empirical survival of $U(6)_W$ (or U(6,6)) for the vertex function after unitarity corrections are taken into account is obscure.

(6) We wish to make the important point that the equality of electric and magnetic form factors is a direct consequence of the U(2, 2) extension of the Lorentz group, and therefore irrespective of whether SU(3) was ever invented or not, the extension of the homogeneous Lorentz group symmetry SL(2, C) to U(2, 2) was long overdue. The U(2, 2) group with translations admits of two spins $\underset{\sim}{k}$ and $\underset{\sim}{\ell}$, $\underset{\sim}{j} = \underset{\sim}{k} + \underset{\sim}{\ell}$ being the Poincaré spin and $k_3 - \ell_3$ presenting a new quantum number which in the past would have been designated "internal", just like the quantum numbers of SU(3); the form factor equality is related to the conservation of $k_3 - \ell_3$. Its appearance however is so analogous to Poincaré spin that the situation seems to call for translations associated not only with Lorentz rotations but also with the full U(2, 2). A study then of [U(6) ⊗ U(6)], and all group theoretical questions connected with it, is facilitated by embedding $\tilde{U}(12)$ theory in a larger framework, the so-called inhomogeneous $\tilde{U}(12)$ theory[18] which is treated in Section 6. The physical structure outlined previously forms a substructure of this wider theory which may provide us with a new method for studying the validity of the approximation procedures. The inhomogenous $\tilde{U}(4)$ or $\tilde{U}(12)$ theories are so elegant that it is maddening one has not been able to make more use of them for physical purposes.

(7) Section 5 summarises the experimental data against which has been checked the prediction of $\tilde{U}(12)$ theory.

2. THE MATHEMATICAL STRUCTURE OF U(6, 6)

The material presented below is covered in great detail. The impatient reader who is not inclined to read through this section is respectfully urged to look up equations (2.1) - (2.4), (2.10) - (2.12), (2.21) - (2.24), (2.41) - (2.47) and (2.50) - (2.52), to acquaint himself with the notation. A knowledge of sections (b) and (e) will also prove rather useful.

(a) The U(2, 2) Algebra

We may define a non-compact U(2, 2) algebra in terms of sixteen 4 x 4 Dirac matrices,

$$\gamma_R = 1 \;,\; \gamma_\mu \;,\; \sigma_{\mu\nu} = \frac{i}{2}\left[\gamma_\mu, \gamma_\nu\right] \;,\; i\gamma_\mu\gamma_5 \;,\; \gamma_5 = \gamma_0\gamma_1\gamma_2\gamma_3$$

$$R = 1, \dots 16 \quad ; \quad \mu, \nu = 0, 1, 2, 3 \tag{2.1}$$

with $\left\{\gamma_\mu, \gamma_\nu\right\} = 2 g_{\mu\nu}$ and $g_{\mu\nu} = (1, -1, -1, -1)_{\text{diag.}}$ (2.2)

The γ_R so defined obey the multiplication rules

$$\gamma_\mu\gamma_\nu = -i\sigma_{\mu\nu} + g_{\mu\nu}$$

$$\gamma_\lambda\sigma_{\mu\nu} = i\left(g_{\lambda\mu}\gamma_\nu - g_{\lambda\nu}\gamma_\mu\right) + \varepsilon_{\kappa\lambda\mu\nu}\, i\gamma_\kappa\gamma_5$$

$$\gamma_\mu \cdot i\gamma_\nu\gamma_5 = i g_{\mu\nu}\gamma_5 + \frac{1}{2}\varepsilon_{\mu\nu\kappa\lambda}\sigma_{\kappa\lambda}$$

$$\sigma_{\kappa\lambda}\sigma_{\mu\nu} = i\left(g_{\kappa\nu}\sigma_{\lambda\mu} + g_{\lambda\mu}\sigma_{\kappa\nu} - g_{\kappa\mu}\sigma_{\lambda\nu} - g_{\lambda\nu}\sigma_{\kappa\mu}\right)$$
$$+ \left(g_{\kappa\mu}g_{\lambda\nu} - g_{\lambda\mu}g_{\kappa\nu} - \varepsilon_{\kappa\lambda\mu\nu}\gamma_5\right)$$

$$\sigma_{\kappa\lambda} \cdot i\gamma_\mu\gamma_5 = i\left(g_{\lambda\mu} \cdot i\gamma_\kappa\gamma_5 - g_{\kappa\mu}\, i\gamma_\lambda\gamma_5\right) - \varepsilon_{\kappa\lambda\mu\nu}\gamma_\nu$$

$$\sigma_{\kappa\lambda}\gamma_5 = \frac{i}{2}\varepsilon_{\kappa\lambda\mu\nu}\sigma_{\mu\nu}$$

$$i\gamma_\mu\gamma_5 \cdot i\gamma_\nu\gamma_5 = i\sigma_{\mu\nu} - g_{\mu\nu} \tag{2.3}$$

We take γ_0 hermitian $(\gamma_0 = \gamma_0^\dagger)$ and $\vec{\gamma}$ anti-hermitian $(\vec{\gamma}^\dagger = -\vec{\gamma})$ so that the set of γ-matrices obeys the defining property

$$\gamma_0\gamma_R\gamma_0 = \gamma_R^\dagger \tag{2.4}$$

of the U(2, 2) algebra; for, in the Pauli representation

$$\gamma_0 = (+1, +1, -1, -1)_{diag} \; . \tag{2.5}$$

Then of the total of sixteen, there are

8 hermitian matrices: $1 \; , \vec{\sigma} \; ; \; \gamma_0 \; , \; \gamma_0 \vec{\sigma} = i \vec{\gamma} \gamma_5$

8 anti-hermitian matrices: $\gamma_5 \; , \; \gamma_5 \vec{\sigma} = -i \vec{\sigma}_0 \; ; \; i \gamma_0 \gamma_5 \; , \; i \gamma_0 \gamma_5 \vec{\sigma} = \vec{\gamma}$

where $\vec{\sigma} = (\sigma_{23} \, , \sigma_{31} \, , \sigma_{12}) \; , \; \vec{\sigma}_0 = (\sigma_{01} \, , \sigma_{02} \, , \sigma_{03})$.

A symbolic way to remember the set of matrices is to note that in Pauli representation

$$i \gamma_5 = \left(\begin{array}{c|c} 0 & 1 \\ \hline 1 & 0 \end{array} \right) \quad , \quad \vec{\sigma} = \left(\begin{array}{c|c} \vec{\sigma} & 0 \\ \hline 0 & \vec{\sigma} \end{array} \right)$$

so that all hermitian matrices are block diagonal and all anti-hermitian matrices are block off-diagonal.

The U(2, 2) algebra will be realized by a set of matrices J_R which satisfy commutation rules analogous to those of the γ_R, in detail these being

$$
\begin{aligned}
&[J, J_R] = [J_{\kappa\lambda}, J_5] = [J_5, J_5] = 0 \\
&[J_\mu, J_\nu] = -[J_{\mu 5}, J_{\nu 5}] = -i J_{\mu\nu} \quad , [J_\mu, J_{\nu 5}] = i g_{\mu\nu} J_5 \\
&[J_\lambda, J_{\mu\nu}] = i (g_{\lambda\mu} J_\nu - g_{\lambda\nu} J_\mu) \\
&[J_{\lambda 5}, J_{\mu\nu}] = i (g_{\lambda\mu} J_{\nu 5} - g_{\lambda\nu} J_{\mu 5}) \\
&[J_{\kappa\lambda}, J_{\mu\nu}] = i (g_{\kappa\nu} J_{\lambda\mu} + g_{\lambda\mu} J_{\kappa\nu} - g_{\kappa\mu} J_{\lambda\nu} - g_{\lambda\nu} J_{\kappa\mu}) \\
&[J_\mu, J_5] = -i J_{\mu 5} \, , \quad [J_{\mu 5}, J_5] = -i J_\mu
\end{aligned}
\tag{2.6}
$$

The normalisation of the structure constants has been adjusted so that $J^R = \frac{1}{2} \gamma^R$ constitutes the fundamental representation.

One can exhibit the structure of the algebra in an equivalent basis which is useful for many purposes. This basis is defined by the alternative set of generators

$$J^{\alpha}_{\beta} = \frac{1}{2} (\gamma^{R})^{\beta}_{\alpha} \; J^{R} \qquad \alpha, \beta = 1, 2, 3, 4 \tag{2.7}$$

summarising the commutation rules to

$$\left[J^{\beta}_{\alpha}, J^{\delta}_{\gamma} \right] = \delta^{\delta}_{\alpha} J^{\beta}_{\gamma} - \delta^{\beta}_{\gamma} J^{\delta}_{\alpha} \tag{2.8}$$

and the defining property of U(2, 2) as the hermiticity condition

$$\left(J^{\beta}_{\alpha} \right)^{T} = (\gamma_{0})^{\gamma}_{\beta} \; J^{\delta}_{\gamma} \; (\gamma_{0})^{\alpha}_{\delta} \tag{2.9}$$

where γ_0 is given by (2.5). In the form (2.7), (2.8) the rules are immediately generalizable to U(n, n) with $\gamma_0 = (\underbrace{1, \ldots 1}_{n} ; \underbrace{-1, \ldots -1}_{n})_{diag}$

There exists one last form of expressing the generators of U(2, 2) which comes from the isomorphism to O(4, 2), the real orthogonal group in six dimensions with metric

$$g_{MN} = (+ - - - - +) \quad ; M N = 0, 1, 2, 3, 5, 6 . \tag{2.10}$$

This isomorphism is peculiar to U(4) only and does not generalize to higher rank "unitary groups". It is best seen if we write the γ-matrices (bar unity) in the form γ_{MN} with the identification

$$\gamma_{\mu\nu} = \sigma_{\mu\nu} , \quad \gamma_{\mu 5} = i \gamma_{\mu} \gamma_{5} \quad , \quad \gamma_{\mu 6} = \gamma_{\mu} \quad , \quad \gamma_{56} = \gamma_{5} \tag{2.11}$$

The table (2.3) is then conveniently summarized by

$$\gamma_{KL} \gamma_{MN} = i \left(g_{KN} \gamma_{LM} + g_{LM} \gamma_{KN} - g_{KM} \gamma_{LN} - g_{LN} \gamma_{KM} \right)$$
$$+ \left(g_{KM} g_{LN} - g_{KN} g_{LM} + \frac{1}{2} \varepsilon_{KLMNIJ} \gamma_{IJ} \right) \tag{2.12}$$

The analogue of (2.7) is

$$J_{MN} = \frac{1}{2}(\gamma_{MN})_\alpha^\beta \, J_\beta^\alpha \tag{2.13}$$

whence

$$\begin{aligned}[J_{KL}, J_{MN}] = i\,(g_{KN} J_{LM} + g_{LM} J_{KN} - \\ - g_{KM} J_{LN} - g_{LN} J_{KM})\end{aligned} \tag{2.14}$$

characteristic of angular momentum operators in 6 dimensions. The associated real parameter rotations are of the types:

06, 12, 13, 23, 15, 25, 35	planes :	Euclidean
01, 02, 03, 05, 16, 26, 36, 56	planes :	Hyperbolic

The Euclidean rotations are related to the hermitian J_R, the hyperbolic rotations to the anti-hermitian J_R.

Irreducible representations of SU(2, 2) are classified by the eigenvalues of the 3 Casimir operators $(J_\alpha^\alpha \equiv 0)$

$$C_1 = J_\alpha^\beta J_\beta^\alpha \;, \quad C_2 = J_\beta^\alpha J_\gamma^\beta J_\alpha^\gamma \;, \quad C_3 = J_\beta^\alpha J_\gamma^\beta J_\delta^\gamma J_\alpha^\delta$$

In the alternative notations

$$\begin{aligned}C_1 = J_R J_R = \frac{1}{2} J_{KL} J_{KL} = J_\mu J_\mu + \frac{1}{2} J_{\mu\nu} J_{\mu\nu} - J_{\mu 5} J_{\mu 5} \\ - J_5 J_5\end{aligned} \tag{2.15}$$

explicitly, while $\varepsilon_{IJKLMN} J_{IJ} J_{KL} J_{MN}$ and $J_{MK} J_{ML} J_{NK} J_{NL}$ are linear combinations of C_2 and C_3.

(b) U(2, 2) Subgroups

We exploit the isomorphism to O(4, 2) for a systematic classification of all possible subgroups, by considering rotations in the various subspaces of the 6-dimensional space.

Subgroup	Tensor	Vector	Scalar
$O(4,2)$	$M,N = 0,1,2,3,5,6$		
$O(4,1)$	$M,N = 0,1,2,3,5$	$M = 6;\ N = 0,1,2,3,5$	
	$= 1,2,3,5,6$	$M = 0;\ N = 1,2,3,5,6$	
$O(3,2)$	$M,N = 0,1,2,3,6$	$M = 5;\ N = 0,1,2,3,6$	
	$= 0,1,2,5,6$	$M = 3;\ N = 0,1,2,5,6$	
	....	....	
$O(4,0)$	$M,N = 1,2,3,5$	$M = 0,6;\ N = 1,2,3,5$	$M,N = 0,6$
$O(3,1)$	$M,N = 0,1,2,3$	$M = 5,6;\ N = 0,1,2,3$	$M,N = 5,6$
	$= 6,1,2,3$	$M = 0,5;\ N = 6,1,2,3$	$M,N = 0,5$
	$= 0,1,2,5$	$M = 3,6;\ N = 0,1,2,5$	$M,N = 3,6$
	....	....	....
$O(2,2)$	$M,N = 0,1,2,6$	$M = 3,5;\ N = 0,1,2,6$	$M = 3, N = 5$
	....	....	....
$O(3,0)$	$M,N = 1,2,3$	$M = 0,5,6;\ N = 1,2,3$	$M,N = 0,5,6$
	$M,N = 1,2,5$	$M = 0,3,6;\ N = 1,2,5$	$M,N = 0,3,6$
	....	....	....
$O(2,1)$	$M,N = 0,1,2$	$M = 6,3,5;\ N = 0,1,2$	$M,N = 6,3,5$
	$M,N = 0,3,5$	$M = 6,1,2;\ N = 0,3,5$	$M,N = 6,1,2$
	$M,N = 0,5,6$	$M = 1,2,3;\ N = 0,5,6$	$M,N = 1,2,3$
	....	....	....

TABLE I.

Our entries in the table refer to J_{MN} and we have listed the tensor, vector and scalar transformation characters relative to each O(i, j) subgroup. Significant subalgebras of U(2, 2) are the following :

Compact (1) 0(4) or $U(2) \otimes U(2)$ composed of the generators $1, \vec{\sigma}, \gamma_0, \gamma_0 \vec{\sigma}$ The combinations $\frac{1}{4}(1 \pm \gamma_0), \frac{1}{4}(1 + \gamma_0)\vec{\sigma}$ refer to the $U^{\pm}(2)$

This is the maximal compact subalgebra and further U(2) subalgebras are

(2) The ordinary spin group : $1, \vec{\sigma}$

(3) The W-spin group [17] : $1, \sigma_{12}, \gamma_0 \sigma_{23}, \gamma_0 \sigma_{31}$

Non-compact (1) SO(4,1) composed of generators

$$\sigma_{MN} = (\sigma_{\mu\nu}, i\gamma_\mu \gamma_5); \; M, N = 0, 1, 2, 3, 5$$

The 5-vector is $\gamma_M = (\gamma_\mu, \gamma_5)$ and the multiplication table of the Dirac algebra reads $(\epsilon_{01235} = i)$

$$\gamma_M \gamma_N = -i\sigma_{MN} + g_{MN}$$

$$\gamma_L \sigma_{MN} = i(g_{LM}\gamma_N - g_{LN}\gamma_M) + \frac{1}{2}\varepsilon_{LMNIJ}\sigma_{IJ}$$

$$\sigma_{KL}\sigma_{MN} = i(g_{KN}\sigma_{LM} + g_{LM}\sigma_{KN} - g_{KM}\sigma_{LN} - g_{LN}\sigma_{KM})$$

$$+ (g_{KM}g_{LN} - g_{KN}g_{LM} + \varepsilon_{KLMNJ}\gamma_J) \tag{2.16}$$

The two Casimir operators are

$$J_M J_M = J_\mu J_\mu - J_5 J_5 \quad , \quad \frac{1}{2} J_{MN} J_{MN} = \frac{1}{2} J_{\mu\nu} J_{\mu\nu} - J_{\mu 5} J_{\mu 5} \tag{2.17}$$

For this algebra there exists an antisymmetric tensor B such that

$$B^T = -B \quad , \quad \gamma_\mu^T = B^{-1} \gamma_\mu B \tag{2.18}$$

Introducing the notation ,

$$B \to B_{\alpha\beta} \quad , \quad B^{-1} \to (B^{-1})^{\alpha\beta} \, , \; (B^{-1})^{\alpha\beta} B_{\beta\gamma} = \delta^{\alpha}_{\gamma}$$

$$(\gamma_\mu)^{\beta}_{\alpha} = (B^{-1})^{\beta\gamma} (\gamma_\mu)^{\delta}_{\gamma} B_{\delta\alpha}$$

we find that in their indices

$$B_{\alpha\beta} \, , (\gamma_M B)_{\alpha\beta} \, , (B^{-1})^{\alpha\beta} , \; (B^{-1}\gamma_M)^{\alpha\beta} \quad \text{are \underline{antisymmetric}}$$

$$(\sigma_{MN} B)_{\alpha\beta} \, , \; (B^{-1} \sigma_{MN})^{\alpha\beta} \qquad\qquad \text{are \underline{symmetric}}$$

(2.19)

Note that the maximal compact subgroup of 0(4, 1) is U(2) ⊗ U(2).

(2) SO(3, 2) composed of the generators

$$\sigma_{MN} = (\sigma_{\mu\nu} \, , \gamma_\mu) \qquad ; \; MN = 0, 1, 2, 3, 6$$

Now $\gamma_M = (i\gamma_\mu \gamma_5 \, , -\gamma_5)$ and the multiplication table is

$$\gamma_M \gamma_N = i \, \sigma_{MN} - g_{MN}$$

$$\gamma_L \sigma_{MN} = i (g_{LM} \gamma_N - g_{LN} \gamma_M) - \tfrac{1}{2} \varepsilon_{LMNIJ} \sigma_{IJ}$$

$$\sigma_{KL} \sigma_{MN} = i (g_{KN} \sigma_{LM} + g_{LM} \sigma_{KN} - g_{KM} \sigma_{LN} - g_{LN} \sigma_{KM})$$

$$+ (g_{KM} g_{LN} - g_{KN} g_{LM} + \varepsilon_{KLMNJ} \gamma_J) \qquad (2.20)$$

with $\varepsilon_{0123 6} = 1$ [Note the identity $\frac{1}{6} \varepsilon_{LMNIJ} \gamma_N \sigma_{IJ} = \sigma_{LM}$ which is valid also for SO(4, 1).] The Casimir operators are

$J_{\mu 5} J_{\mu 5} + J_5 J_5$ and $J_\mu J_\mu + \frac{1}{2} J_{\mu\nu} J_{\mu\nu}$

and analogously to SO(4, 1) we define[20] $C_{\alpha\beta} = -(\gamma_5 B)_{\alpha\beta}$

for which $(C^{-1})^{\alpha\beta} (\gamma_\mu)^{\gamma}_{\beta} C_{\gamma\delta} = -(\gamma_\mu)^{\alpha}_{\delta} \; , (C^{-1})^{\alpha\beta} C_{\beta\gamma} = \delta^{\alpha}_{\gamma}$

Thus as before,

$C_{\alpha\beta}$, $(\gamma_M C)_{\alpha\beta}$, $(C^{-1})^{\alpha\beta}$, $(C^{-1}\gamma_M)^{\alpha\beta}$ are <u>antisymmetric</u>

while $(\sigma_{MN} C)_{\alpha\beta}$, $(C^{-1}\sigma_{MN})$ are <u>symmetric</u>

(2.21)

Unlike the previous case the maximal compact subgroup is now $U(2) \otimes U(1)$.

(3) Cases (1) and (2) corresponded to the maximal non-compact subalgebras. These intersect to form the familiar homogeneous Lorentz group $\mathscr{L}_4$ or $0(3,1)$ or $SL(2,C)$ with generators $\sigma_{\mu\nu}$.

(4) $SO(2,2)$ with generators of the type

$$\gamma_0, \ \sigma_{12}, \ \gamma_5\sigma_{23}, \ \gamma_5\sigma_{31}, \ i\gamma_0\gamma_5\sigma_{23}, \ i\gamma_0\gamma_5\sigma_{31}$$

(5) $SO(2,1)$, the 2+1 Lorentz groups, such as the $\mathcal{B}$-spin[17] subgroup with generators γ_0, $\gamma_5\sigma_{12}$, $i\gamma_0\gamma_5\sigma_{12}$

(c) <u>Multispinor Representations of $U(2,2)$</u>

In the fundamental (quark)[21] representation of $U(2,2)$ an infinitesimal transformation of the 4-component spinor ψ_α $(\alpha = 1,2,3,4)$ is given by

$$\delta\psi_\alpha = i\,\varepsilon_R\,(\gamma_R)_\alpha^{\ \beta}\,\psi_\beta$$

$$= i\left(\varepsilon + \varepsilon_\mu\gamma_\mu + \tfrac{1}{2}\varepsilon_{\mu\nu}\sigma_{\mu\nu} + \varepsilon_{\mu5}\,i\gamma_\mu\gamma_5 + \varepsilon_5\gamma_5\right)_\alpha^{\ \beta}\psi_\beta \qquad (2.22)$$

The important point is the reality of the parameters ε_R. With

$$\bar{\psi}^\beta \equiv (\psi_\alpha)^\dagger\,(\gamma_0)_\alpha^{\ \beta} \qquad (2.23)$$

this definition of the $U(2,2)$ group has the property of leaving $\bar{\psi}^\alpha\psi_\alpha$ invariant since

$\delta\bar{\psi}^\alpha = -i\,\varepsilon_R\,\bar{\psi}^\beta(\gamma_R)_\beta^{\ \alpha}$ by virtue of (2.4).

All finite-dimensional (and therefore non-unitary) representations can be obtained by constructing multispinors which transform as direct products of quarks, viz.

$$\psi^{\gamma\delta\ldots}_{\alpha\beta\ldots} \rightarrow S_\alpha^{\alpha'} S_\beta^{\beta'} \ldots (S^{-1})^{\gamma}_{\gamma'} (S^{-1})^{\delta}_{\delta'} \psi^{\gamma'\delta'}_{\alpha'\beta'} \qquad (2.24)$$

where

$$S = \exp(i\varepsilon_R J_R) = \exp(i\varepsilon_\alpha^\beta J_\beta^\alpha) \qquad (2.25)$$

The irreducible representations of SU(2,2) will correspond to traceless tensors of well-defined symmetry characters and for convenience we list typical low-dimensional representations, introducing brackets [ ] and { } to denote antisymmetry and symmetry in the enclosed indices.

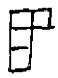
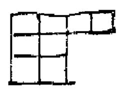
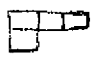
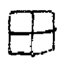
Dimensionality		Young Tableau
<u>4</u>	ψ_α	
<u>6</u>	$\Phi_{[\alpha\beta]}$	
<u>10</u>	$\Phi_{\{\alpha\beta\}}$	
<u>15</u>	Φ_α^β	
<u>20</u>	$\Psi_{\{\alpha\beta\gamma\}}$	
<u>20'</u>	$\Psi_{[\alpha\beta]\gamma}$	
<u>4*</u>	$\Psi_{[\alpha\beta\gamma]}$	
<u>84</u>	$\Phi^{\{\gamma\delta\}}_{\{\alpha\beta\}}$	
<u>45</u>	$\Phi^{[\gamma\delta]}_{\{\alpha\beta\}}$	
<u>20''</u>	$\Phi^{[\gamma\delta]}_{[\alpha\beta]}$	

TABLE II

The Casimir operators C_1, C_2, C_3 are connected to the associated Young tableaux. Letting (λ, μ, ν) label the number of boxes in the first second and third rows so that the dimensionality of the representation is

$$\frac{1}{12}(\lambda-\mu+1)(\lambda-\nu+2)(\mu-\nu+1)(\lambda+3)(\mu+2)(\nu+1)$$

we have typically

$$C_1 = \lambda^2+\mu^2+\nu^2+3\lambda+\mu-\nu \qquad \text{etc.}$$

These multispinors may be reduced with respect to any particular subgroup of U(2, 2) and we shall choose to do so under the largest subgroup, the non-compact $\check{S}p$ (4) or 0(3, 2) (and O(4, 1)) groups. For these subgroups in addition to $\bar{\psi}\psi$ we also have $C^{-1}\psi^T\psi$ (and $B^{-1}\psi^T\psi$) invariant under an infinitesimal transformation i.e. $C_{\alpha\beta}$ (and $B_{\alpha\beta}$) act like metric tensors. We now examine in detail the decomposition of SU(2, 2) multispinors relative to SO(3, 2). By substituting $M = \mu, 5$ for the 5-component indices we discover the reduction relative to the homogeneous Lorentz group immediately. (Also the decomposition of the multispinors relative to SO(4, 1) follows by replacing C with B everywhere below.)

(1) Multispinors of rank 2

$$\Phi_\alpha^{\ \beta} = (\gamma_M)_\alpha^{\ \beta}\,\phi_M + \frac{1}{2}(\sigma_{MN})_\alpha^{\ \beta}\,\phi_{MN}\ ; \quad \phi_\alpha^{\ \alpha} \equiv 0 \tag{2.26}$$

or $15 \to 10 \oplus 5$

$$\phi_{\alpha\beta} = \phi_{[\alpha\beta]} + \phi_{\{\alpha\beta\}} = C_{\alpha\beta} + (\gamma_M C)_{\alpha\beta}\,\phi_M + \tfrac{1}{2}(\sigma_{MN}C)_{\alpha\beta}\,\phi_{MN} \tag{2.27}$$

or $16 = 10 \oplus 6 \to 10 \oplus 5 \oplus 1$

(2) Multispinors of rank 3

The completely symmetric tensor decomposes as

$$\Psi_{\{\alpha\beta\gamma\}} = \frac{1}{2}(\sigma_{MN}C)_{\alpha\beta}\,\psi_{\gamma MN} \tag{2.28}$$

with the symmetry constraints

$$(C^{-1})^{\beta\gamma}\Psi_{\{\alpha\beta\gamma\}} = (C^{-1}\gamma_K)^{\beta\gamma}\Psi_{\{\alpha\beta\gamma\}} = 0$$

or

$$\sigma_{MN}\psi_{MN} = 0$$

and $\sigma_{MN}\gamma_K\psi_{MN} = 0$ reducing to the set of 20 conditions

$$\gamma_M\psi_{MN} = \frac{i}{4}\,\varepsilon_{JKLMN}\,\sigma_{JK}\,\psi_{LM}. \tag{2.29}$$

In the mixed symmetry case,

$$\Psi_{[\alpha\beta]\gamma} = C_{\alpha\beta}\psi_\gamma + (\gamma_M C)_{\alpha\beta}\psi_{\gamma M} \tag{2.30}$$

the irreducibility condition

$$\Psi_{[\alpha\beta]\gamma} + \Psi_{[\beta\gamma]\alpha} + \Psi_{[\gamma\alpha]\beta} = 0 \tag{2.31}$$

gives 4 subsidiary equations

$$\psi = \gamma_M\psi_M \tag{2.32}$$

Finally for the case of complete antisymmetry

$$\Psi_{[\alpha\beta\gamma]} = C_{\alpha\beta}\psi_\gamma + (\gamma_M C)_{\alpha\beta}\psi_{\gamma M} \tag{2.33}$$

again, but $(C^{-1}\sigma_{KL})^{\beta\gamma}\Psi_{[\alpha\beta\gamma]} = 0$ gives 16 constraints

$$\psi_M = -\gamma_M\psi \tag{2.34}$$

Observe that the subsidiary conditions on the $\tilde{Sp}(4)$ tensors always give the correct number of independent quantities to comply with the SU(2, 2) irreducibility.

(3) Multispinors of rank 4

With completely symmetric upper and lower indices,

$$\Phi^{\{\gamma\delta\}}_{\{\alpha\beta\}} = \frac{1}{4}(\sigma_{KL}C)_{\alpha\beta}\,\varphi_{KL,MN}\,(C^{-1}\sigma_{MN})^{\gamma\delta} \tag{2.35}$$

the tracelessness property $\Phi^{\{\gamma\beta\}}_{\{\alpha\beta\}} = 0$ giving $\sigma_{KL}\sigma_{MN}\,\varphi_{KL,MN} = 0$

or

$$\phi_{KL,KL} = \phi_{KL,KM} - \phi_{KM,KL} = \varepsilon_{KLMNJ}\,\varphi_{KL,MN} = 0 \tag{2.36}$$

These comprise 16 equations leaving 84 independent fields as required.

The mixed symmetry situation corresponds to

$$\Phi^{[\gamma\delta]}_{\{\alpha\beta\}} = \frac{1}{2}(\sigma_{KL}C)_{\alpha\beta}\left[\phi_{KL}(C^{-1})^{\gamma\delta} + \phi_{KL,M}(C^{-1}\gamma_M)^{\gamma\delta}\right] \tag{2.37}$$

Tracelessness gives $\sigma_{KL}\,\phi_{KL} + \sigma_{KL}\gamma_M\,\phi_{KL,M} = 0$ or

$$\phi_{KL,K} = \phi_{KL} + \frac{1}{2}\varepsilon_{KLMNJ}\,\phi_{MN,J} = 0 \tag{2.38}$$

These are 15 conditions giving us the 45 independent quantities.

Finally for the case of complete antisymmetry

$$\Phi^{[\gamma\delta]}_{[\alpha\beta]} = C_{\alpha\beta}\,\phi\,(C^{-1})^{\gamma\delta} + (\gamma_M C)_{\alpha\beta}\,\phi_{M,}(C^{-1})^{\gamma\delta} + C_{\alpha\beta}\,\phi_{,M}(C^{-1}\gamma_M)^{\gamma\delta} + (\gamma_M C)_{\alpha\beta}\,\phi_{M,N}(C^{-1}\gamma_N)^{\gamma\delta} \tag{2.39}$$

The 16 relations

$$\phi_{M,} + \varphi_{,M} = \phi + \phi_{M,M} = \phi_{M,N} - \phi_{N,M} = 0 \qquad (2.40)$$

follow from $\Phi^{[\alpha\delta]}_{[\alpha\beta]} = 0$ and reduce the number of independent fields to 20.

(d) <u>The U(6, 6) algebra</u>

The basic apparatus for U(2, 2) having been set up in the previous subsections, the passage to U(6, 6) becomes straightforward. Analogously to (2.7) and (2.8) the generators J_A^B (A, B, = 1, . . 12) of the U(6, 6) algebra will obey the standard commutation rules,

$$\left[J^A_B , J^C_D \right] = \delta^C_B J^A_D - \delta^A_D J^C_B \qquad (2.41)$$

with

$$\left(J^A_B \right)^\dagger = \left(\gamma_0 J \gamma_0 \right)^B_A \qquad (2.42)$$

One may pass to a "hermitian" basis by making use of the γ_R defined earlier together with the unitary spin matrices T^i $(i = 0, \dots 8)$ which define the fundamental U(3) representation. Our convention for the latter is as follows:

$T^i = \frac{1}{2} \lambda^i$ $(i = 1, \dots 8)$ with λ^i defined by GELL-MANN[22] and $T^0 = 1/\sqrt{6}$. Thus

$$T^i T^j = \frac{1}{2} \left(d^{ijk} + i f^{ijk} \right) T^k ; \quad i, j, k = 0, \dots 8 \qquad (2.43)$$

where d^{ijk}, f^{ijk} $(i, j, k = 1, \dots 8)$ have been given by Gell-Mann and $f^{0jk} = 0$, $d^{0jk} = \delta^{jk} \sqrt{2/3}$.

It follows that

$$\mathrm{Tr} \left(T^i T^j \right) = \frac{1}{2} \delta^{ij} \qquad (2.44)$$

In the alternative OKUBO basis[23] the nine U(3) generators T^s_r $(r, s = 1, 2, 3)$ obey

$$[T^p_q, T^r_s] = \delta^p_s \delta^r_q - \delta^r_q T^p_s, \quad Tr(T^q_p T^s_r) = \delta^q_r \delta^s_p \tag{2.45}$$

and in the fundamental (3x3) representation possess the matrix elements $(T^q_p)^s_r = \delta^q_r \; \delta^s_p$ from which the connection with the T^i is readily established.

We now construct the generators

$$J^i_R = (\gamma_A)^\beta_\alpha (T^i)^s_r \; J^{\alpha r}_{\beta s}$$
$$A = \alpha, r \;, \; B = \beta s \;; \; \alpha, \beta = 1, \ldots 4 \;; \; r, s = 1, 2, 3 \tag{2.46}$$

The transformed commutation rules read

$$[J^i, J^j] = if^{ijk} J^k$$
$$[J^i, J^j_5] = if^{ijk} J^k_5$$
$$[J^i_5, J^j_5] = -if^{ijk} J^k$$
$$[J^i, J^j_{\mu\nu}] = if^{ijk} J^k_{\mu\nu}$$
$$[J^i_5, J^j_{\mu\nu}] = \tfrac{1}{2} i f^{ijk} \epsilon_{\mu\nu\kappa\lambda} J^k_{\kappa\lambda}$$
$$[J^i_{\kappa\lambda}, J^j_{\mu\nu}] = id^{ijk} \left(g_{\kappa\nu} J^k_{\lambda\mu} + g_{\lambda\mu} J^k_{\kappa\nu} - g_{\kappa\mu} J^k_{\lambda\nu} - g_{\lambda\nu} J^k_{\kappa\mu} \right)$$
$$+ if^{ijk} \{ (g_{\kappa\mu} g_{\lambda\nu} - g_{\lambda\mu} g_{\kappa\nu}) J^k - \epsilon_{\kappa\lambda\mu\nu} J^k_5 \}$$

$$[J^i_\mu, J^j_\nu] = if^{ijk} g_{\mu\nu} J^k - id^{ijk} J^k_{\mu\nu}$$
$$[J^i_\mu, J^j_{\nu 5}] = id^{ijk} g_{\mu\nu} J^k_5 + \tfrac{1}{2} i f^{ijk} \epsilon_{\mu\nu\kappa\lambda} J^k_{\kappa\lambda}$$
$$[J^i_{\mu 5}, J^j_{\nu 5}] = -if^{ijk} g_{\mu\nu} J^k + id^{ijk} J^k_{\mu\nu}$$

and

$$[J^i, J^j_\mu] = if^{ijk} J^k_\mu$$
$$[J^i, J^j_{\mu 5}] = if^{ijk} J^k_{\mu 5}$$
$$[J^i_5, J^j_\mu] = id^{ijk} J^k_{\mu 5} \tag{2.47}$$

$$[J_5^i, J_{\mu 5}^j] = i d^{ijk} J_\mu^k$$
$$[J_\lambda^i, J_{\mu\nu}^j] = i d^{ijk} (g_{\lambda\mu} J_\nu^k - g_{\lambda\nu} J_\mu^k) - i f^{ijk} \epsilon_{\lambda\mu\nu\kappa} J_{\kappa 5}^k$$
$$[J_{\lambda 5}^i, J_{\mu\nu}^j] = i d^{ijk} (g_{\lambda\mu} J_{\nu 5}^k - g_{\lambda\nu} J_{\mu 5}^k) - i f^{ijk} \epsilon_{\lambda\mu\nu\kappa} J_\kappa^k$$

These commutators are conveniently remembered in the 0(4, 2) notation of (2.12) if we use (J^i, J_{MN}^i) in place of the J_R^i Then,

$$[J^i, J^j] = i f^{ijk} J^k$$

$$[J^i, J_{MN}^j] = i f^{ijk} J_{MN}^k$$

$$[J_{KL}^i, J_{MN}^j] = i f^{ijk} [g_{KM} g_{LN} - g_{KN} g_{ML}) J^k + \\ + \tfrac{1}{2} \varepsilon_{KLMNIJ} J_{IJ}^k] + i d^{ijk} [g_{KN} J_{LM}^k + \\ + g_{LM} J_{KN}^k - g_{KM} J_{LN}^k - g_{LN} J_{KM}^k] \tag{2.48}$$

The eleven Casimir operators of SU(6, 6) are easily built up in the form

$$C_1 = J_B^A J_A^B, \quad C_2 = J_B^A J_C^B J_A^C, \ldots \quad C_{11} = J_B^A \ldots J_A^C$$

However, their appearance is considerably complicated when expressed in terms of the J_R^i for instance,

$$C_1 = 2 (J^i J^i - J_5^i J_5^i - J_{\lambda 5}^i J_{\lambda 5}^i + J_\lambda^i J_\lambda^i + \tfrac{1}{2} J_{\kappa\lambda}^i J_{\kappa\lambda}^i)$$
$$C_2 = \tfrac{2}{3} f^{ijk} [3 J_\mu^i J_\nu^j J_{\mu\nu}^k + 3 J_{\mu 5}^i J_{\mu\nu}^j J_{\nu 5}^k + 6 J_5^i J_\mu^j J_{\mu 5}^k \\ + J_{\mu\nu}^i J_{\nu\lambda}^j J_{\lambda\mu}^k] \\ + d^{ijk} [3 \varepsilon_{\kappa\lambda\mu\nu} (J_\kappa^i J_{\lambda\mu}^j J_{\nu 5}^k + \tfrac{1}{4} J_{\kappa\lambda}^i J_{\mu\nu}^j J_5^k) + J^i J^j J^k \\ + 3 J^i (J_\lambda^j J_\lambda^k + \tfrac{1}{2} J_{\kappa\lambda}^j J_{\kappa\lambda}^k - J_{\lambda 5}^j J_{\lambda 5}^k - J_5^j J_5^k]$$

(2.49)

(e) Important Subalgebras of U(6, 6)

The systematic classification cannot be carried out as readily as for U(2, 2) because there no longer exists an isomorphism with a rotation group[19]. We will therefore concentrate only on those subalgebras which have conceivable physical application.

Compact (1) Non-chiral[13] U(6) ⊗ U(6) composed of the generators

$$\tfrac{1}{2}(1 \pm \gamma_0) T^i \;,\; \tfrac{1}{2}(1 \pm \gamma_0)\vec{\sigma}\, T^i .$$

This is the maximal compact subgroup and contains the further important subalgebras

(2) $U(6)_W$[17] : $T^i,\ \sigma_{12} T^i,\ \gamma_0 \sigma_{23} T^i,\ \gamma_0 \sigma_{31} T^i$

(3) $U(6)$[1] : $T^i,\ \vec{\sigma}\, T^i$ and its own subgroups such as

(4) $U(3) \otimes U(2)$: $T^i\ ;\ 1,\ \vec{\sigma}$

(5) $U_I(4) \otimes U_Y(2)$: $1,\ \vec{\sigma},\ \vec{\tau},\ \vec{\sigma}\vec{\tau}\ ;\ Y,\ Y\vec{\sigma}$

where $\vec{\tau} = (T^1, T^2, T^3)$, $Y = \sqrt{\tfrac{2}{3}}\,(T^0 - \sqrt{2}\, T^8)$

We recognize $U_I(4)$ as Wigner's original supermultiplet group[24] and $U_Y(2)$ as the strange quark spin introduced by LIPKIN[25].

Non-compact (1) The simplest relativistic extension of the U(6) group, SL(6,C)[2] with generators

$$T^i,\ \gamma_5 T^i,\ \vec{\sigma}\, T^i,\ \gamma_5 \vec{\sigma}\, T^i$$

By performing a unitary trick the non-unitary representations can be obtained as those of a chiral[2] U(6) ⊗ U(6) having generators $\tfrac{1}{2}(1 \pm i\gamma_5) T^i$, $\tfrac{1}{2}(1 \pm i\gamma_5)\vec{\sigma}\, T^i$

(2) The subgroup $\tilde{U}(4) \otimes U(3)$: $\gamma_R\ ;\ T^i$

and its own subgroup SL(2, C) ⊗ U(3) in terms of which we carry out all our reductions later on.

(3) Algebras of the type

$$U_I(4,4) \otimes U_Y(2,2) \ :\ \gamma_R,\ \gamma_R \vec{\tau}\ ;\ Y\gamma_R$$

This is significant to the extent that SU(3) may be badly broken and one is generalising WIGNER'S U_I (4) group[26].

(4) $\widetilde{Sp}(12)$ subgroups consisting of 78 generators. It is worthwhile to stress that the subgroup[27] $\widetilde{Sp}(4) \otimes U(3)$ is <u>not</u> contained in $\widetilde{Sp}(12)$.

(f) <u>Multispinor Representations of U(6, 6)</u>

Under a homogeneous U(6, 6) transformation the quark spinor undergoes the change $\psi_A \rightarrow S_A^B \psi_B$ with

$$S = \exp\left(i \varepsilon_R^i J_R^i\right) = \exp\left(i \varepsilon_B^A J_A^B\right) \tag{2.50}$$

The adjoint (antiquark) spinor is defined by

$$\bar{\psi}^A = \bar{\psi}^{\alpha r} = \left(\psi_{\beta r}\right)^{\dagger} \left(\gamma_0\right)_{\beta}^{\alpha} \tag{2.51}$$

transforms under S^{-1} and leaves $\bar{\psi}^A \psi_A$ invariant because the parameters ε_R^i are real. Finite-dimensional non-unitary representations of U(6, 6) may be built up from products of quarks and antiquarks. These multispinors transform according to the general rule,

$$\Psi_{AB\ldots}^{CD\ldots} \rightarrow S_A^{A'} S_B^{B'} \ldots (S^{-1})_{C'}^{C} (S^{-1})_{D'}^{D} \ldots \Psi_{A'B'\ldots}^{C'D'\ldots} \tag{2.52}$$

where S is given by (2.50). Relevant representations with their dimensionalities enclosed in brackets are

$$\Phi_A^B\,(143),\ \Phi_{[CD]}^{[AB]}\,(4212),\ \Phi_{\{CD\}}^{[AB]}\,(5005),\ \Phi_{\{CD\}}^{\{AB\}}\,(5940)$$

$$\psi_{\{ABC\}}\,(364),\ \Psi_{[AB]C}\,(572),\ \Psi_{[ABC]}\,(220),\ \Psi_{[BCDE)}^{A}\,(5720) \tag{2.53}$$

Of especial importance are the reductions of the products

$$
\begin{aligned}
12 \otimes \overline{12} &= 1 \oplus 143 \\
12 \otimes 12 \otimes 12 &= 220 \oplus 364 \oplus 572 \oplus 572 \\
143 \otimes 143 &= 1 \oplus 143 \oplus 143 \oplus 4212 \oplus 5005 \oplus \overline{5005} \oplus 5940 \\
364 \otimes \overline{364} &= 1 \oplus 143 \oplus 5940 \oplus 126412 \\
143 \otimes 364 &= 364 \oplus 572 \oplus 16016 \oplus 35100
\end{aligned}
\tag{2.54}
$$

For convenience we collect below the decomposition of U(6, 6) multiplets with respect to the subgroups SU(2, 2) $\otimes$ SU(3) and U(6) $\otimes$ U(6) since these are the only ones to hold physical interest. Because the further reduction under SO(3, 2), SL(2, C), subgroups of SU(2, 2) has effectively been carried out already in section (c) we are only left to include the reduction of SU(6) multiplets relative to its own subgroups SU(3) $\otimes$ SU(2) and $SU_I(4) \otimes SU_Y(2)$, which we do further on.

(1) SU(2, 2) $\otimes$ SU(3) Decomposition of some SU(6, 6) multiplets

$$
\begin{aligned}
143 &= (15,8) \oplus (15,1) \oplus (1,8) \\
4212 &= (84,8) \oplus (84,1) \oplus (45,\overline{10}) \oplus (\overline{45},10) \oplus (45,8) \oplus (\overline{45},8) \\
&\quad \oplus (20'',27) \oplus (20'',8) \oplus (20'',1) \oplus (15,27) \oplus (15,10) \oplus (15,\overline{10}) \\
&\quad \oplus 3(15,8) \oplus (15,1) \oplus (1,27) \oplus (1,8) \oplus (1,1) \\
5940 &= (84,27) \oplus (84,8) \oplus (84,1) \oplus (45,10) \oplus (\overline{45},\overline{10}) \oplus (45,8) \\
&\quad \oplus (\overline{45},8) \oplus (20'',8) \oplus (20'',1) \oplus (15,27) \oplus (15,10) \oplus (15,\overline{10}) \\
&\quad \oplus 3(15,8) \oplus (15,1) \oplus (1,27) \oplus (1,8) \oplus (1,1) \\
220 &= (20',8) \oplus (20,1) \oplus (\overline{4},10) \\
572 &= (20',10) \oplus (20',8) \oplus (20,8) \oplus (20',1) \oplus (\overline{4},8) \\
364 &= (20,10) \oplus (20',8) \oplus (\overline{4},1)
\end{aligned}
\tag{2.55}
$$

(2) SU(6) ⊗ SU(6) Decomposition of some SU(6, 6) multiplets

$$143 = (6,\bar{6}) \oplus (\bar{6},6) \oplus (1,1) \oplus (35,1) \oplus (1,35)$$

$$4212 = (15,\overline{15}) \oplus (\overline{15},15) \oplus (6,\bar{6}) \oplus (\bar{6},6) \oplus (35,35) \oplus (1,1)$$
$$\oplus (84,\bar{6}) \oplus (6,\overline{84}) \oplus (\overline{84},6) \oplus (\bar{6},84) \oplus (35,1) \oplus (1,35)$$
$$\oplus (189,1) \oplus (1,189)$$

$$5940 = (21,\overline{21}) \oplus (\overline{21},21) \oplus (6,\bar{6}) \oplus (\bar{6},6) \oplus (35,35) \oplus (1,1)$$
$$\oplus (120,\bar{6}) \oplus (6,\overline{120}) \oplus (\overline{120},6) \oplus (\bar{6},120) \oplus (35,1) \oplus (1,35)$$
$$\oplus (405,1) \oplus (1,405)$$

$$220 = (20,1) \oplus (1,20) \oplus (15,6) \oplus (6,15)$$

$$572 = (70,1) \oplus (1,70) \oplus (21,6) \oplus (6,21) \oplus (15,6) \oplus (6,15)$$

$$364 = (56,1) \oplus (1,56) \oplus (21,6) \oplus (6,21) \tag{2.56}$$

(3) SU(3) ⊗ SU(2) Decomposition of some SU(6) multiplets

$$35 = (8,3) \oplus (1,3) \oplus (8,1)$$

$$189 = (8,5) \oplus (1,5) \oplus (10,3) \oplus (\overline{10},3) \oplus 2(8,3)$$

$$\oplus (27,1) \oplus (8,1) \oplus (1,1)$$

$$405 = (27,5) \oplus (8,5) \oplus (1,5) \oplus (27,3) \oplus (10,3)$$

$$\oplus (\overline{10},3) \oplus 2(8,3) \oplus (27,1) \oplus (8,1) \oplus (1,1)$$

$$20 = (1,4) \oplus (8,2)$$

$$70 = (8,4) \oplus (10,2) \oplus (8,2) \oplus (1,2)$$

$$56 = (10,4) \oplus (8,2) \tag{2.57}$$

(4) $SU_J(4) \otimes SU_Y(2)$ Decomposition of some SU(6) multiplets

The third entry in the brackets below refers to the hypercharge

$$35 = (4,2,1) \oplus (15,1,0) \oplus (1,3,0) \oplus (1,1,0) \oplus (\overline{4},2,-1)$$

$$189 = (6,1,2) \oplus (\overline{20}',2,1) \oplus (4,2,1) \oplus (20'',1,0)$$

$$\oplus (15,3,0) \oplus (15,1,0) \oplus (1,1,0) \oplus (20',2,-1)$$

$$\oplus (\overline{4},2,-1) \oplus (\overline{6},1,-2)$$

$$
\begin{aligned}
405 &= (10,3,2) \oplus (\overline{36},2,1) \oplus (4,4,1) \oplus (4,2,1) \oplus (84,1,0) \\
&\oplus (15,3,0) \oplus (15,1,0) \oplus (1,5,0) \oplus (1,3,0) \\
&\oplus (\bar{4},4,-1) \oplus (\bar{4},2,-1) \oplus (36,-2,1) \oplus (\overline{10},3,-2) \\
20 &= (\bar{4},1,1) \oplus (6,2,0) \oplus (4,1,-1) \\
70 &= (20',1,1) \oplus (6,2,0) \oplus (10,2,0) \oplus (4,1,-1) \\
&\oplus (4,3,-1) \oplus (1,2,-2) \\
56 &= (20,1,1) \oplus (10,2,0) \oplus (4,3,-1) \oplus (1,4,-2)
\end{aligned}
\tag{2.58}
$$

This completes all that we need to know about the homogeneous U(6, 6) group and its subgroups for the forthcoming development.

3. FREE PARTICLE FORMALISM

(a) Bargmann-Wigner Equations as Relativistic Boosts

Since the homogeneous group U(6, 6) is non-compact its finite-dimensional representations are not unitary and so cannot be associated directly with physical particles. These representations in fact are realized in complex vector spaces with indefinite metric. It is possible, however, to project out the definite sectors in a simple way and with these there is no obstacle to making a physical interpretation.

Let us begin with a multispinor $\phi^{B_1 \ldots}_{A_1 A_2 \ldots}$ belonging to some irreducible representation of U(6, 6). For the invariant scalar product we must take

$$
\phi^{B_1 \ldots}_{A_1 A_2 \ldots} \bar{\phi}^{A_1 A_2 \ldots}_{B_1 \ldots} = \phi^{B_1 \ldots}_{A_1 A_2 \ldots} (\gamma_0)^{B_1'}_{B_1} \cdots \left(\phi^{B_1' \ldots}_{A_1' A_2' \ldots} \right)^* (\gamma_0)^{A_1}_{A_1'} (\gamma_0)^{A_2}_{A_2'} \cdots \tag{3.1}
$$

where the asterisk denotes complex conjugation and

$$
(\gamma_0)^{B}_{A} = (\gamma_0)^{\beta}_{\alpha} \, \delta^{s}_{r} \tag{3.2}
$$

In the Pauli representation of the Dirac matrices we have

$$(\gamma_0)_\alpha^{\beta} = \operatorname{diag}(+1, +1, -1, -1)$$

The form (3.1) is evidently not positive definite. A definite subspace can be projected out simply by restricting the set $\phi_{A_1 A_2 \ldots}^{B_1}$ to those on which γ_0 takes one value, for example

$$(\gamma_0)_{A_1}^{A_1'} \phi_{A_1' A_2}^{B_1 \ldots} = \phi_{A_1 A_2 \ldots}^{B_1 \ldots},$$

$$(\gamma_0)_{A_2}^{A_2'} \phi_{A_1 A_2' \ldots}^{B_1 \ldots} = \phi_{A_1 A_2 \ldots}^{B_1 \ldots}, \ldots \tag{3.3}$$

for lower indices and

$$(\gamma_0)_{B_1'}^{B_1} \phi_{A_1 A_2}^{B_1'} = -\phi_{A_1 A_2 \ldots}^{B_1 \ldots}, \ldots \tag{3.4}$$

for upper indices. The special virtue in this choice of signs will be argued shortly.

The set of multispinors satisfying (3.3) and (3.4) is not invariant under the full group. Rather, we have reduced the U(6, 6) representation under the subgroup consisting of those matrices S_A^B for which

$$S \gamma_0 S^{-1} = \gamma_0 \tag{3.5}$$

i.e. for which

$$S^{-1} = S^{\dagger}$$

This is the maximal compact subgroup, U(6) ⊠ U(6). The indefinite space of the original representation breaks up in this fashion into a collection of subspaces, invariant under U(6) ⊠ U(6), each of which is definite. The prescriptions (3.3) and (3.4) single out a particular one of these.

It is easy to verify that under U(6) ⊗ U(6) the basic spinors ψ_A and $\bar{\psi}^A$ reduce in the following way*

$$\psi_A = (6,1)_+ \oplus (1,6)_-$$

$$\bar{\psi}^A = (\bar{6},1)_+ \oplus (1,\bar{6})_- \qquad (3.6)$$

where the appended sign indicates the value taken by γ_0 in the respective subspaces. If we apply antiparticle conjugation,

$$\psi_{\alpha r} \rightarrow C_{\alpha\beta} \bar{\psi}^{\beta r} \qquad (3.7)$$

and notice the property $C^{-1}\gamma_0 C = -\gamma_0$ then it follows that under this operation

$$\left.\begin{aligned} (6,1) &\rightarrow (1,\bar{6}) \\ \text{and} \qquad (1,6) &\rightarrow (\bar{6},1) \end{aligned}\right\} \qquad (3.8)$$

The relative signs chosen in (3.3) and (3.4), it now appears, assure the invariance of these equations under antiparticle conjugation. The spinors ψ_A and $\bar{\psi}^A$ in particular become associated with (6,1) and (1,$\bar{6}$), or quark and antiquark, respectively. The other set (1,6) and ($\bar{6}$,1) are excluded[28] by (3.3), (3.4). From the products of (6,1) and (1,$\bar{6}$) alone we shall not be able to construct all the representations of U(6) ⊗ U(6). Equations (3.3) and (3.4) are thus seen to constitute a restrictive assumption. Insofar as the accommodating of known particles is concerned, it has not proved unduly restrictive[29].

If the multispinors discussed above are taken to represent the possible states of particles at rest then it is possible to set them in motion by applying the appropriate relativistic boost[30]. Denote by $(L_p)_\alpha^\beta$ a family of Lorentz matrices for which

$$L_p \gamma_0 L_p^{-1} = \frac{1}{m}\not{p} = \frac{1}{m}(p_0\gamma_0 - p_1\gamma_1 - p_2\gamma_2 - p_3\gamma_3) \qquad (3.6)$$

where m denotes the rest mass, $p^2 = m^2$. We can then define the state with momentum p_μ by

$$\phi^{B_1 \ldots}_{A_1 A_2 \ldots}(p) = (L_p)^{A_1'}_{A_1} (L_p)^{A_2'}_{A_2} \ldots \phi^{B_1' \ldots}_{A_1' A_2' \ldots} (L_p^{-1})^{B_1}_{B_1'} \ldots \quad (3.7)$$

In analogy with the reduction mentioned above of a U(6, 6) representation with respect to U(6) ⊠ U(6) we can do the same with respect to $(U(6) \otimes U(6))_p$ defined as the subgroup for which

$$S \not{p} S^{-1} = \not{p} \quad (3.8)$$

It is just the little group[30]. There is, of course, one subspace which, in the limit $\vec{p} \to 0$ goes into the space picked out by equations (3.3) and (3.4). This subspace consists of the set of $\phi^{B_1 \ldots}_{A_1 A_2 \ldots}(p)$ satisfying the relations,

$$(\not{p})^{A_1'}_{A_1} \phi^{B_1 \ldots}_{A_1' A_2 \ldots}(p) = m\, \phi^{B_1 \ldots}_{A_1 A_2 \ldots}(p)$$

$$(\not{p})^{A_2'}_{A_2} \phi^{B_1 \ldots}_{A_1 A_2' \ldots}(p) = m\, \phi^{B_1 \ldots}_{A_1 A_2 \ldots}(p), \ldots \quad (3.9)$$

for lower indices, and

$$(\not{p})^{B_1}_{B_1'} \phi^{B_1' \ldots}_{A_1 A_2 \ldots}(p) = -m\, \phi^{B_1 \ldots}_{A_1 A_2 \ldots}(p) \quad (3.10)$$

for upper indices. The prescriptions (3.9) and (3.10) will be referred to as the BARGMANN-WIGNER equations[7].

Defining a linearly independent set of positive and negative energy solutions of the Dirac equation by

$$u^a_A(p) = (L_p)^a_A \,, \quad \bar{u}^A_a(-p) = (L_p^{-1})^A_a \,; \; p_0 > 0 \,, \; a = 1, \ldots 6 \quad (3.11)$$

we can rewrite (3.7) in the form

$$\phi^{B_1 \ldots}_{A_1 A_2 \ldots}(p) = u^{a_1}_{A_1}(p)\, u^{a_2}_{A_2}(p) \ldots \phi^{b_1 \ldots}_{a_1 a_2 \ldots}\, \bar{u}^{B_1}_{b_1}(-p) \ldots \quad (3.12)$$

which exhibits the multispinor $\phi^{B_1}_{A_1 A_2}(p)$ explicitly as a solution of the B.W. equations. Evidently a transformation of U(6) ⊠ U(6) on $\phi^{b_1 \ldots}_{a_1 a_2 \ldots}$ induces on $\phi^{B_1}_{A_1 A_2}(p)$ a transformation of the little group $U(6) \otimes U(6)\,)_p$.

In summary, the scheme for associating physical states with an irreducible finite-dimensional representation of the non-compact group U(6, 6) lies in breaking up the indefinite space into a set of subspaces each of which is definite and, moreover, invariant under the compact group $(U(6) \otimes U(6))_p$. To characterize the physical states one keeps only those vectors whose components vanish outside the subspace specified by the B.W. equations (3.9), (3.10). Alternatively, one could reduce to subspaces invariant under the U(6) ⊠ U(6) specified by (3.5) and require that the components of a physical state in each of these be related through equation (3.12).

(b) Many-particle states

The group of transformations applicable on a 1-particle state with momentum p_μ consists of all those transformations in U(6, 6) which commute with $\not{p}$, namely $(U(6) \otimes U(6))_{\not{p}}$. The same would be true of many-particle states provided there was no relative momentum.

For a 2-particle state with distinct momenta p_1, p_2 there are two independent boosts operating and the only U(6, 6) transformations which leave the momenta unaffected are those which commute with $\not{p}_1$ and $\not{p}_2$. Choosing the co-ordinate system such that p_1, p_2 lie in the 0-3 plane we require all those matrices of U(6, 6) which commute with γ_0 and γ_3. These are generated by

$$\sigma_{12} T^i, \quad \sigma_{25} T^i, \quad \sigma_{51} T^i \qquad i = 0, 1, \ldots, 8$$

where $\sigma_{i5} = i\gamma_i\gamma_5, \ldots$ etc. This group may be labelled $(U(6))_{p_1 p_2}$. It is otherwise known as $U(6)_W$,[17] $\tilde{PU}(12)$, the "hybrid[11] U(6)", or the "lesser" group[18].

Similarly, for the 3-particle state with momenta p_1, p_2, p_3 confined to the 023 space the transformation group is confined to those matrices which commute with γ_0, γ_2 and γ_3, namely

$$\sigma_{i5}\, T^i \,, \quad i = 0, \ldots, 8$$

which group may be labelled[11] $(U(3) \otimes U(3))_{p_1 p_2 p_3}$.

Finally, states with four or more independent momenta can be subjected only to those transformations which commute with γ_μ namely those generated by T^i or simply U(3).

Thus we have a hierarchy of "little groups"

$$U(6) \otimes U(6) \rightarrow U(6) \rightarrow U(3) \otimes U(3) \rightarrow U(3)$$

which can be applied, for example, in S-matrix or form-factor calculations[4]. Mass spectra which involve one 4-momentum should be classified under $U(6) \otimes U(6)$. Coupling constants, involving 2 momenta, should be classified under U(6) and scattering amplitudes, involving 3 momenta, under $U(3) \otimes U(3)$. These considerations of course ignore the unitarity contributions of the many-particle intermediate states which can have at most U(3) symmetry.

(c) Physical Multiplets

The association of $U(6) \otimes U(6)$ multiplets with U(6, 6) representations is given explicitly and in general by formula (3.12). For practical calculations however there is an alternative formulation which is more useful. Since it involves a manipulation of the Dirac indices only we shall for the moment suppress the SU(3) indices.

It is a simple matter to verify the equivalence between the following two ways of writing the second-rank multispinor ϕ_α^β which satisfies the B.W. equations,

$$(\not{p} - m)_{\alpha}^{\alpha'} \phi_{\alpha'}^{\beta}(p) = 0 \quad \text{and} \quad (\not{p} + m)_{\beta'}^{\beta} \phi_{\alpha}^{\beta'}(p) = 0 \tag{3.13}$$

namely,

$$\phi_{\alpha}^{\beta}(p) = u_{\alpha}^{a}(p)\ \phi_{a}^{b}(p)\ \bar{u}_{b}^{\beta}(-p) \tag{3.14}$$

and

$$\phi_{\alpha}^{\beta}(p) = \left[(\not{p} + m)(\gamma_\mu \phi_\mu - \gamma_5 \phi_5)\right]_{\alpha}^{\beta} \tag{3.15}$$

In fact the connection is given by

$$\phi_\mu(p) = \frac{1}{4m}\ \bar{u}_b(-p)\gamma_\mu\ \bar{u}^a(p)\ \phi_a^b(p)$$

$$\phi_5(p) = \frac{1}{4m}\ \bar{u}_b(-p)\ \gamma_5\ \bar{u}^a(p)\ \phi_a^b(p) \tag{3.16}$$

and, conversely

$$\phi_a^b(p) = \frac{1}{2m}\ \bar{u}_a(p)\ (\gamma_\mu \phi_\mu - \gamma_5 \phi_5)\ u^b(-p) \tag{3.17}$$

Notice in particular that $p_\mu \phi_\mu = 0$ so that the vector ϕ_μ characterizes the spin-parity 1^- part of the multiplet and ϕ_5 characterizes the 0^- part[31].

The multispinor $\psi_{\alpha\beta}$ satisfying the equations

$$(\not{p} - m)_{\alpha}^{\alpha'} \psi_{\alpha'\beta} = 0 \quad \text{and} \quad (\not{p} - m)_{\beta}^{\beta'} \psi_{\alpha\beta'} = 0 \tag{3.18}$$

which can be reduced into a symmetrical piece $\psi_{(\alpha\beta)}$ and an anti-symmetrical piece $\psi_{[\alpha\beta]}$ may be dealt with in a similar fashion giving[4]

$$\psi_{(\alpha\beta)}(p) = \left((\not{p} + m)\gamma_\mu C\right)_{\alpha\beta} \psi_\mu(p) \tag{3.19}$$

$$\psi_{[\alpha\beta]}(p) = \left((\not p + m)\gamma_5 C\right)_{\alpha\beta}\psi(p) \tag{3.20}$$

where $p_\mu \psi_\mu = 0$ and where $C_{\alpha\beta}$ is the matrix introduced in Section 2. One can verify that $\psi(p)$ and $\psi_\mu(p)$ correspond to the 0^+ and 1^+ parts respectively.

Let us apply these results to several of the smaller representations[4].

(1) Meson 143^- or $(6,\bar{6})^-$

The traceless U(6, 6) tensor ϕ_A^B when subjected to the B.W. relations simplifies as follows

$$\begin{aligned} \phi_A^B &= \phi_{\alpha r}^{\beta s} \\ &= \frac{1}{2\mu}\left((\not p + \mu)(\gamma_\mu \phi_{\mu r}^{s} - \gamma_5 \phi_{5 r}^{s})\right)_\alpha^\beta \end{aligned} \tag{3.21}$$

where $\phi_{\mu r}^{s}$ and ϕ_{5r}^{s} correspond to the 1^- and 0^- parts of the multiplet respectively. From each can be separated an SU(3) singlet and octet part,

$$\begin{aligned} \phi_{\mu r}^{s} &= \phi_\mu(1) \oplus \phi_\mu(8) \\ \phi_{5 r}^{s} &= \phi_5(1) \oplus \phi_5(8) \end{aligned} \tag{3.22}$$

(2) Meson 4212^+ or $(15,\overline{15})^+$

The representation 4212 consists of traceless tensors with a pair of lower indices anti-symmetrisized and a pair of upper indices also anti-symmetrized. After applying the B.W. equations it can be written in the form

$$(g\mu)^2 \phi^{[B_1 B_2]}_{[A_1 A_2]} = \left((\not{p}+\mu)\gamma_\mu C\right)_{\alpha_1\alpha_2} \phi^{[s_1 s_2]}_{\mu\nu[r_1 r_2]} \left(C^{-1}\gamma_\nu(\not{p}-\mu)\right)^{\beta_1\beta_2}$$

$$+\left((\not{p}+\mu)\gamma_\mu C\right)_{\alpha_1\alpha_2} \phi^{(s_1 s_2)}_{\mu 5[r_1 r_2]} \left(C^{-1}\gamma_5(\not{p}-\mu)\right)^{\beta_1\beta_2}$$

$$+\left((\not{p}+\mu)\gamma_5 C\right)_{\alpha_1\alpha_2} \phi^{[s_1 s_2]}_{5\mu(r_1 r_2)} \left(C^{-1}\gamma_\mu(\not{p}-\mu)\right)^{\beta_1\beta_2}$$

$$+\left((\not{p}+\mu)\gamma_5 C\right)_{\alpha_1\alpha_2} \phi^{(s_1 s_2)}_{55(r_1 r_2)} \left(C^{-1}\gamma_5(\not{p}-\mu)\right)^{\beta_1\beta_2} \qquad (3.23)$$

The SU(3)-irreducible parts can be separated out in the usual way by extracting traces etc; the content is as follows:

$$\phi^{[s_1 s_2]}_{\mu\nu[r_1 r_2]} = \phi_{\mu\nu}(1) \oplus \phi_{\mu\nu}(8)$$

$$\phi^{(s_1 s_2)}_{\mu 5[r_1 r_2]} = \phi_{\mu 5}(8) \oplus \phi_{\mu 5}(10)$$

$$\phi^{[s_1 s_2]}_{5\mu(r_1 r_2)} = \phi_{5\mu}(8) \oplus \phi_{5\mu}(\overline{10})$$

$$\phi^{(s_1 s_2)}_{55(r_1 r_2)} = \phi_{55}(1) \oplus \phi_{55}(8) \oplus \phi_{55}(27) \qquad (3.24)$$

The Lorentz scalar, ϕ_{55} corresponds to a 0^+ particle, the axial vectors $\phi_{\mu 5}$ and $\phi_{5\mu}$ to 1^+ particles and the tensor $\phi_{\mu\nu}$ to 0^+, 1^+, and 2^+ particles which can be separated without difficulty.

$$\phi_{\mu\nu}(0^+) = \frac{1}{3}\left(g_{\mu\nu} - \frac{p_\mu p_\nu}{\mu^2}\right)\phi_{(\lambda\lambda)}$$

$$\phi_{\mu\nu}(1^+) = \phi_{[\mu\nu]}$$

$$\phi_{\mu\nu}(2^+) = \phi_{(\mu\nu)} - \frac{1}{3}\left(g_{\mu\nu} - \frac{p_\mu p_\nu}{\mu^2}\right)\phi_{(\lambda\lambda)} \qquad (3.25)$$

where $\phi_{(\mu\nu)}$ and $\phi_{[\mu\nu]}$ denote the symmetrical and anti-symmetrical parts of $\phi_{\mu\nu}$ respectively.

The meson 5940^+ or $(21, \overline{21})^+$ characterized by the U(6, 6) tensor $\phi^{(B_1 B_2)}_{(A_1 A_2)}$ can be handled in a similar fashion.

(3) Baryon 364^+ or $(56, 1)^+$

The fully symmetrical tensor $\Psi_{(A_1 A_2 A_3)}$ may be written

$$\Psi_{(A_1 A_2 A_3)} = \frac{1}{2m}\sqrt{\frac{3}{2}}\left((\not{p}+m)\gamma_\mu C\right)_{\alpha_1\alpha_2} D_{\mu\alpha_3(r_1 r_2 r_3)}$$

$$+ \frac{1}{2m}\sqrt{\frac{1}{6}}\left[\left((\not{p}+m)\gamma_5 C\right)_{\alpha_1\alpha_2}\varepsilon_{r_1 r_2 s} N^{s}_{\gamma r_3} + \text{cycl.}(1,2,3)\right] \qquad (3.26)$$

where D_μ satisfies (in addition to the Dirac equation)

$$(\gamma_\mu)^\beta_\alpha D_{\mu\beta(r_1 r_2 r_3)} = 0 \qquad (3.27)$$

(4) Baryon 572^+ or $(70, 1)^+$

The tensor of mixed symmetry type, $\Psi_{[A_1 A_2] A_3}$, satisfying

$$\Psi_{[A_1 A_2]A_3} + \Psi_{[A_2 A_1]A_3} = 0 \quad \text{and} \quad \Psi_{[A_1 A_2]A_3} + \Psi_{[A_2 A_3]A_1} + \Psi_{[A_3 A_1]A_2} = 0 \qquad (3.28)$$

may be written as[32]

$$\bar{\Psi}_{[A_1 A_2] A_3} = \frac{1}{2m}\frac{1}{2}\left((\not p + m)\gamma_\mu C\right)_{\alpha_1\alpha_2} \varepsilon_{r_1 r_2 s} \mathcal{N}_{\mu\alpha_3, r_3}^{s}$$

$$+\frac{1}{2m}\sqrt{\frac{1}{6}}\left[\varepsilon_{r_2 r_3 s}\left((\not p + m)\gamma_5 C\right)_{\alpha_2\alpha_3}\mathcal{N}_{\alpha_1, r_1}^{s} - \varepsilon_{r_3 r_1 s}\left((\not p + m)\gamma_5 C\right)_{\alpha_3\alpha_1}\mathcal{N}_{\alpha_2, r_2}^{s}\right.$$

$$+\frac{1}{2m}\frac{1}{6}\varepsilon_{r_1 r_2 r_3}\left[\left((\not p + m)\gamma_5 C\right)_{\alpha_1\alpha_3}\mathcal{Y}_{\alpha_2} + \left((\not p + m)\gamma_5 C\right)_{\alpha_2\alpha_3}\mathcal{Y}_{\alpha_1}\right]$$

$$+\frac{1}{2m}\sqrt{\frac{1}{2}}\left((\not p + m)\gamma_5 C\right)_{\alpha_1\alpha_2}\mathcal{D}_{\alpha_3 (r_1 r_2 r_3)} \qquad (3.29)$$

where

$$(\gamma_\mu)_\alpha^\beta \mathcal{N}_{\mu\beta, r}^{s} = 0 \qquad (3.30)$$

(5) <u>Baryon 220^+ or $(20,1)^+$</u>

The fully anti-symmetrical tensor $\bar{\Psi}_{[A_1 A_2 A_3]}$ may be written

$$\bar{\Psi}_{[A_1 A_2 A_3]} = \frac{1}{2m}\sqrt{\frac{1}{12}}\varepsilon_{r_1 r_2 r_3}\left((\not p + m)\gamma_\mu C\right)_{\alpha_1\alpha_2}\Delta_{\mu\alpha_3}$$

$$+\frac{1}{2m}\frac{1}{3}\sqrt{\frac{1}{6}}\left[\left((\not p + m)\gamma_5 C\right)_{\alpha_1\alpha_2}\left(\varepsilon_{r_1 r_2 s}\mathcal{V}_{\alpha_3, r_2}^{s} + \varepsilon_{r_2 r_3 s}\mathcal{V}_{\alpha_3 r_1}^{s}\right) + \text{cyc}\right] \qquad (3.31)$$

where

$$(\gamma_\mu)_\alpha^\beta \Delta_{\mu\beta} = 0$$

(d) Kinetic Supermultiplets

A scheme has been proposed[33] wherein higher multiplets are regarded as angular momentum enhanced recurrences of lower ones. The so-called "kinetic supermultiplet" is described in terms of a reducible tensor which is obtained as a product of a basic U(6, 6) irreducible tensor and kinetic tensor components of the regular representation. The meson 143-fold leads, for example, to the fourth-rank tensor

$$\Phi_{AC}^{BD}(p) = (\gamma_\mu)_A^B \, \phi_{\mu C}^{\;D}$$

where

$$p_\mu \, \phi_{\mu C}^{\;D} = 0$$

The new tensor contains states which are obtainable from those in the 143^- by compounding one unit of orbital angular momentum with them, namely singlet and octet of spins $0^+, 1^+, 1^+$, and 2^+.

(e) Mass Splitting

To conclude this section we give a brief discussion of the modifications that are necessary when the masses are not degenerate. The general procedure is to subdivide further the $U(6) \otimes U(6)$ invariant spaces until the stage is reached where, in the subspaces so defined, the masses are completely degenerate. This reduction can be carried out in the rest-frame to begin with - the corresponding reduction for states with momentum $\underset{\sim}{p} \neq 0$ being deduced from it by applying appropriate relativistic boosts. The new feature is that the boosts, being mass dependent, will differ from one subspace to the next. The resulting multispinor will satisfy B. W.-like equations.

Given the reduction to mass-degenerate subspaces in the rest-frame,

$$\phi^{B_1 \cdots}_{A_1 A_2 \cdots} = \phi^{B_1 \cdots}_{A_1 A_2 \cdots}(m_1) + \phi^{B_1 \cdots}_{A_1 A_2 \cdots}(m_2) + \cdots \tag{3.32}$$

where each of the multispinors on the right-hand side satisfies the equations (3.3) and (3.4), we can make the boost to states with momentum $\underset{\sim}{p}$ through an obvious generalization of (3.12), namely

$$\phi^{B_1 \cdots}_{A_1 A_2 \cdots}(p) = \phi^{B_1 \cdots}_{A_1 A_2 \cdots}(p, m) + \phi^{B_1 \cdots}_{A_1 A_2 \cdots}(p, m_2) + \cdots \tag{3.33}$$

where

$$\phi^{B_1 \cdots}_{A_1 A_2 \cdots}(p, m_i) = u^{a_1}_{A_1}(p, m_i)\, u^{a_2}_{A_2}(p, m_i) \cdots \phi^{b_1 \cdots}_{a_1 a_2 \cdots}(p, m_i)\, \bar{u}^{B_1}_{b_1}(-p, m_i) \cdots \tag{3.34}$$

with

$$\phi^{b_1 \cdots}_{a_1 a_2 \cdots}(p, m_i) = \delta(p^2 - m_i^2)\, \varphi^{b_1 \cdots}_{a_1 a_2 \cdots}(\underset{\sim}{p}, m_i) \tag{3.35}$$

Applying $\not{p}$ to (3.33) we get, for example

$$(\not{p})^{A_1'}_{A_1} \phi^{B_1 \cdots}_{A_1' A_2 \cdots}(p) = m_1 \phi^{B_1 \cdots}_{A_1 A_2 \cdots}(p, m_1) + m_2 \phi^{B_1 \cdots}_{A_1 A_2 \cdots}(p, m_2)$$

In order to find the equations satisfied by $\phi(p)$ it is necessary to eliminate the $\phi(p, m_i)$ from the right-hand side, that is, to find some (p-independent) projections $E(m_i)$ such that

$$\phi^{B_1 \cdots}_{A_1 A_2 \cdots}(p, m_i) = \left(E(m_i)\, \phi\right)^{B_1 \cdots}_{A_1 A_2 \cdots}(p) \tag{3.36}$$

for then

$$(\not p)_{A_1}^{A_1'} \phi_{A_1' A_2 \ldots}^{B_1 \ldots}(p) = \sum_i m_i \left(E(m_i)\phi\right)_{A_1 A_2 \ldots}^{B_1 \ldots}(p)$$

$$(\not p)_{A_2}^{A_2'} \phi_{A_1 A_2' \ldots}^{B_1 \ldots}(p) = \sum_i m_i \left(E(m_i)\phi\right)_{A_1 A_2 \ldots}^{B_1 \ldots}(p) \qquad (3.37)$$

for lower indices, and

$$(\not p)_{B_1'}^{B_1} \phi_{A_1 A_2 \ldots}^{B_1' \ldots}(p) = -\sum_i m_i \left(E(m_i)\phi\right)_{A_1 A_2 \ldots}^{B_1 \ldots}(p) \qquad (3.38)$$

for upper indices. These equations represent the generalization to the case of non-degenerate masses of the B.W equations (3.9) and (3.10). The problem lies in constructing, for a given situation, the projections $E(m_i)$

The projection E appropriate to a given case can be produced by the application of some simple rules which we now derive. Since the projections depend only on the masses they must be Lorentz invariant and, in the rest frame, they must leave invariant the equations (3.3) and (3.4). Replacing the 12-valued indices A, B, ... by the Dirac-SU(3) pairs αr, βs etc., we see that the SU(3) structure of the projections is unrestricted but that the Dirac structure must be built up from the invariants δ_α^β, $C_{\alpha\beta}$ and $(C^{-1})^{\alpha\beta}$. Explicitly, for the Dirac multispinor ϕ_α^β we can project out just the singlet and triplet parts,

$$E_{\alpha\alpha'}^{\beta\beta'}(1)\, \phi_{\beta'}^{\alpha'} = \frac{1}{2}\left(\phi_\alpha^\beta + (C^{-1})^{\beta\beta'} \phi_{\beta'}^{\alpha'} C_{\alpha'\alpha}\right)$$

$$(3.39)$$

$$E_{\alpha\alpha'}^{\beta\beta'}(3)\, \phi_{\beta'}^{\alpha'} = \frac{1}{2}\left(\phi_\alpha^\beta - (C^{-1})^{\beta\beta'} \phi_{\beta'}^{\alpha'} C_{\alpha'\alpha}\right)$$

which are easily verified by reference to (3.15).

Now in reducing out a particular component from the U(6) ⊠ U(6) multispinor

$$\phi^{b_1 \dots}_{a_1 a_2 \dots} = \phi^{j_1 s_1 \dots}_{i_1 r_1 i_2 r_2 \dots}$$

(where i, j, ... = 1, 2 denote SU(2) indices) one performs a succession of trace and symmetrization or anti-symmetrization operations which we need not go into. To reduce out the analogous component of the corresponding U(6, 6) multispinor,

$$\phi^{B_1 \dots}_{A_1 A_2 \dots} = \phi^{\beta_1 s_1 \dots}_{\alpha_1 r_1 \alpha_2 r_2}$$

one performs exactly the same operations except that in the case of the SU(2) trace

$$\phi^{j \dots}_{i \dots} \longrightarrow \frac{1}{2}\, \delta^{j}_{i}\, \phi^{k \dots}_{k \dots} \qquad (3.40)$$

one does instead

$$\phi^{\beta \dots}_{\alpha \dots} \longrightarrow E^{\beta \beta'}_{\alpha \alpha'}(1)\, \phi^{\alpha' \dots}_{\beta' \dots} \qquad (3.41)$$

Thus the problem of constructing the projections needed for the generalized B.W equations (3.37), (3.38) is reduced to an analogous but manageable problem of reducing a U(6) ⊠ U(6) multispinor under the SU(2) spin-group.

By way of example consider the mass splitting that comes about when the U(6) ⊠ U(6) symmetry is reduced to U(6). For example

$(6,\bar{6}) = \underline{1} + \underline{35}$ for mesons,

and $(56,1) = \underline{56}$ for baryons.

Hence this symmetry allows the mass of the meson singlet (X^0) to be split from the others and nothing more. The projection is simply

$$\phi^{\beta s}_{\alpha r}(X^0) = \frac{1}{3} E^{\beta\beta'}_{\alpha\alpha'}(1)\, \delta^s_r \phi^{\alpha' t}_{\beta' t}$$

$$= \frac{1}{6}\delta^s_r \left(\phi^{\beta t}_{\alpha t} + (C^{-1})^{\beta\beta'} \phi^{\alpha' t}_{\beta' t} C_{\alpha'\alpha}\right) \qquad (3.42)$$

The B.W equations are thus

$$(\not{p})^{\alpha'}_{\alpha} \phi^{\beta s}_{\alpha' r}(p) = m_{35}\left(\phi^{\beta s}_{\alpha r} - \phi^{\beta s}_{\alpha r}(X^0)\right) + m_1 \phi^{\beta s}_{\alpha r}(X^0)$$

$$= m_{35}\phi^{\beta s}_{\alpha r} + \frac{m_1 - m_{35}}{6} \delta^s_r \left(\phi^{\beta t}_{\alpha t} + (C^{-1})^{\beta\beta'} \phi^{\alpha' t}_{\beta' t} C_{\alpha'\alpha}\right) \qquad (3.43)$$

and

$$(\not{p})^{\beta}_{\beta'} \phi^{\beta' s}_{\alpha r}(p) = -m_{35}\phi^{\beta s}_{\alpha r}$$

$$- \frac{m_1 - m_{35}}{6} \delta^s_r \left(\phi^{\beta t}_{\alpha t} + (C^{-1})^{\beta\beta'} \phi^{\alpha' t}_{\beta' t} C_{\alpha'\alpha}\right) \qquad (3.44)$$

Further mass-splittings can be effected in the same way[34].

4. SUPERMULTIPLET FIELD THEORY AND THE POSSIBLE DYNAMICAL ORIGIN OF THE $\tilde{U}(12)$

So far we have considered free particle states and incorporated phenomenologically into them a $U(6) \otimes U(6)$ symmetry, in the rest frame. To see whether this higher symmetry can persist after all manner of relativistic interactions one is faced with solving a dynamical problem. There are essentially just two types of dynamical procedures known for attempting a solution of this: one is to begin with some sort of fundamental Lagrangian theory with some built-in symmetry. The bound states, with their associated composite fields, will of course exhibit this symmetry but may, for peculiar dynamical reasons exhibit a symmetry which is still higher. The alternative procedure is the exploration of a bootstrap dynamics, which has in fact a similar raison d'être in that one starts with a consistency postulate coupled with a primitive symmetry and shows that dynamical accidents (like dominance of single-particle exchanges) lead consistently to an effective higher symmetry for the particle multiplets. The current algebra approach is similar in character.

We shall adopt the first approach in sections (a) and (e), by seeking a basis for a $U(6) \otimes U(6)$ multiplet structure of the known particles within the dynamics of a quark Lagrangian, assuming these arise as quark-antiquark composites. Thus we start with the assumption of 3 Dirac quarks ψ_1, ψ_2, ψ_3, (p', n', λ' say) and write the conventional free Lagrangian assuming only substitution invariance S_3, $\psi_1 \leftrightarrow \psi_2 \leftrightarrow \psi_3$. This implies mass degeneracy which then is the only postulate needed at this stage. It has been recognized by YAMAGUCHI[36] and others that the resulting permutation-invariant Lagrangian happens also to possess the continuous Lie-group symmetry of SU(3). We show in section (a) that the symmetry is in fact much higher: it is the symmetry $(U(6) \otimes U(6))_p$. We next write down the <u>only possible Lorentz invariant</u> interactions for the three fields $\psi_r(x)$ consistent with U(6) ⊗ U(6), and find that it must be full U(6, 6) invariant. As emphasized by Schwinger this is where the strength of the local field concept comes in.

In sections (b)-(d) we discuss the problem of writing phenomenological free and interaction Lagrangians for the physical multiplets[4], postponing to section (e) the problem of showing how under suitable assumptions they can arise as quark composites from the Lagrangian model of section (a), viz. how the composite multiplets at rest again possess the U(6) ⊗ U(6) structure and the effective quark-composite interaction is U(6, 6) invariant to a good approximation. Clearly the free field equations of the <u>physical</u> multiplets must respect this U(6) ⊗ U(6) structure while for the interaction terms we once again postulate a U(6, 6) symmetry, both for internal consistency and by analogy with the quark problem.

By $\widetilde{U}(12)$ we mean then the theory possessing for its free particle multiplets the U(6) ⊗ U(6) structure relativistically boosted through B. W. equations, together with a U(6, 6) invariant $\mathcal{L}_{int}(x)$.

The dynamical calculations of quark binding in (e) are naturally approximate because of intrinsic difficulties in obtaining the complete solution to a Bethe-Salpeter type of equation. The work is nonetheless quite suggestive with regard to the possible origin of $\widetilde{U}(12)$.

(a) Quark Lagrangians

The free Lagrangian density of 3 Dirac quarks $\psi_A = \psi_{\alpha r}$ (r = 1, 2, 3), which exhibits substitution invariance $\psi_1 \leftrightarrow \psi_2 \leftrightarrow \psi_3$ is evidently

$$\mathcal{L}_f(x) = \bar{\psi}^A(x)\,(i\partial\!\!\!/ - m)_A^B\,\psi_B(x) \qquad ; \quad \partial\!\!\!/_A^{\,B} = \gamma_\alpha^\beta\,\delta_r^s \qquad (4.1)$$

and the only transformations, not involving derivatives, that L_f is invariant under are U(3) ⊗ $I\mathcal{L}_4$. However there exist transformations which change $\mathcal{L}_f$ by a divergence term (leaving the field equations intact). These are obtained by writing the PAULI-LUBANSKI[37] and CALOGERO[38] spin operators with the unitary spin matrices, and form the $[U(6) \otimes U(6)]_{*c}$ group. To understand this we note that $[U(6) \otimes U(6)]_{p}$ has the generators T^j, $\not{p}\, T^j/m$ and

$$\omega_{5\mu}(p)\, T^j = \epsilon_{\mu\nu\kappa\lambda}\, \sigma_{\nu\kappa}\, p_\lambda\, T^j/2m = \gamma_5 \sigma_{\mu\nu} p_\nu\, T^j/m \tag{4.2}$$

$$\omega_{\mu\nu}(p)\, T^j = -i\, \epsilon_{\mu\nu\kappa\lambda}\, \gamma_\kappa \gamma_5\, p_\lambda\, T^j/m \;, \tag{4.3}$$

$$\left[\, p_\mu\, \omega_{5\mu}(p) = p_\mu\, \omega_{\mu\nu}(p) \equiv 0 \tag{4.4} \right]$$

which in the rest frame ($\underline{p} = 0$) reduce to T^j, $\gamma_0 T^j$ and $\omega_{50} T^j = 0$, $\omega_{5r} T^j = -\sigma_r T^j$

$$\omega_{r0}\, T^j = 0 \;, \qquad \omega_{rs}\, T^j = -\gamma_0\, \epsilon_{rst}\, \sigma_t\, T^j$$

characteristic of $U(6) \otimes U(6)$. For a general p the commutation rules of $[U(6) \otimes U(6)]_p$ are

$$[\omega_{5\mu} T^j, \omega_{5\nu} T^k] = i\, d^{ijk}\, \epsilon_{\mu\nu\kappa\lambda}\, p_\kappa\, \omega_{5\lambda} T^i/m + 2i\, f^{ijk}\, T^i\, (p_\mu p_\nu - m^2 g_{\mu\nu})/m^2$$

$$[\omega_{5\lambda} T^j, \omega_{\mu\nu} T^k] = i\omega_{\rho\sigma} \begin{pmatrix} g_{\lambda\mu}\, \epsilon_{\nu\kappa\rho\sigma}\, p_\kappa - g_{\lambda\nu}\, \epsilon_{\mu\kappa\rho\sigma}\, p_\kappa \\ + \epsilon_{\mu\nu\rho\sigma}\, p_\lambda \end{pmatrix} \frac{d^{ijk} T^i}{m} + 2i\, \epsilon_{\kappa\lambda\mu\nu}\, p_\kappa\, f^{ijk}\, T^i\, \not{p}/m^2$$

$$[\omega_{\kappa\lambda} T^j, \omega_{\mu\nu} T^k] = \frac{i\, d^{ijk}\, T^i\, p_\rho}{m} \begin{pmatrix} \epsilon_{\kappa\lambda\nu\rho}\, \omega_{5\mu} - \epsilon_{\kappa\lambda\mu\rho}\, \omega_{5\nu} \\ + \epsilon_{\kappa\rho\mu\nu}\, \omega_{5\lambda} - \epsilon_{\lambda\rho\mu\nu}\, \omega_{5\kappa} \end{pmatrix} + \frac{2i f^{ijk}\, T^i}{m^2} \begin{pmatrix} p_\kappa p_\mu g_{\lambda\nu} - p_\lambda p_\mu g_{\kappa\nu} + p_\nu p_\lambda g_{\kappa\mu} \\ - p_\nu p_\kappa g_{\lambda\mu} + m^2 g_{\kappa\nu} g_{\lambda\mu} - m^2 g_{\kappa\mu} g_{\lambda\nu} \end{pmatrix} \tag{4.5}$$

An infinitesimal $[U(6) \otimes U(6)]_p$ transformation causes

$$\delta\psi(p) = i T^j \left[\eta^j \frac{\not p}{m} + \eta^j_{5\mu}\, \omega_{5\mu}(p) + \tfrac{1}{2}\eta^j_{\mu\nu}\, \omega_{\mu\nu}(p) \right] \psi(p) \qquad (4.6)$$

and will of course commute with the Dirac operator $(\not p - m)$. Hence in momentum space $\delta\mathcal{L}_f(p) = \delta[\bar\psi(p)(\not p - m)\psi(p)] = 0$. The corresponding changes in co-ordinate space have the form

$$\begin{aligned}\delta\mathcal{L}_f(x) = {} & \eta^j_{5\mu}\, \partial_\nu \left[\bar\psi(x)\gamma_5 \sigma_{\mu\nu} T^j (i\not\partial - m)\psi(x)\right] \\ & + \eta'^j \partial_\mu \left[\bar\psi(x)\gamma_\mu T^j (i\not\partial - m)\psi(x)\right] \\ & - \tfrac{1}{2}\eta^j_{\mu\nu}\partial_\lambda \left[\bar\psi(x)\epsilon_{\mu\nu\kappa\lambda}\gamma_\kappa\gamma_5 (i\not\partial - m)\psi(x)\right]\end{aligned} \qquad (4.7)$$

and since these represent divergence terms the physical content is unaltered. Observe that written in the U(6, 6) form,

$$\delta\psi(p) = iT^j\left[\epsilon^j + \epsilon^j_\mu \gamma_\mu + \tfrac{1}{2}\epsilon^j_{\mu\nu}\sigma_{\mu\nu} + \epsilon_{\mu 5}\, i\gamma_\mu\gamma_5\right]\psi(p) \qquad (4.8)$$

with $\epsilon^j = \eta^j$, $\epsilon^j_\mu = p_\mu \eta'^j / m$

$$\epsilon^j_{\mu\nu} = \epsilon_{\mu\nu\kappa\lambda}\, \eta_{5\kappa}\, p_\lambda \; , \qquad \epsilon^j_{\mu 5} = -\tfrac{1}{2}\epsilon_{\mu\nu\kappa\lambda}\, p_\nu\, \eta_{\kappa\lambda}$$

equation (4.6) can be recognized as Barnes' infinitesimal PU(6, 6) transformation[17].

The nature of the U(6) ⊗ U(6) 'invariance group' can be appreciated by writing the equal time commutator,

$$\{\psi_A(x), \bar\psi^B(y)\}\,\delta(x_0 - y_0) = (\gamma_0)^B_A\, \delta^4(x-y) \qquad (4.9)$$

We notice that not only is (4.9) invariant under $U(3) \otimes I\mathcal{L}_4$ but also under the wider group[14] $U(6) \otimes U(6)$. Indeed if we regard the physical particle fields as quark composites constructed effectively as highly localised products of ψ and $\bar{\psi}$ then a $U(6) \otimes U(6)$ multiplet classification is called for.

Coming to interaction Lagrangians, we know that there exist 10 independent 4-Fermi couplings that are $SU(3) \otimes I\mathcal{L}_4$ invariant. Now the basic expectation of a supermultiplet theory is that many of these (large) coupling constants are in fact equal. Indeed if we postulate that the interaction is at least $U(6) \otimes U(6)$ invariant, then the extra requirement of Lorentz invariance leads us to a scalar coupling under $U(6,6)$ as the smallest possible group of invariance[40]. i.e. we are led to

$$\mathcal{L}_{int}(x) = g\, \bar{\psi}^A(x)\, \psi_A(x)\, \bar{\psi}^B(x)\, \psi_B(x) \tag{4.10}$$

Had we instead demanded SU(6) invariance only in the static limit, or SL(6, C) invariance as the relativistic generalization[2]. we should have the two Lagrangians

$$\mathcal{L}_1 = g_1 \left[(\bar{\psi} \gamma_\mu T^j \psi)^2 - (\bar{\psi} \gamma_\mu \gamma_5 T^j \psi)^2 \right]$$

$$\mathcal{L}_2 = g_2 \left[(\bar{\psi} T^j \psi)^2 - (\bar{\psi} \gamma_5 T^j \psi)^2 + \tfrac{1}{2} (\bar{\psi} \sigma_{\mu\nu} T^j \psi)^2 \right]$$

$$\tag{4.11}$$

By the Fierz rearrangement theorem, the larger group $U(6,6)$ implies $\mathcal{L}_{int} = \mathcal{L}_1 + \mathcal{L}_2$ and $g = g_1 = g_2$ and so on.

Without further discussion we will take (4.1) and (4.10) to define our model Lagrangian theory. The interaction and mass term are $U(6,6)$ invariant while the kinetic term $\bar{\psi}\, i\partial\!\!\!/\, \psi$ is not. This is a general feature of the $\tilde{U}(12)$ Lagrangian models for the physical multiplets that we assume later. The equal time commutation relation of the unrenormalised Heisenberg fields is given by (4.9) and the propagator in the interaction picture,

$$(\psi_A, \bar{\psi}^B)_+ = (\not{p} + m)_A^B / (p^2 - m^2) \tag{4.12}$$

shows explicitly how the U(6,6) invariance is destroyed by the "kineton" term[40] $\not{p}_A^B$ in the numerator.

From (4.1) and (4.10) we obtain the equation of motion

$$(i\not{\partial} - m)_A^B \psi_B(x) = 2g\psi_A(x)\bar{\psi}^C(x)\psi_C(x) \tag{4.13}$$

Various other equations of motion are easily derived if we remember that under an infinitesimal U(6,6) transformation there can only be a change in the kinetic part of the free Lagrangian

$$\delta\mathcal{L}_f = -\epsilon_R^j \bar{\psi}[i\not{\partial}, \gamma_R]T^j\psi = -\epsilon_R^j \partial_\lambda g_{\lambda,R}^j \tag{4.14}$$

where

$$i g_{\lambda,R}^j = \frac{\delta\mathcal{L}}{\delta(\partial_\lambda\psi)}\gamma_R T^j\psi = i\bar{\psi}\gamma_\lambda\gamma_R T^j\psi \tag{4.15}$$

We therefore have typical divergence equations,

$$\partial_\lambda g_{\lambda,0}^j = 0$$

$$\begin{aligned}\partial_\lambda g_{\lambda,5}^j &= 2\bar{\psi}\not{\partial}\gamma_5 T^j\psi \\ &= 2im\bar{\psi}\gamma_5 T^j\psi + 2ig\bar{\psi}\gamma_5 T^j\psi\bar{\psi}\psi\end{aligned} \tag{4.16}$$

$$\partial_\lambda g_{\lambda,\mu}^j = 2i\bar{\psi}\sigma_{\mu\nu}\partial_\nu T^j\psi \quad , \text{ etc} \tag{4.17}$$

In that $(\bar{\psi}\gamma_k \Gamma^i \psi)$ is equivalent to a meson field Φ_R^i the above could be used to provide GOLDBERGER-TREIMAN[41] like relations.

It is a straightforward matter to develop the rest of the field theory by constructing the energy-momentum tensor, the total angular momentum operator, etc. Indeed we shall return to the Lagrangian (4.1)-(4.10) to discuss the question of bound states and the U(6) ⊗ U(6) structure of the composite fields. However, as an immediate and important example we cite the U(6,6) generators which can be constructed from the quark Lagrangian. Following GELL-MANN[2,11], their expressions are :-

$$J_R^i(t) = \int_t \bar{\psi}(x)\gamma_0\gamma_R \Gamma^i \psi(x)\, d^3\underset{\sim}{x} \tag{4.18}$$

It is easy to verify from (4.9) that the time-dependent expressions J_R^i obey the commutation rules of the non-compact U(6,6) group.

(b) Free Lagrangians for the 143 and 364 Multiplets

As applied to multispinors with more than two indices the B.W. equations form a largely redundant set. This makes the problem of construcing free Lagrangians quite difficult; therefore we prefer to follow the conventional wave formulation and set down Lagrangians for the Lorentz fields that appear in the multispinor decomposition, Lagrangians which give rise to equations of motion that have exactly the same content as the B.W. equations. The only point of ambiguity of this approach is the non-uniqueness of the contact terms[42] which appear in the propagators.

We briefly recall the reduction of the 143 and 364 multiplets relative to SU(3) ⊗ $I\mathcal{L}_4$ and the consequence of the B.W. equations[4] (see sections 2.c and 2.f).

$$\bar{\Phi}_A^{\;B} = (T^i)_r^{\;s}\left[\phi^i + \gamma_5\phi_5^i + i\gamma_\mu\gamma_5\phi_{\mu 5}^i + \gamma_\mu\phi_\mu^i + \tfrac{1}{2}\sigma_{\mu\nu}\phi_{\mu\nu}^i\right]_\alpha^{\;\beta} \tag{4.19}$$

$$\Psi_{\{ABC\}} = \epsilon_{rst}\, V_{[\alpha\beta\gamma]} + D_{\{rst\}\{\alpha\beta\gamma\}} + $$

$$+ \left[\epsilon_{rsu}\, N_{[\alpha\beta]\gamma,\,t}^{\;u} + \epsilon_{stu}\, N_{[\beta\gamma]\alpha,\,r}^{\;u} + \epsilon_{tru}\, N_{[\gamma\alpha]\beta,\,s}^{\;u}\right] \tag{4.20}$$

with

$$V_{[\alpha\beta\gamma]} = C_{\alpha\beta}\, V'_\gamma + (\gamma_5 C)_{\alpha\beta}\, V_\gamma + (i\gamma_\mu\gamma_5 C)_{\alpha\beta}\, V_{\mu\gamma}$$

$$N_{[\alpha\beta]\gamma} = C_{\alpha\beta}\, N'_\gamma + (\gamma_5 C)_{\alpha\beta}\, N_\gamma + (i\gamma_\mu\gamma_5 C)_{\alpha\beta}\, N_{\mu\gamma}$$

$$D_{\{\alpha\beta\gamma\}} = (\gamma_\mu C)_{\alpha\beta}\, D_{\mu\gamma} + \tfrac{1}{2}(\sigma_{\mu\nu} C)_{\alpha\beta}\, D_{\mu\nu\gamma} \tag{4.21}$$

and

$$V = \gamma_5 V', \qquad V_\mu = -i\gamma_\mu\gamma_5 V'$$

$$N' = \gamma_5 N + i\gamma_\mu\gamma_5 N_\mu$$

$$\gamma_\mu D_\mu = \gamma_\mu D_{\mu\nu} + iD_\nu = 0 \tag{4.22}$$

apart from overall normalisation factors that have been set right in (3.21) and (3.26). Upon application of B.W. equations one finds[4],

$$\phi = N^{i} = V \equiv 0$$

$$(\not{p} - m) N = 0 , \qquad i m N_\mu = p_\mu N$$

$$(\not{p} - m) D_\mu = 0 , \qquad i m D_{\mu\nu} = p_\mu D_\nu - p_\nu D_\mu$$

$$i p_\mu \phi_{\mu 5} = \mu \phi_5 , \qquad i \mu \phi_{\mu 5} = p_\mu \phi_5$$

$$i p_\nu \phi_{\nu\mu} = \mu \phi_\nu , \qquad i \mu \phi_{\mu\nu} = p_\mu \phi_\nu - p_\nu \phi_\mu$$

$$(4.23)$$

By a straightforward generalization of the KEMMER formalism[43] it is a simple matter to set down the Lagrangian of the 143-fold involving the multispinor as a whole:

$$\mathcal{L}_f(143) = \bar{\Phi}^A_C (i\overrightarrow{\not{\partial}})^B_A \Phi^C_B - \bar{\Phi}^A_C (i\overleftarrow{\not{\partial}})^B_A \Phi^C_B - 2m \bar{\Phi}^B_A \bar{\Phi}^A_B \qquad (4.24)$$

This is because the equations

$$i\overrightarrow{\not{\partial}} \bar{\Phi} + \bar{\Phi} i\overleftarrow{\not{\partial}} = 0$$

which complement the equations

$$i\overrightarrow{\not{\partial}} \bar{\Phi} - \bar{\Phi} i\overleftarrow{\not{\partial}} = 2m \bar{\Phi}$$

to form the complete B.W. set, are redundant. However, when we proceed to the higher multispinors such as the 364 fold it is only possible to write an $\mathcal{L}_f$ of the type by introducing auxiliary 572 and 220 fields which greatly complicates the problem[44]; we will adopt the conventional and easier approach of constructing $\mathcal{L}_f$ as functionals of the wave fields ϕ_R ,

N, N_μ, D_μ, $D_{\mu\nu}$ to dispense with redundant field variables. An additional advantage of this method is that whereas the usual gauge generation of electromagnetic interactions by changing $\not\partial \to \not\partial - e\not A$ makes the B. W. set inconsistent there is apparently no contradiction when this replacement is carried out for the Lagrangians below. The free Lagrangians are the following[4]:

$$\mathcal{L}_f(143) = \tfrac{1}{2}\mu\left[(\partial_\lambda \phi_{\lambda 5})\phi_5 - \phi_{\lambda 5}(\partial_\lambda \phi_5) + \phi_{\kappa\lambda}(\partial_\kappa \phi_\lambda - \partial_\lambda \phi_\kappa)\right]$$
$$- \tfrac{1}{2}\mu^2\, \Phi_A^B\, \Phi_B^A$$

(4.25)

$$\begin{aligned}\mathcal{L}_f(364) = {} & \bar N i\not\partial N + \bar N_\mu \cdot \partial_\mu N - \bar N \partial_\mu N_\mu - \partial_\mu \bar N \partial_\mu N / m - \\ & - \bar D_\mu i\not\partial D_\mu + \tfrac{i}{3}\bar D_\mu(\gamma_\mu \partial_\nu + \gamma_\nu \partial_\mu) D_\nu - \\ & - \tfrac{1}{3}\bar D_\mu \gamma_\mu (i\not\partial + m)\gamma_\nu D_\nu + \\ & + (\partial_\nu \bar D_\mu \partial_\nu D_\mu - \partial_\mu \bar D_\mu \partial_\nu D_\nu)/m \\ & - \tfrac{1}{2}\bar D_{\mu\nu}(\partial_\mu D_\nu - \partial_\nu D_\mu) - \tfrac{1}{2}(\bar D_\mu \partial_\nu - \bar D_\nu \partial_\mu) D_{\mu\nu} \\ & - m\, \bar\Psi^{\{ABC\}}\, \Psi_{\{ABC\}}\end{aligned}$$

(4.26)

$$\Phi_A^B \Phi_B^A = \phi^2 - \phi_5^2 - \phi_{\mu 5}^2 + \phi_\mu^2 + \tfrac{1}{2}\phi_{\mu\nu}^2$$

(4.27)

$$\bar\Psi^{\{ABC\}}\Psi_{\{ABC\}} = \bar V V + \bar N_\mu N_\mu + \bar N N - \tfrac{1}{2}\bar D_{\mu\nu} D_{\mu\nu} - \bar D_\mu D_\mu$$

(4.28)

where a trace over unitary spin indices is implied. The above clearly show how the derivative terms upset the U(6, 6) symmetry.

At this point we derive the free particle propagators by the functional differentiation method of GLASHOW[45], where source terms of the type $\frac{1}{2}\mu^2 j^A_B \Phi^B_A + \frac{1}{2} m j^{ABC} \Psi_{\{ABC\}}$ are introduced. Leaving out the unitary spin factors of the type δ^{ij} these Green's functions read

$$(\phi\,,\,\phi)_+ = \frac{1}{\mu^2}$$

$$(\phi_5\,,\,\phi_5)_+ = \frac{1}{p^2-\mu^2}$$

$$(\phi_{\mu 5}\,,\,\phi_5)_+ = -\frac{i p_\mu}{\mu\,(p^2-\mu^2)}$$

$$(\phi_{\mu 5}\,,\,\phi_{\nu 5})_+ = \frac{p_\mu p_\nu}{\mu^2\,(p^2-\mu^2)} - \frac{g_{\mu\nu}}{\mu^2}$$

$$(\phi_\mu\,,\,\phi_\nu)_+ = \frac{-g_{\mu\nu} + p_\mu p_\nu/\mu^2}{p^2-\mu^2}$$

$$(\phi_{\lambda\mu}\,,\,\phi_\nu)_+ = \frac{i\,(p_\lambda g_{\mu\nu} - p_\mu g_{\lambda\nu})}{\mu\,(p^2-\mu^2)}$$

$$(\phi_{\kappa\lambda}\,,\,\phi_{\mu\nu})_+ = \frac{p_\kappa p_\nu g_{\lambda\mu} + p_\lambda p_\mu g_{\kappa\nu} - p_\kappa p_\mu g_{\lambda\nu} - p_\lambda p_\nu g_{\kappa\mu}}{\mu^2\,(p^2-\mu^2)} +$$

$$+ \frac{1}{\mu^2}\,(g_{\kappa\mu} g_{\lambda\nu} - g_{\kappa\nu} g_{\lambda\mu}) \,, \qquad (4.29)$$

$$(V\,,\,\bar{V})_+ = -\frac{1}{m}$$

$$(N, \bar{N})_+ = \frac{1}{\not{p} - m}$$

$$(N_\mu, \bar{N})_+ = \frac{-i p_\mu}{\not{p} - m}$$

$$(N_\mu, \bar{N}_\nu)_+ = \frac{p_\mu p_\nu}{m(\not{p} - m)} - \frac{g_{\mu\nu}}{m}$$

$$(D_\mu, \bar{D}_\nu)_+ = \frac{\nabla_{\mu\nu}}{p^2 - m^2}$$

$$(D_{\lambda\mu}, \bar{D}_\nu)_+ = \frac{i(p_\mu \nabla_{\lambda\nu} - p_\lambda \nabla_{\mu\nu})}{m(p^2 - m^2)}$$

$$(D_{\kappa\lambda}, \bar{D}_{\mu\nu})_+ = \frac{p_\lambda p_\nu \nabla_{\kappa\mu} + p_\kappa p_\mu \nabla_{\lambda\nu} - p_\kappa p_\nu \nabla_{\lambda\mu} - p_\lambda p_\mu \nabla_{\kappa\nu}}{m^2(p^2 - m^2)}$$

$$+ \frac{1}{m}(g_{\kappa\mu} g_{\lambda\nu} - g_{\kappa\nu} g_{\lambda\mu})$$

(4.30)

where

$$\nabla_{\mu\nu}(p) = (\not{p} + m)\left[-g_{\mu\nu} + \tfrac{1}{3}\gamma_\mu\gamma_\nu + \tfrac{1}{3m}(\gamma_\mu p_\nu - \gamma_\nu p_\mu) + \frac{2p_\mu p_\nu}{3m^2}\right]$$

$$- \frac{2}{3m^2}(p^2 - m^2)\left[\gamma_\mu p_\nu - \gamma_\nu p_\mu + (\not{p} - m)\gamma_\mu\gamma_\nu\right]$$

(4.31)

Note the uniform appearance of contact terms involving $g_{\mu\nu}$ and $g_{\kappa\mu} g_{\lambda\nu}$ etc. in the propagators. These extra terms are characteristic of all linearised field theories. For gauge theories it is easy to understand their significance but for spin 1/2 and 3/2 particles their appearance (for the first time in the present theory) presents a new feature. It is worthwhile to mention that these terms are <u>unique</u> for the 143 field and that in this case the propagator is conveniently summarized by

$$(\bar{\Phi}_A^{\ B}, \Phi_C^{\ D})_+ = \frac{(\not{p}-\mu)_C^{\ B}(\not{p}+\mu)_A^{\ D}}{2\mu^2(p^2-\mu^2)} + \frac{\delta_A^D\delta_C^B}{2\mu^2} \qquad (4.32)$$

On the other hand for the 364 field, although the contact term is rather arbitrary[4] the numerator of the pole contribution is well defined by unitarity and

$$(\Psi_{\{ABC\}}, \bar{\Psi}^{\{DEF\}})_+ = \sum_{\substack{ABC \\ \text{perms}}} (\not{p}+m)_A^{\ D}(\not{p}+m)_B^{\ E}(\not{p}+m)_C^{\ F}/(p^2-m^2)$$

$$+ \text{ contact terms} . \qquad (4.33)$$

(c) Interaction Lagrangians

As stated previously we adopt U(6,6) invariant interaction terms and will limit our considerations to the 3-point couplings of the 143 and 364 multiplets, though the methods generalize trivially to other multiplets and interactions. By hypothesis then, with g and h dimensionless coupling constants,

$$\mathcal{L}_{int}(x) = \tfrac{1}{6} h\mu\, \Phi_A^{\ B}(x)\Phi_B^{\ C}(x)\Phi_C^{\ A}(x) + g\,\bar{\Psi}^{\{ABC\}}(x)\,\Phi_C^{\ D}(x)\,\Psi_{\{ABD\}}(x)$$

$$(4.34)$$

Note that the three meson term is unique, as a consequence of charge conjugation invariance[46]

$$\Phi_{\alpha r}^{\ \beta s} \rightarrow C_{\alpha\alpha'}\,\Phi_{\beta' s}^{\ \alpha' r}\,(C^{-1})^{\beta'\beta} \qquad (4.35)$$

whereby $\phi_5 \rightarrow \phi_5$, $\phi_\mu \rightarrow -\phi_\mu$ the normal charge parity situation viz.

$$\pi^0 \rightarrow \pi^0 \quad \text{and} \quad \rho_\mu^0 \rightarrow -\rho_\mu^0 \qquad (4.36)$$

Thus $\mathrm{Tr}\,[\bar{\Phi}\,\Phi\,\Phi]$ may be replaced by $\frac{1}{2}\,\mathrm{Tr}\,[\bar{\Phi}\,\{\Phi,\Phi\}]$. It should be emphasized that U(6, 6) is automatically parity-conserving as γ_0 is part of the U(6, 6) transformation, while charge conjugation invariance is external to the theory and must be invoked separately to limit the possible interactions and associated matrix elements. A further discussion of charge conjugation is given in section 6 to clarify some recent controversy on this matter[12].

Expanding out the multispinors Φ_A^B, $\Psi_{\{ABC\}}$ in (4.34) we obtain for $\mathcal{L}(\bar{\Phi}\Phi\Phi)$ effectively the Casimir operator (2.49) with J^i_R replaced by Φ^i_R. Also,

$$g^{-1}\mathcal{L}(\bar{D}D\phi) = \frac{3}{2}\,\bar{D}^{rst}_{\lambda}\,(T^i)^u_r\,(\phi^i_R\,\gamma_R)\,D_{\lambda,ust} + \frac{3}{4}\,\bar{D}^{rst}_{\kappa\lambda}\,(T^i)^u_r\,(\phi^i_R\,\gamma_R)\,D_{\kappa\lambda,ust} \qquad (4.37)$$

$$g^{-1}\mathcal{L}(\bar{D}N\phi) = -\frac{i}{2}\,\bar{D}^{rst}_{\lambda}\,\gamma_5\,(T^i)^u_r\,(\gamma_R\,\phi^i_R)\,\epsilon_{usv}\,N^{v}_{\lambda t} + \frac{1}{4}\,\bar{D}^{rst}_{\kappa\lambda}\,\gamma_5\,(T^i)^u_r\,(\gamma_R\,\phi^i_R)\,\epsilon_{usv}\,N^{v}_{t} + \frac{1}{4}\,\epsilon_{\mu\nu\kappa\lambda}\,\bar{D}^{rst}_{\mu\nu}\,(T^i)^u_r\,\epsilon_{usv}\,\gamma_\kappa\,N^{v}_{\lambda t}$$

(4.38)

$$g^{-1}\mathcal{L}(\bar{N}N\phi) = \frac{1}{24}\left(\bar{N}^{[\beta\alpha]\gamma}\,\phi^{\delta}_{\alpha}\,(143)\,N_{[\beta\delta]\gamma}\right)_{12S+3D+5F} + \frac{1}{12}\left(\bar{N}^{[\beta\alpha]\gamma}\,\phi^{\delta}_{\alpha}\,(143)\,N_{[\delta\gamma]\beta}\right)_{3S+3D+2F}$$

where

$$(\bar{N}\phi N)_S = \mathrm{Tr}\,(\bar{N}\,T^0\,N)\,\phi^0$$

$$(\bar{N}\phi N)_D = \mathrm{Tr}\,(\bar{N}\,\{T^i, N\})\,\phi^i \qquad (i = 1,\ldots 8)$$

$$(\bar{N}\phi N)_F = \mathrm{Tr}\,(\bar{N}\,[T^i, N])\,\phi^i \qquad (i = 1\ldots 8)$$

(4.39)

and

$$
\begin{aligned}
\tfrac{1}{4}\,\bar N^{[\beta\alpha]\gamma}\,\phi_\alpha^{\alpha'}\,N_{[\alpha'\beta]\gamma} =\; & 2\bar N\,(\phi+\gamma_5\phi_5)\,N \\
& + i\bar N_\mu\,(\phi\gamma_\mu - \phi_\mu - i\phi_{\mu 5}\gamma_5 + \gamma_\mu\gamma_5\phi_5)\,N \\
& + i\bar N\,(-\phi\gamma_\mu + \phi_\mu - i\phi_{\mu 5}\gamma_5 - \gamma_5\gamma_\mu\phi_5)\,N_\mu \\
& + \bar N_\mu \left[\begin{array}{l} (g_{\mu\nu} + \gamma_\mu\gamma_\nu)\,\phi + i\phi_{\mu\nu} \\ + i\phi_{\mu 5}\gamma_\nu\gamma_5 + i\gamma_\mu\gamma_5\phi_{\nu 5} \end{array} \right] N_\nu
\end{aligned}
\tag{4.40}
$$

$$
\begin{aligned}
\tfrac{1}{4}\,\bar N^{[\beta\alpha]\gamma}\,\phi_\alpha^{\alpha'}\,N_{[\alpha'\gamma]\beta} =\; & \bar N\,(\phi + \tfrac{1}{2}\sigma_{\mu\nu}\phi_{\mu\nu} + \gamma_5\phi_5)\,N \\
& + \tfrac{1}{2} i\bar N_\mu \left(\begin{array}{l} \gamma_\mu\phi - \phi_\nu\gamma_\nu\gamma_\mu - \tfrac{1}{2}\phi_{\nu\lambda}\gamma_\mu\sigma_{\nu\lambda} \\ + i\phi_{\nu 5}\gamma_\nu\gamma_5\gamma_\mu + \gamma_\mu\gamma_5\phi_5 \end{array} \right) N \\
& - \tfrac{1}{2} i\bar N \left(\begin{array}{l} \gamma_\mu\phi - \gamma_\mu\gamma_\nu\phi_\nu - \tfrac{1}{2}\phi_{\nu\lambda}\sigma_{\nu\lambda}\gamma_\mu \\ + i\phi_{\nu 5}\gamma_\mu\gamma_5\gamma_\nu + \gamma_5\gamma_\mu\phi_5 \end{array} \right) N_\mu \\
& + \tfrac{1}{2}\bar N_\mu \left[\begin{array}{l} (g_{\mu\nu} + \gamma_\mu\gamma_\nu)\,\phi + \phi_\lambda(\gamma_\mu\gamma_\lambda\gamma_\nu - g_{\mu\nu}\gamma_\lambda) \\ + \phi_5\,(2\gamma_\mu\gamma_5\gamma_\nu + g_{\mu\nu}\gamma_5) \\ + \tfrac{1}{2}\phi_{\kappa\lambda}\,(\gamma_\mu\sigma_{\kappa\lambda}\gamma_\nu + g_{\mu\nu}\sigma_{\kappa\lambda} + 4i g_{\mu\lambda}g_{\kappa\nu}) \\ + \phi_{\kappa 5}\,(i\gamma_\mu\gamma_5 g_{\kappa\nu} + i\gamma_\nu\gamma_5 g_{\kappa\mu} - i\gamma_\kappa\gamma_5 g_{\mu\nu}) \end{array} \right] N_\nu
\end{aligned}
\tag{4.41}
$$

Note carefully the combination 3D + 2F that appears throughout.

The complete Lagrangian of $\tilde{U}(12)$ is the sum of (4.25), (4.26) and (4.34), and typical equation of motions are

$$
\mu^2\phi^i = \tfrac{1}{2} h_\mu\, d^{ijk}\,(\phi^j\phi^k + \phi^j_\mu\phi^k_\mu + \tfrac{1}{2}\phi^j_{\mu\nu}\phi^k_{\mu\nu} - \phi^j_{\mu 5}\phi^k_{\mu 5} - \phi^j_5\phi^k_5) + \bar\psi\psi \text{ terms}
\tag{4.42}
$$

$$
\begin{aligned}
(\partial^2 + \mu^2)\,\phi_5^i = {} & h\mu\left[-f^{ijk}\,\phi_\mu^j\,\phi_{\mu 5}^k + d^{ijk}\left(\phi^j\phi_5^k - \tfrac{1}{8}\epsilon_{\kappa\lambda\mu\nu}\,\phi_{\kappa\lambda}^j\phi_{\mu\nu}^k\right)\right] \\
& + h\,\partial_\mu\left[\,f^{ijk}\left(\phi_{\mu\nu}^j\,\phi_{\nu 5}^k + \phi_5^j\,\phi_\mu^k\right) - \right. \\
& \left. - d^{ijk}\left(\phi^j\,\phi_{\mu 5}^k - \tfrac{1}{4}\epsilon_{\kappa\lambda\nu\mu}\,\phi_\kappa^j\,\phi_{\lambda\nu}^k\right)\right] \\
& + \bar{\Psi}\psi \quad \text{terms.}
\end{aligned}
\tag{4.43}
$$

Thus, with interactions switched on, the trivial component ϕ^i is no longer zero (similarly for the V-field of the 364) and the static equation (4.42) provides a definition of it in terms of other fields. Wishing to treat the ϕ^i as spurions[47] we can look for an unsymmetrical solution, $\phi^8 \neq 0$ to obtain a natural mechanism for SU(3) breaking. Thus a crude unsymmetric solution of (4.42) is $\phi^8 \approx 6\mu / h\sqrt{3}$, all other $\phi^i = 0$.

(d) Meson Currents

We shall now present the currents of the 143 multiplet as arising from 143-143 and 364-364 states in the lowest perturbation approximation. This is the goal we have been striving for

$$
\begin{aligned}
j_R^i &= \langle p'(364)\,|\,j_R^i(0)\,|\,p(364)\rangle + \langle p'(143)\,|\,j_R^i(0)\,|\,p(143)\rangle \\
&= g\,\bar{\Psi}^{\{ABC\}}(p')(\gamma_R T^i)_C^D\,\Psi_{\{ABD\}}(p) + \\
&\quad + h\mu\,(\gamma_R T^i)_C^D\,\{\bar{\Phi}(-p'),\,\Phi(p)\}_D^C
\end{aligned}
\tag{4.44}
$$

In stating the results we use the abbreviations (4.39) and

$$
\begin{aligned}
(\bar{\Phi}\,\phi)_F^i &= \mathrm{Tr}\,(\bar{\Phi}\,[T^i,\,\phi]) \\
(\bar{\Phi}\,\phi)_D^i &= \mathrm{Tr}\,(\bar{\Phi}\,\{T^i,\,\phi\})
\end{aligned}
$$

$$\bar{D}\,T^i\,N = \bar{D}^{rst}\,(T^i)_t^{\;u}\,\epsilon_{usv}\,N^v_{\;r}$$

$$\bar{D}\,T^i\,D = \bar{D}^{rst}\,(T^i)_t^{\;u}\,D_{rsu}$$

$$\bar{D},\ \bar{N},\ \bar{\phi} = \bar{D}(p'),\ \bar{N}(p'),\ \phi(-p')$$

$$D,\ N,\ \phi = D(p),\ N(p),\ \phi(p)$$

$$q = p - p', \quad P = p + p', \quad r_\mu = \epsilon_{\mu\nu\kappa\lambda}P_\nu q_\kappa \gamma_\lambda \gamma_5 \tag{4.45}$$

with p, p' denoting incoming and outgoing particle momenta. Also coupling constant factors g and h are to be understood as multiplying the 364 and 143 field contributions.

$$\begin{aligned} j^i = {} & \frac{P^2}{4m^2}\left[(\bar{N}N)^i_{3S+F} + 3\,\bar{D}_\lambda T^i D_\lambda\right] + \frac{3}{2m^2} q_\kappa \bar{D}_\kappa T^i q_\lambda D_\lambda \\ & + \frac{q_\kappa q_\lambda}{\mu}(\bar{\phi}_\kappa \phi_\lambda)^i_D + \frac{P^2}{2\mu}(\bar{\phi}_\lambda \phi_\lambda - \bar{\phi}_5\phi_5)^i_D \end{aligned} \tag{4.46}$$

$$\begin{aligned} j^i_5 = {} & \frac{P^2}{4m^2}\left[(\bar{N}\gamma_5 N)_{-S+D+\frac{2}{3}F} + 3\,\bar{D}_\lambda \gamma_5 T^i D_\lambda\right] \\ & + \frac{3}{2m^2} q_\kappa \bar{D}_\kappa \gamma_5 T^i q_\lambda D_\lambda + \frac{q_\mu}{m}\bar{D}_\mu T^i N + h.c \\ & + i q_\lambda(\bar{\phi}_\lambda \phi - \bar{\phi}\phi_\lambda)^i_F - \frac{1}{2\mu}\epsilon_{\kappa\lambda\mu\nu}P_\kappa q_\lambda(\bar{\phi}_\mu \phi_\nu)^i_D \end{aligned} \tag{4.47}$$

$$\begin{aligned} j^i_{\mu 5} = {} & \frac{P^2}{4m^2}\left[i(\bar{N}\gamma_\mu\gamma_5 N)_{-S+D+\frac{2}{3}F} + 3i\,\bar{D}_\lambda \gamma_\mu\gamma_5 T^i D_\lambda\right] \\ & + \frac{3i}{2m^2} q_\kappa \bar{D}_\kappa \gamma_\mu\gamma_5 T^i q_\lambda D_\lambda - \frac{iP^2}{2m^2}\bar{D}_\mu T^i N + i\frac{p'_\mu}{m} q_\lambda \bar{D}_\lambda T^i N + h.c \\ & - \frac{P^2}{2\mu}(\bar{\phi}_5\phi_\mu - \bar{\phi}_\mu\phi_5)^i_F + \frac{q_\mu q_\lambda}{2\mu}(\bar{\phi}_\lambda\phi_5 - \bar{\phi}_5\phi_\lambda)^i_F \\ & + \frac{P_\mu}{\mu} q_\lambda(\bar{\phi}_\lambda\phi_5 + \bar{\phi}_5\phi_\lambda)^i_F - \frac{i}{2}\epsilon_{\mu\nu\kappa\lambda}P_\nu(\bar{\phi}_\kappa\phi_\lambda)^i_D \end{aligned} \tag{4.48}$$

$$j_\mu^i = \frac{P_\mu}{2m}(\bar N N)^i_{3S+F} + \frac{1}{4m^2}(\bar N \gamma_\mu N)^i_{-S+D+\frac{2}{3}F} + \frac{3P^2}{4m^2}\bar D_\lambda \gamma_\mu T^i D_\lambda$$

$$+ \frac{3}{2m^2} q_\kappa \bar D_\kappa \gamma_\mu T^i q_\lambda D_\lambda + \frac{1}{2m^2}\epsilon_{\mu\nu\kappa\lambda} P_\kappa q_\lambda \bar D_\nu T^i N + h.c$$

$$+ iP_\mu(\bar\phi_5 \phi_5 - \bar\phi_\lambda \phi_\lambda)^i_F - iq_\lambda(\bar\phi_\lambda \phi_\mu - \bar\phi_\mu \phi_\lambda)^i_F$$

$$+ \frac{1}{2\mu}\epsilon_{\mu\nu\kappa\lambda} P_\nu q_\kappa (\bar\phi_\lambda \phi_5 - \bar\phi_5 \phi_\lambda)^i_D \tag{4.49}$$

$$j_{\mu\nu}^i = \frac{P^2}{4m^2}\left[(\bar N \sigma_{\mu\nu} N)^i_{-S+D+\frac{2}{3}F} + 3\bar D_\lambda \sigma_{\mu\nu} T^i D_\lambda\right]$$

$$+ \frac{3}{2m^2} q_\kappa \bar D_\kappa \sigma_{\mu\nu} T^i q_\lambda D_\lambda + \frac{i}{4m^2}(P_\mu q_\nu - P_\nu q_\mu)(\bar N N)^i_{4S-D+\frac{1}{3}F}$$

$$- \frac{i}{m}\epsilon_{\mu\nu\kappa\lambda} P_\kappa \bar D_\lambda T^i N + h.c$$

$$- \frac{1}{2\mu}(P_\mu q_\nu - P_\nu q_\mu)(\bar\phi_5 \phi_5 - \bar\phi_\lambda \phi_\lambda)^i_F - \frac{P^2}{2\mu^2}(\bar\phi_\mu \phi_\nu - \bar\phi_\nu \phi_\mu)^i_F$$

$$+ \frac{q_\lambda}{\mu}\left[\bar\phi_\lambda(p'_\mu \phi_\nu - p'_\nu \phi_\mu) + (p_\mu \bar\phi_\nu - p_\nu \bar\phi_\mu)\phi_\lambda\right]^i_F$$

$$+ i\epsilon_{\mu\nu\kappa\lambda} P_\kappa (\bar\phi_\lambda \phi_5 - \bar\phi_5 \phi_\lambda)^i_D \tag{4.50}$$

From (4.23) we know that the free fields satisfy $\mu\phi_5^i = ip_\mu \phi_{\mu 5}^i$, $\mu\phi_\nu = ip_\nu \phi_{\nu\mu}$. Consequently we have as our total effective pseudo-scalar and vector currents (on the mass shell)

$$j^i_{5,tot} = j^i_5 + iq_\mu j^i_{\mu 5}/\mu$$

$$j^i_{\mu\, tot} = j^i_\mu + iq_\nu j^i_{\nu\mu}/\mu \tag{4.51}$$

In detail, with coupling constants included,

$$j_{5\,\mathrm{eff}} = g\left(1+\frac{2m}{\mu}\right)\left[\frac{P^2}{4m^2}\left\{(\bar{N}\gamma_5 N)_{D+\frac{2}{3}F-S} + 3\bar{D}_\lambda\gamma_5 D_\lambda\right\} + \frac{3}{2m^2}q_\lambda\bar{D}_\lambda\gamma_5 q_\kappa D_\kappa + \frac{q_\lambda\bar{D}_\lambda}{m}N + h.c\right]$$

$$-\frac{3ih}{2\mu}\epsilon_{\kappa\lambda\mu\nu}q_\kappa P_\lambda(\bar{\phi}_\mu\phi_\nu)_D - 3h\,q_\lambda(\bar{\phi}_\lambda\phi_5 - \bar{\phi}_5\phi_\lambda)_F \tag{4.52}$$

$$j_{\mu\,\mathrm{eff}} = g\frac{P_\mu}{2m}\left(1+\frac{q^2}{2m\mu}\right)(\bar{N}N)_{F+3S} + g\left(1+\frac{2m}{\mu}\right)\left(\bar{N}\frac{r_\mu}{4m^2}N\right)_{D+\frac{2}{3}F-S}$$

$$+\frac{3gP^2}{4m^2}\bar{D}_\lambda\left[\left(1+\frac{2m}{\mu}\right)\gamma_\mu - \frac{P_\mu}{\mu}\right]D_\lambda + \frac{3g}{2m^2}q_\lambda\bar{D}_\lambda\left[\left(1+\frac{2m}{\mu}\right)\gamma_\mu - \frac{P_\mu}{\mu}\right]q_\kappa D_\kappa$$

$$+\frac{g}{2m^2}\left(1+\frac{2m}{\mu}\right)\epsilon_{\mu\nu\kappa\lambda}P_\nu q_\kappa\bar{D}_\lambda N + h.c$$

$$+h\left[\left(1+\frac{q^2}{2\mu^2}\right)P_\mu(\bar{\phi}_5\phi_5 - \bar{\phi}_\lambda\phi_\lambda)_F - P_\mu q_\nu q_\lambda(\bar{\phi}_\nu\phi_\lambda) + 3q_\nu(\bar{\phi}_\nu\phi_\mu - \phi_\nu\bar{\phi}_\mu)_F\right]$$

$$+\frac{ih}{4\mu}\epsilon_{\mu\kappa\lambda\nu}P_\lambda q_\nu(\bar{\phi}_\kappa\phi_5 + \bar{\phi}_5\phi_\kappa)_D \tag{4.53}$$

It is in extrapolating (4.52) and (4.53) off the mass shell - we have purposely not set $q^2 = \mu^2$ for this reason - that possible differences with other authors have arisen[10]. This matter will be discussed in section 6 with reference to e.m.form factors.

Some remarks about the implication of (4.52) and (4.53) and the question of universality. We observe firstly the characteristic factor $\left(1+\frac{2m}{\mu}\right)$ for baryons and 3/2 for mesons. Secondly the D + 2/3F combination (well-known in SU(6)) for the pseudoscalar interaction; regarding the vector coupling we note the F coupling of the charge form factor (P_μ) and the (D + 2/3F) coupling of the magnetic form factor (r_μ). The universality hypothesis as it is commonly understood requires equality of all charge couplings in the limit $q \to o$ i.e. $g = h$; as far as the <u>mass shell</u> constants $(q^2 = \mu^2)$ are concerned it would then follow that

$$g^{ch}_{NNV} = g\left(1+\frac{\mu}{2m}\right) \quad , \quad g^{ch}_{PPV} = g^{ch}_{VVV} = \frac{3}{2} g \tag{4.54}$$

The free particle Lagrangians (4.25), (4.26), the propagators (4.29), (4.30), and the U(6,6) invariant interaction Lagrangians (2.49), (4.37) - (4.41) provide the starting point of all perturbation calculations of the S-matrix (actually there may also exist basic effective 4-point interactions of the baryons and mesons for all we know). The lowest order currents (4.46)-(4.53) are extremely basic in this connection as a knowledge of them represents a radical step in the computations, whether using perturbation theory or the in-put amplitude of an S-matrix calculation.

(e) Quark-Antiquark scattering in a Model Theory

The Lagrangians for the phenomenological 143 and 364 fields with their associated symmetries were invoked to provide the simplest physical description for the interactions of most of the particles and resonances so far observed. We shall now attempt to provide dynamical reasons which suggest this choice of multiplet structure and U(6,6) invariance by considering the simplest possible model theory of quarks:

$$\mathcal{L} = \bar{\psi}^{\alpha}(i\partial\!\!\!/ - m)^{\beta}_{\alpha}\psi_{\beta} + G(\bar{\psi}^{\alpha}\psi_{\alpha})^2/m^2 \tag{4.55}$$

We neglect unitary indices for simplicity and have introduced a U(2,2) invariant $\mathcal{L}_{int}$ as the starting point of the discussion.

We approximate the potential V in the Bethe-Salpeter equation for quark-antiquark scattering,

$$M^{\alpha\delta}_{\beta\gamma}(P;q,q') = V^{\alpha\delta}_{\beta\gamma}(P;q,q') - \frac{i}{(2\pi)^4}\int dq''\, V^{\alpha\delta'}_{\beta\gamma'}(P;q,q'')\, S^{\gamma'}_{\alpha'}\left(\frac{P}{2}+q''\right). S^{\beta'}_{\delta'}\left(-\frac{P}{2}+q''\right) M^{\alpha'\varepsilon}_{\beta'\gamma}(P,q'',q') \tag{4.56}$$

by the lowest order perturbation value

$$V^{\alpha\delta}_{\beta\gamma} = \frac{G}{m^2}\left(\delta^{\alpha}_{\gamma}\,\delta^{\delta}_{\beta} - \delta^{\alpha}_{\beta}\,\delta^{\delta}_{\gamma}\right) \tag{4.57}$$

By rearranging the iterated series arising from (4.57) it is easily shown that

$$M^{\alpha\delta}_{\beta\gamma} = T^{\alpha\delta}_{\beta\gamma} - R^{\alpha}_{\beta}\,D\,R^{\delta}_{\gamma} \tag{4.58}$$

where

$$T^{\alpha\delta}_{\beta\gamma} = \frac{G}{m^2}\left[\delta^{\alpha}_{\gamma}\,\delta^{\delta}_{\beta} - \frac{i}{(2\pi)^4}\int dk\; S^{\alpha}_{\alpha'}\left(\frac{P}{2}+k\right)S^{\beta'}_{\beta}\left(-\frac{P}{2}+k\right)T^{\alpha'\delta}_{\beta'\gamma}\right] \tag{4.59}$$

$$R^{\alpha}_{\beta} = \frac{2\sqrt{G}}{m^2}\left[\delta^{\alpha}_{\beta} - \frac{i}{(2\pi)^4}\int dk\; T^{\alpha\delta}_{\beta\gamma}\; S^{\gamma}_{\epsilon}\left(\frac{P}{2}+k\right)S^{\epsilon}_{\delta}\left(-\frac{P}{2}+k\right)\right] \tag{4.60}$$

$$D^{-1} = 1 - \frac{i}{(2\pi)^4}\,\frac{G}{m^2}\int dk\; S^{\beta}_{\alpha}\left(\frac{P}{2}+k\right)S^{\alpha}_{\beta}\left(-\frac{P}{2}+k\right)$$
$$+ \frac{i}{(2\pi)^8}\int \frac{G}{m^2}\,dk\,dk'\; T^{\alpha\delta}_{\beta\gamma}\; S^{\alpha'}_{\alpha}\left(\frac{P}{2}+k\right)S^{\beta}_{\alpha'}\left(-\frac{P}{2}+k\right)\cdot$$
$$S^{\gamma}_{\gamma'}\left(\frac{P}{2}+k'\right)S^{\gamma'}_{\delta}\left(-\frac{P}{2}+k'\right) \tag{4.61}$$

Thus the problem is reduced to solving the equation for T, which, we notice, is independent (in this approximation) of q and q' and is therefore purely algebraic:

$$T^{\alpha\delta}_{\beta\gamma}(P) = \frac{G}{m^2}\left[\delta^{\alpha}_{\gamma}\,\delta^{\delta}_{\beta} + m^2 K^{\alpha\beta'}_{\beta\alpha'}(P)\, T^{\alpha'\delta}_{\beta'\gamma}(P)\right]$$

$$K^{\alpha\beta'}_{\beta\alpha'} = \frac{-i}{(2\pi)^4}\int dk\, \frac{\left(\frac{P}{2}+k+m\right)^{\alpha}_{\alpha'}\left(-\frac{P}{2}+k+m\right)^{\beta'}_{\beta}}{m^2\left[\left(\frac{P}{2}+k\right)^2-m^2\right]\left[\left(\frac{P}{2}-k\right)^2-m^2\right]}$$

(4. 61)

Rearrangement of (4. 61) allows us to write

$$G^{-1}m^2 T^{\alpha\delta}_{\beta\gamma} = \delta^{\alpha}_{\gamma}\,\delta^{\delta}_{\beta} + m^2\left[\begin{array}{l} K_0\,\delta^{\alpha}_{\alpha'}\delta^{\beta'}_{\beta} + K_1\left(\frac{P^{\alpha'}_{\alpha}}{2m}\delta^{\beta'}_{\beta} - \delta^{\alpha'}_{\alpha}\frac{P^{\beta'}_{\beta}}{2m}\right) \\ + K_2\,\frac{P^{\alpha}_{\alpha'}P^{\beta'}_{\beta}}{4m^2} + K'\,(\gamma_\mu)^{\alpha}_{\alpha'}(\gamma_\mu)^{\beta'}_{\beta}\end{array}\right] T^{\alpha'\delta}_{\beta'\gamma}$$

(4. 62)

where

$$K_0(s) = \frac{1}{(4\pi)^2}\int_1^\infty dx\,\frac{\rho(x)}{x-s}\left(1-\frac{1}{x}\right)^{\frac{1}{2}}, \quad K_1(s) = K_0(s)$$

$$K_2(s) = \frac{1}{(4\pi)^2}\int_1^\infty dx\,\frac{\rho(x)}{x-s}\left(1-\frac{1}{x}\right)^{\frac{1}{2}}\left[1-\frac{1}{3}\left(1-\frac{1}{x}\right)\right]$$

$$K'(s) = -\frac{1}{(4\pi)^2}\int_1^\infty dx\,\frac{\rho(x)}{x-s}\left(1-\frac{1}{x}\right)^{\frac{1}{2}}\cdot\frac{1}{3}(x-1)$$

(4. 63)

$s = P^2/4m^2$ and we have introduced an additional cut-off function $\rho(x)$ in the spectral representations (4. 63) to ensure that all integrals converge. Thus $\rho(1) = 0$ but $\rho(s) \to 0$ as $s \to \infty$ sufficiently rapidly. The common terminology for the kernels in (4. 62) is that K_0 is U(6, 6) invariant, K_1 is the first derivative (one kineton[41] or first-type spurion[48]) kernel, K_2 the double derivative kernel, K' the spin-splitting kernel (second-type spurion[49]).

To solve (4.62) we pass to the 'hermitian' basis:

$$T^{\alpha\delta}_{\ \beta\gamma} = \tfrac{1}{4}(\gamma_R)^{\alpha}_{\ \beta}(\gamma_S)^{\delta}_{\ \gamma}K_{RS}\,, \qquad K^{\alpha\beta'}_{\ \beta\alpha'} = \tfrac{1}{4}(\gamma_R)^{\alpha}_{\ \beta}(\gamma_S)^{\beta'}_{\ \alpha'}K_{RS}$$

for which (4.62) reads

$$m^2\left(G^{-1}g_{RR'} - K_{RR'}\right)T_{R'S} = g_{RS} \tag{4.64}$$

$$\therefore \quad m^2\,T_{RS} = \left(G^{-1} - K\right)^{-1}_{RS} \tag{4.65}$$

Expressing $G^{-1}-K$ in block diagonal form,

$$(G^{-1}-K) = \begin{array}{c}
\left[\begin{array}{ccccc}
G^{-1}-K_0-\Delta K_2-4K' & 0 & 0 & 0 & 0 \\
0 & \begin{array}{l}(G^{-1}-K_0)g_{\kappa\mu}\\ +\Delta K_2\left(g_{\kappa\mu}-\frac{2P_\kappa P_\mu}{p^2}\right)\\ +2g_{\kappa\mu}K'\end{array} & \frac{iK_1}{m}(g_{\kappa\mu}P_\nu - g_{\kappa\nu}P_\mu) & 0 & 0 \\
0 & \frac{iK_1}{m}\left(\begin{array}{l}g_{\kappa\mu}P_\lambda - \\ -g_{\lambda\mu}P_\kappa\end{array}\right) & \begin{array}{l}(G^{-1}-K_0)(g_{\kappa\mu}g_{\lambda\nu}-g_{\kappa\nu}g_{\lambda\mu})\\ -\Delta K_2\left\{\begin{array}{l}(g_{\kappa\mu}g_{\lambda\nu}-g_{\kappa\nu}g_{\lambda\mu})\\ +\frac{2}{p^2}\left(\begin{array}{l}P_\mu P_\lambda g_{\kappa\nu}+P_\kappa P_\nu g_{\lambda\mu}\\ -P_\mu P_\kappa g_{\lambda\nu}-P_\lambda P_\nu g_{\kappa\mu}\end{array}\right)\end{array}\right\}\\ -8K'(g_{\kappa\mu}g_{\lambda\nu}-g_{\kappa\nu}g_{\lambda\mu})\end{array} & 0 & 0 \\
0 & 0 & 0 & \begin{array}{l}-(G^{-1}-K_0)g_{\kappa\mu}\\ +\Delta K_2\left(g_{\kappa\mu}-\frac{2P_\kappa P_\mu}{p^2}\right)\\ +2g_{\kappa\mu}K'\end{array} & \frac{iK_1P_\kappa}{m} \\
0 & 0 & 0 & -\frac{iK_1P_\mu}{m} & \begin{array}{l}-G^{-1}+K_0\\ -\Delta K_2-4K'\end{array}
\end{array}\right]
\begin{array}{c} S \\ V \\ T \\ A \\ P \end{array}
\\
\begin{array}{ccccc} S & V & T & A & P \end{array}
\end{array}$$

(4.66)

Thus, remarkably, the matrix is block diagonal in $SO(3, 2)$ space and we have only to invert two block matrices $\begin{pmatrix} AA & AP \\ PA & PP \end{pmatrix}$, $\begin{pmatrix} VV & VT \\ TV & TT \end{pmatrix}$ which contain $(0^- 1^+)$ and $(1^-, 0^+, 1^+)$ projections. We present the inverse of the first matrix as the second follows the same pattern:

$$\begin{array}{ccc} & A & P \\ A & \multicolumn{2}{c}{\multirow{2}{*}{$\left(\begin{array}{c|c} -\alpha g_{\kappa\mu} + (\alpha+\beta)\dfrac{P_\kappa P_\mu}{P^2} & i\gamma P_\kappa \\ \hline -i\gamma P_\mu & \delta \end{array}\right)^{-1}$}} \\ P & & \end{array} = \left(\begin{array}{c|c} -A g_{\kappa\mu} + (A+B)\dfrac{P_\kappa P_\mu}{P^2} & iCP_\kappa \\ \hline -iCP_\mu & D \end{array}\right)$$

with

$$\alpha = A^{-1}, \quad \beta = \frac{D}{P^2C^2 - BD}, \quad \gamma = \frac{C}{P^2C^2 - BD}, \quad \delta = \frac{B}{P^2C^2 - BD} \tag{4.67}$$

The possible poles of T_{RS} occur as solutions to the equations

$$\begin{array}{ll} 0^+ \ (SS) & G^{-1} - K_0 - sK_2 - 4K' = 0 \\ 0^+ \ (VV) & G^{-1} - K_0 - sK_2 + 2K' = 0 \\ 1^+ \ (AA) & G^{-1} - K_0 - sK_2 - 2K' = 0 \\ 1^+ \ (TT) & G^{-1} - K_0 - sK_2 - 8K' = 0 \\ 0^- \ (PP, AA) & (G^{-1} - K_0 + sK_2 - 2K')(G^{-1} - K_0 + sK_2 + 4K') = 4sK_1^2C^2 \\ 1^- \ (VV, TT) & (G^{-1} - K_0 + sK_2 + 2K')(G^{-1} - K_0 + sK_2 - 8K') = 4sK_1^2C^2 \end{array}$$

(4.68)

We shall solve these equations in perturbative fashion, taking into account the various contributions to the kernel:

(1) If derivative and spin splitting kernels were absent, $K_1 = K_2 = K' = 0$ we should have a pure U (6, 6) invariant situation with all parity $(\pm)$ mesons arising as a common pole[50] at $G^{-1} = K_0(\Delta)$

(2) If spin splitting only is neglected viz, $K' = 0$ the $0^- 1^-$ poles coincide at $(G^{-1} - K_0 - \Delta K_2)^2 = 4\Delta K_0^2$ while the $0^+ 1^+$ poles coincide at $(G^{-1} - K_0 - \Delta K_2) = 0$. ie K_0, K_2 split the mesons of opposite parities.

(3) Finally, if K' is included the $0^- 1^-$ mesons split as do the four 0^+ and 1^+ mesons.

We are looking for the dynamical circumstance which should guarantee that the dominant poles are $0^- 1^-$ mesons and therefore lead to a U(2) ⊗ U(2) multiplet structure i.e. where situation (1) is absent but (2) arises and (3) represents a perturbation of it. To secure this, first let us assume that the quark-mass m is very high compared to the masses μ of the $0^-, 1^-$ composites. Also assume that $\rho(x)$ is strongly peaked near this high threshold vanishing rapidly for higher x, i.e. take

$$\rho(x) = \frac{\rho}{16\pi^3} \frac{\epsilon}{(x-1)^2 + \epsilon^2} \quad \text{where} \quad \epsilon \approx \frac{\mu}{2m} \ll 1 \tag{4.69}$$

The precise shape of $\rho(x)$ is unimportant for the qualitative results below. As stated before the desired case (2) amounts to setting K' at zero. i.e. retaining terms to order $\sqrt{\epsilon}$. This gives all $(+)$ parity mesons poles at

$$\Delta_+ = 1 + \frac{\epsilon}{\eta}(1+\eta) \tag{4.70}$$

where η is the parameter

$$\eta = \frac{1}{G\rho\sqrt{\epsilon}} - 1 . \tag{4.71}$$

and the amplitude equals

$$T^{-1}(s) \approx \frac{\eta^2}{\epsilon G}(1+\eta)(s-s_+) + \frac{\epsilon}{\eta}(1+\eta)\ \mathrm{Im}\,k_0 \quad (4.72)$$

in the vicinity of the resonance. All (-) mesons bind at

$$s_- = \left(\frac{\eta}{\eta+2}\right)^2 < 1 \quad (4.73)$$

and

$$T^{-1}(s) \approx \eta^{-1} G^{-1}(s-s_-) \quad (4.74)$$

We wish to ensure that only the (-) mesons dominate the scattering amplitude at low energies . This is assured provided we impose one further condition on the coupling constant i.e. let $0 < \eta = \left(\frac{1}{G\rho\sqrt{\epsilon}} - 1\right) \ll 1$ for in that case $s_+/s_- \gg 1$. In this situation the (+) mesons occur as broad resonances at high energies (greater than 2 quark masses) while the $0^- 1^-$ bound mesons with their equal residues show the characteristic U(2) ⊗ U(2) multiplet structure together with the U(2, 2) invariant coupling to the quarks. The condition imposed above - $\eta \ll 1$ - means $G^{-1} \approx \rho\sqrt{\epsilon}$ consistently with the picture that "the stronger the coupling G, the lower the mass of the bound state μ ".

Let us now include the terms of order $\epsilon^{3/2}$ to take account of the spin splitting kernel K'. We find that the 0^- pole is shifted down by $\frac{7}{6}\epsilon(\epsilon+\eta)$ while the 1^- pole is shifted up by $\frac{1}{6}\epsilon(\epsilon+\eta)$ We wish to stress the correct qualitative nature of this result, especially the fact that the magnitude of the shift is comparable to the unperturbed mass even though the kernel has received a small perturbation.

(1) The pure U(6, 6) situation where $K_1 = K_2 = K'$ are set at zero and all meson poles coincide at $s_0 = \epsilon + \eta(\eta+1)^{-1}$ is unrealistic and incompatible with our assumptions[49].

At low energy therefore we see that the $0^- 1^-$ particles of the 15 dimensional U(2, 2) multiplet, coupled invariantly to the quarks (to a good approximation whereby K' is neglected) will dominate quark-antiquark scattering. To carry out a similar dynamical calculation for baryons one would need to solve a three-body problem which is certainly beyond our scope at the present time. It would be surprising nevertheless if we did not recover the $\tilde{U}(4)$ character of the strong interactions in the spirit of the approximations carried out earlier. At the same time these considerations clearly generalize to $\tilde{U}(12)$ once we include unitary indices and use the Lagrangian of section (a) as the starting point of the calculations. Moreover if the complete kernel to the Bethe-Salpeter equation is used so that the relative momentum dependence of the scattering amplitude cannot be neglected, we conjecture the presence of $U(6) \otimes U(6)$ multiplets and kinetic supermultiplets[33] thereof, exhibiting a $U(6) \otimes U(6) \otimes U(3)$ structure.

Field theoretically, for the purposes of explicitly including the bound states into any computation, we introduce composite ("quasi-particle") fields ϕ for the physical particles and, as suggested by earlier considerations, construct local Lagrangians for these $U(6) \otimes U(6)$ multiplets with interactions that are $U(6,6)$ invariant.

$$\mathcal{L} = \mathcal{L}_f(\psi) + \mathcal{L}_f(\phi) + \mathcal{L}_{int}(\phi\,\phi) + \mathcal{L}_{int}(\psi, \phi) \quad (4.75)$$

The final abstraction of the situation would be if the quark mass is so extremely high that the quark field ψ disappears to all intents and purposes, so far as low-energy effects are concerned. Now, SCHWINGER[14], using local field theory concepts has shown that at least for the case of 143 mesons, the requirement that the mesons exhibit a $U(6) \otimes U(6)$ structure implies that $\mathcal{L}_{int}(\phi, \phi)$ has to be invariant for its relativistic completion $U(6,6)$. From the dynamical viewpoint sketched above we would arrive at the same result from a consideration of the self-energy graphs corresponding to 0^-1^- mesons together with a strong coupling condition which, analogously to the quark case treated above, pushes the possible 0^+1^+ poles to the very high energy region.

5. $\tilde{U}(12)$ CALCULATIONS AND EXPERIMENTS

Computations of S-matrix elements based on the material in the last sections may now be carried out on two different lines. In the first and traditional approach one follows the canons of perturbation theory and proceeds from basic $\tilde{U}(12)$ Lagrangians, evaluating higher order corrections to the lowest order phenomenological point vertices. There is no question but that this method produces a unitary (though not a U(6, 6) symmetric) S-matrix up to any given order in the coupling constant. The application of a perturbation approach for a strong coupling theory is however questionable. The second is the S-matrix approach which specifically relies on the N/D-like methods for calculating scattering amplitudes, and makes convenient "input" approximations to the left-hand cut, for instance that the main contribution to the potential comes from an exchange supermultiplet, perhaps together with a four-point contact interaction. Unitarity is then forced on the S-matrix as a basic physical requirement.

The "input" approximation in these calculations may possess a higher symmetry but one would naturally inquire what vestige of it finally remains at the end of the calculation. It is obvious that, in general, the overall S-matrix symmetry must reduce to SU(3) $\otimes I\mathcal{L}_4$ by the intrinsic breaking from kinetic energy terms both in the asymptotic and intermediate states. Nevertheless certain situations could well arise where an approximate higher symmetry persists, and this is most clearly seen by the following simple argument:

The completeness relation for free particle states reads

$$I = \sum_n |n\rangle\langle n| = \sum_n \int \prod_{i=1}^{n} dp_i \, \Lambda(p_i) \, \delta(p_i^2 - m_i^2) \qquad (5.1)$$

where $\Lambda(p_i)$ is a spin-function and I is the identity operator for the Poincaré group $\otimes$ U(3), the assumed internal symmetry. Even with all masses m_i in a supermultiplet taken degenerate, I cannot be the identity operator for any higher symmetry[50] group such as U(6, 6) on account of the presence of momentum terms in $\Lambda(p_i)$ Indeed we recall from section 3 that the maximal symmetry one may expect for (i) one-particle states is $U(6) \otimes U(6)$, (ii) two-particle states is $U(6)_W$, (iii) three-particle states is $U(3) \otimes U(3)$, when these momenta are fixed. In Eq. (5.1), except for one-particle states, the momenta are not fixed but integrated over so that the maximal symmetry of this unitarity relations is even further circumscribed.

Now write the unitarity condition on the T-matrix in the symbolic form

$$\mathrm{Im}\; T^{-1} = I \tag{5.2}$$

Clearly the symmetry of the discontinuity of T^{-1} is restricted to the maximal permissible symmetry for $\mathrm{Im}\, T^{-1}$ The reflexion of this on $\mathrm{Re}\, T^{-1}$ is visible when we write the symbolic dispersion relation

$$T^{-1}(s) = B^{-1}(s) + \int \frac{\mathrm{Im}\, T^{-1}(x)}{x - s}\, dx \tag{5.3}$$

To take an example, even if B(s), the pole term, may show $U(6)_W$ symmetry[17] the maximal symmetry of $\mathrm{Re}\, T^{-1}$ will be limited through (5.3).

A large number of calculations have been performed where the assumption has been made that T is $U(6)_W$ or more restrictively, U(6, 6) invariant, like the fundamental interaction Lagrangians postulated. This is equivalent to the (unwarranted) assumption that T is dominated by B and therefore possesses the symmetry of its contact and pole contributions. The expectation that these calculations would agree with experiment (in the physical region of scattering) have been largely disappointed (at least for U(6, 6)) as we shall see later and hardly surprisingly in view of the blatant contradiction with unitarity. At the very least these (zeroeth order) expressions should have been supplemented with the proper unitarity correction like (5.3).

As stated in the introduction (see back, p.3) it so happens that the zeroeth order U(6,6)-invariant expressions for the three-point function and the form obtained from $U(6)_W$ predict similar results which agree with experiment, indicating that apart from mass renormalization of external particles there is an (unexplained) unitarity suppression for the vertex function. We therefore believe that heuristically one can obtain a decent expression for a four-point (or higher) function T by taking for B the one supermultiplet-exchange diagrams using U(6,6) invariant residues. Such calculations have not yet been carried out. We feel these are urgently needed.

At all events, with regard to the structure of the S-matrix, it proves convenient to decompose our full $(U(3) \otimes IL_4)$ invariant matrix elements into three parts

$$M = M_o + M_1 + M' \tag{5.4}$$

Except for intrinsic breaking by B.W. equations on the external lines, M_o has the full homogeneous U(6,6) symmetry. Thus it possesses no derivatives and has been called the "regular" amplitude[51]. M_1 contains derivative terms (otherwise known as kineton terms[14], first type spurion[48] terms or irregular couplings[52]) where the external momenta make their appearance as factors $\mathcal{P}_A^B = p_\mu (\gamma_\mu)_\alpha^\beta \, \delta_\gamma^\delta$; as we shall see in section 6, these M_1 terms are as important as the M_o terms from the point of view of the inhomogeneous U(6,6) group[18]. Finally M' supplements M_o and M_1 to give the totality of $IL_4 \otimes U(3)$ amplitudes, by what we may term "unitarity corrections". Such M' amplitudes could involve "spurions of the second kind"[48] $M_R \, (\gamma_R)_\alpha^\beta \, (\gamma_R)_\delta^\gamma$. These are also the terms which split masses of particles within a super-multiplet. Most calculations that we shall review below have assumed dominance of M_o terms. Certain results of these computations (such as total cross section predictions) survive the inclusion of M_1 terms such as the special case of forward scattering or scattering at threshold - in this circumstance the

$(U(6)_W)$ amplitudes essentially reduce to U(6, 6) form because of the fact that there are at most two momenta available to provide $\not{p}$ terms[53].

(a) Two-point Functions

There is good evidence that the better established particles and resonances can be grouped into $U(6) \otimes U(6)$ multiplets. The $(56,1)^+$ GÜRSEY-RADICATI identification[1] for baryons is generally accepted. Also the $(6,6)^-$ for mesons[13], seems well supported by the occurrence of an SU(6) singlet, the X^0 (950 MeV) meson, in addition to the 35^- mesons[1] of SU(6). The classification of other resonances to higher representations is much more fluid; thus positive parity mesons, which account for the bulk of new resonances, could belong to $(15,15)^+$, $(21,21)^+$, $(21,15)^+$, $(15,21)^+$, while baryon resonances could fall in $(70,1)^+$, $(20,1)^+$, $(126,6)^-$, $(210,21)^+$ etc. In the sequel we shall use the economical choices[54] $(15,15)^+$ for mesons which accomodates a nonet of 2^+ particles among other members[55], and $(126,6)^- = 700 \oplus 56^-$ for baryons which contains $\frac{3}{2}^-$ and $\frac{5}{2}^-$ octets[31,32]. We shall therefore be concentrating on the U(6, 6) multiplets 143^-, 4212^+, 364^+, 5720^-.

Mass splitting of the multiplet members (other than SU(3) breaking) will arise via the unitarity corrections M'. In the rest frame we can expect to reproduce little more than the SU(6) results of BÈG and SINGH in connection with mass formulae[56]. The only slight extension is the presence of the SU(6) scalar mass operator

$$\frac{1}{2} J^k_{rs} J^k_{rs} - J^k_{r6} J^k_{r6} = J^{(1)}$$

that separates the X^0 from the SU(6) 35 multiplet[57]. Already without such an SU(6) scalar, $m(X^0) = 780$ MeV is high, and with a mixing angle of about 0.2 with the η (for a good fit of the G.M.O. formula for the 0^- mesons), $m(X^0)$ is pushed up to 800 MeV. $J^{(1)}$ accounts for the remaining 150 MeV mass shift. It would be interesting to see whether the parameters involved in the various SU(6) representations for the mass operator are the same for the higher representations $(15, \overline{15})^+$ etc.

Similar considerations apply to the electromagnetic mass splitting within any multiplet. In fact for the 143 and 364 no new relation can be derived that is not already known[58] on the basis of SU(6).

(b) Three-Point Functions

The verified successes of U(6, 6) rest here[4]. In the following we neglect mass differences between supermultiplet members (i.e. neglect the T^8 operator SU(3) that is chiefly responsible for mass splitting) and focus our attention on the strong and electromagnetic interactions. We let $\mathcal{B} = D(3/2) \oplus N(1/2)$, $M = P(0^-) \oplus V(1^-)$, and $\mathcal{M}$ represent the 364^+, 143^- and 4212^+ multiplets.

(i) Vertex MBB

With all particles on the mass shell,

$$M_0 = g\, \overline{\Phi}^{\{ABC\}}(-p_2)\, \Phi_{\{ABD\}}(p_1)\, \Phi^C_D(p_3)\,; \quad p_1+p_2+p_3=0$$

$$M_1 = g'\, \overline{\Phi}^{\{ABC\}}(-p_2)\, \Phi_{\{ABC\}}(p_1)\, (p_2-p_1)^D_E\, \phi^E_D(p_3)/m\,,$$

and the meson currents can be immediately read off from (4.53) with $q^2 = \mu^2$ if we notice that M_1 affects the vector singlet current alone. Summarizing the results for the octet contributions,

$$g_{PNN} : g^{ch}_{VNN} : g^{mag}_{VND} : g_{PND} : g_{VND} : g_{PDD} : g^{ch}_{VDD}$$
$$= \left(1+\frac{2m}{\mu}\right)_{D+\frac{2}{3}F} : \left(1-\frac{\mu^2}{4m^2}\right)_F : \left(1+\frac{2m}{\mu}\right)_{D+\frac{2}{3}F} : \left(1+\frac{2m}{\mu}\right) : \left(1+\frac{2m}{\mu}\right) : \left(1+\frac{2m}{\mu}\right) : \left(1-\frac{\mu^2}{4m^2}\right) \tag{5.5}$$

We need not elaborate on the (3D + 2F) pseudoscalar coupling which has received ample experimental and theoretical confirmation, or on the same combination for the magnetic interaction[16]. We have the additional important prediction that

$$g_{N^*N\pi} = \frac{3}{5}\left(1+\frac{2m}{\mu}\right) g_{NN\pi} = g_{N^*N^*\pi} \qquad (5.6)$$

Using $\langle\mu\rangle \approx 700$ and $\langle m\rangle \approx 1300$ we obtain the N^* decay width to be $\Gamma_{N^*N\pi} \approx 110$ MeV, a considerable improvement over the SU(6) value of about 80 MeV obtained with an effective kineton interaction[59] $\frac{2m}{\mu}\bar{\Psi} i \gamma_\mu \gamma_5 \psi \partial_\mu \phi$ that has the effect of eliminating the unity from equation (5.6). However the ratio $\Gamma_{Y^*\Sigma\pi} / \Gamma_{Y^*\Lambda\pi}$ remains small, but this is simply the failure of SU(3). Finally there is the prediction obtained on the basis of $SU(6)_W$ by HARARI and LIPKIN[11] that the only allowed amplitude for DNV is of M1 type in accordance with the STODOLSKY-SAKURAI peripheral model[60] with vector meson exchange.

(ii) Vertex MMM

Charge conjugation invariance provides the unique couplings

$$M_0 = \frac{1}{6} h\mu \left[\Phi_A^B(p_1)\, \Phi_B^C(p_2) + \Phi_A^B(p_2)\, \Phi_B^C(p_1) \right] \Phi_C^A(p_3)$$
$$M_1 = \frac{1}{6} h' \left[\Phi_A^D(p_1)\, \Phi_B^A(p_2)\, (p_2-p_1)_E^D\, \Phi_D^E(p_3) + 1,2,3 \text{ perms} \right] \qquad (5.7)$$

on the mass shell, and M_1 just affects the vector singlet currents of (4.52) and (4.53). Observe the F-type coupling of VVV, VPP and the D-type coupling of VVP as prescribed by charge conjugation. Also note the large magnetic coupling (corresponding to a moment of 3) and quadrupole constant 4. These interactions are expected to seriously modify many peripheral calculations with ρ exchange that have been recently carried out. There are otherwise few verifiable predictions. One is the correct absence of the $\varphi \to \rho\pi$ mode because $g_{\varphi\rho\pi} \equiv 0$ with the LIPKIN identifications[61] of the physical φ and ω :

$$\omega = \frac{1}{\sqrt{3}}\left(\phi^0 + \sqrt{2}\,\phi^8\right) \quad , \quad \varphi = \frac{1}{\sqrt{3}}\left(\phi^0 - \sqrt{2}\,\phi^8\right) .$$

Another is the ratio $\frac{g_{\rho\omega\pi}}{g_{\rho\pi\pi}} = \frac{2}{\mu}$ first stated by SAKITA and

and WALI[5]. With $\mu \approx 700$ MeV this compares favourably with the ratio $g_{\rho\omega\pi}/g_{\rho\pi\pi} \approx 2.4/\mu$ obtained by GELL-MANN, SHARP and WAGNER[62] for ω decay. Finally we may determine the extent to which the universality hypothesis holds[59]. Thus from the ρ decay width, $g_{\rho\pi\pi}(q^2=\mu^2) \approx 2.8$ giving $g_{\rho\pi\pi}(0) \approx \frac{2}{3} g_{\rho\pi\pi}(\mu) \approx$ ≈ 1.9 if we extrapolate the perturbation result (4.53). On the other hand $g^{ch}_{\rho NN} = 3\mu\, g_{\pi NN}/5(\mu+2m) \approx 2.0$, so the universality condition $g^{ch}_{\rho NN} = g_{\rho\pi\pi}$ is well satisfied.

(iii) Vertices γBB, γMM

We assume electromagnetic interactions to proceed through ρ, φ intermediate states; in a Lagrangian model this corresponds to introducing the coupling $\frac{e\mu^2}{g} A_\mu \phi_\mu$ in the U-spin scalar projection $\left(\varphi^3 + \frac{1}{\sqrt{3}}\varphi^8\right)$. To lowest order in e this gives the following results for the SACHS form factors[63]:

$$\left(1-\frac{q^2}{4m^2}\right)^{-1} G_E(q^2) = \left[\left(1+\frac{q^2}{2m\mu}\right)\mu^2 F(q^2)\right]/(\mu^2-q^2), \text{ and F-type} \tag{5.8}$$

$$\left(1-\frac{q^2}{4m^2}\right)^{-1} G_M(q^2) = \left[\left(1+\frac{2m}{\mu}\right)\mu^2 F(q^2)\right]/(\mu^2-q^2) \text{ and } (D+\tfrac{2}{3}F)\text{-type} \tag{5.9}$$

where we have introduced an ad hoc form factor $F(q^2)$ into our strong interaction U(6,6) invariant vertex. On account of pole dominance at $q^2 = \mu^2$ one is justified[10] in replacing the factors within [ ] in (5.8) and (5.9) by their residue at μ^2 giving

$$\left(1-\frac{q^2}{4m^2}\right)^{-1} G_E(q^2) \to \left(1+\frac{\mu}{2m}\right)\mu^2/(\mu^2-q^2) \text{ and F-type} \tag{5.10}$$

$$\left(1-\frac{q^2}{4m^2}\right)^{-1} G_M(q^2) \to \left(1+\frac{2m}{\mu}\right)\mu^2/(\mu^2-q^2) \text{ and } (D+\tfrac{2}{3}F)\text{-type} \tag{5.11}$$

Adopting this pole-dominance approximation, we have[64]

$$G_E^n = 0$$

$$G_M^p : G_M^n : G_E^p = \frac{2m}{\mu} : -\frac{2}{3}\left(\frac{2m}{\mu}\right) : 1 \qquad (5.12)$$

We thus obtain the BARNES' result[17] that $\mu^p = 2m/\mu$ and satisfy the threshold condition[65] $G_E(4m^2) = 2m\, G_M(4m^2) = 0$. Also $\langle \frac{r^2}{6} \rangle \approx \frac{(\mu^p)^2}{2m}$. in our earlier paper I the pole-dominance approximation leading to (5.10) and (5.11) was not taken and the results were stated in the form (5.8) and (5.9). This apparently is not favoured by the experiments[15] for the higher values of $-q^2$. Note that kineton terms of the now $U(6)_W$ invariant vertex modify Eqs. (5.8) and (5.9) to [49]

$$G_E(q^2) \propto \left(1 + \frac{q^2}{2m\mu}\right) G_0(q^2) - G_1(q^2) \quad \text{and F-type}$$

$$G_M(q^2) \propto \left(1 + \frac{2m}{\mu}\right) G_0(q^2) - \frac{2m}{\mu} G_1(q^2) \quad \text{and } (D+\tfrac{2}{3}F) \text{ type}$$

Thus $\mu^p/\mu^n = -3/2$ survives, analogously to SU(6) with all manner of kineton couplings[66].

As far as γND interactions are concerned it has been shown by HARARI and LIPKIN[11] that N^* photoproduction through a pure M1 transition follows directly from the assumption that a real photon transforms as a 35 fold of $SU(6)_W$. PAPASTAMATIOU[67] considering the same process has obtained theoretical values $G = 4.44$, $C_2 = 0.41$ - these parameters appear in GOURDIN and SALIN's isobar model[60] where they are estimated as $C_1 = 5.6$, $C_2 = 0.37$ from the experiments. Little that can be said about γDD interactions is subject to verification. The same Lagrangian model as applied to the mesons predicts[4] a magnetic moment of 3 and a quadrupole moment of -4 for the ρ meson.

(iv) <u>Vertex $\mathcal{M}$MM</u>

Considerable data on the decay modes of 2^+ and some 1^+ mesons is now available. A provisional assignment of these resonances to the 4212^+ multiplet can be made[55], and the following $U(6)_W$ couplings constructed[69]

$$
\begin{aligned}
M_0 &= h_\mu \, \Phi^{[AB]}_{[CD]}(p_1)\, \Phi^C_A(p_2)\, \Phi^D_B(p_3)\,, \\
M_1 &= h' \, \Phi^{[AB]}_{[CD]}(p_1)\, (p_2-p_3)^C_A \left[\Phi^E_B(p_2)\, \Phi^D_E(p_3) - \Phi^E_B(p_3)\, \Phi^D_E(p_2) \right] \\
&\quad + h'' \, \Phi^{[AB]}_{[CD]}(p_1)\, (p_2-p_3)^C_A\, (p_2-p_3)^D_B\, \Phi^E_F(p_2)\, \Phi^F_E(p_3)\,.
\end{aligned}
\qquad (5.13)
$$

In the limit of strict U(6,6), viz. retaining M_0 only, the reactions $2^+ \to 0^- 0^-$ and $2^+ \to 0^- 1^-$ are forbidden[55,70] whereas in fact they constitute the decay modes.

The kineton amplitudes M_1 will allow these processes to go and good agreement with the experiments is thereby obtained[69]. Thus the theoretical ratios obtained for the 2^+ octet modes,

$$
\Gamma_{A\rho\pi} : \Gamma_{AK\bar{K}} : \Gamma_{A\eta\pi} : \Gamma_{K^{**}K\pi} : \Gamma_{K^{**}\pi K^*} : \Gamma_{K^{**}K\eta}
$$

$$
\approx 6 : 1 : 2 : 11 : 2 : 1/2
$$

compare quite favourably with the observed values,

$$
7 : 1 : 2 : 12 : ? : ?
$$

The f_0 is mixed with another isosinglet and the resulting modes depend on the mixing angle. Similar calculations have been performed assuming these same mesons fall into a kinetic supermultiplet[54] of the 143 fold. The conclusions are identical to the above.

(v) Because the parity of many baryon resonances remains ambiguous, it is difficult to make multiplet assignments for them with any degree of certainty. Hence few calculations for the decay properties have as yet been undertaken[32, 71].

(c) Four-point Functions

The following amplitudes have been studied on the basis of strict U(6, 6) : MB MB, BB BB, and MMMM. The irregular amplitudes have been neglected as their large number (running into the hundred) make calculations exceedingly difficult. At forward elastic scattering this neglect does not matter[51, 52] and, as it happens, the total cross-section comparisons are effected in this limit by use of the optical theorem. However, we must again emphasize that the discussion given at the beginning of this section casts serious doubt on the validity of the results obtained from M_o alone, if unitarity corrections are neglected. A further cautionary word which would apply even to an amplitude which satisfies unitarity, has to do with the breakdown of the symmetry at the SU(3) level itself[72] because of mass differences both for external and intermediate resonances as well as the effect of the Okubo spurion T^3_3. This compels us to move to higher energies for meaningful comparisons.

(i) MBMB Scattering

There are only four independent U(6, 6) invariant amplitudes if one restricts ones self (in spite of admonitions above) to M_o types of terms:-

$$\begin{aligned} M_0 = & \mathcal{A}\ \bar{\Psi}^{ABC}(p')\,\Psi_{ABC}(p)\,\Phi^D_E(-k')\,\Phi^E_D(k) \\ & + \mathcal{B}\ \bar{\Psi}^{ABC}(p')\,\Psi_{ABD}(p)\,\Phi^D_E(-k')\,\Phi^E_C(k) \\ & + \tilde{\mathcal{B}}\ \bar{\Psi}^{ABC}(p')\,\Psi_{ABD}(p)\,\Phi^D_E(k)\,\Phi^E_C(-k') \\ & + \mathcal{C}\ \bar{\Psi}^{ABC}(p')\,\Psi_{ADE}(p)\,\Phi^D_B(-k')\,\Phi^E_C(k) \end{aligned} \tag{5.14}$$

as can be seen from the reduction of the product 143 $\otimes$ 364 in the direct channel. c.f. equation (3.54). Crossing symmetry tells us that

$$\mathcal{A}(s,t,u) = \mathcal{A}(u,t,s), \quad \mathcal{B}(s,t,u) = \tilde{\mathcal{B}}(u,t,s) \qquad (5.15)$$
$$\mathcal{C}(s,t,u) = \mathcal{C}(u,t,s)$$

where

$$s = (p+k)^2, \quad t = (p-p')^2, \quad u = (p-k')^2$$

Hence one obtains large numbers of relations among the processes

PN $\rightarrow$ PN, VN, PD, VD etc.

Even for the first set there are 7 x 2 = 14 SU(3) $\otimes$ $\mathcal{IL}_4$ amplitudes expressed in terms of only four; so where previously we were hardly able to obtain experimentally feasible comparisons, we now expect to find many new relationships[73]. The following are the significant conclusions

(1) In addition to the well-known SU(3) relation

$$d\sigma(K^-p \rightarrow K^0\Xi^0) = d\sigma(K^-p \rightarrow \Sigma^-\pi^+)$$

there exists $d\sigma(\pi^-p \rightarrow K^+\Sigma^-) = d\sigma(K^-p \rightarrow \Sigma^-\pi^+)$

(2) From the forward scattering limit one deduces the JOHNSON-TREIMAN[74] relations (true even with irregular couplings M_1) :

$$\frac{1}{2}\left[\sigma(K^+p) - \sigma(K^-p)\right] = \sigma(K^0p) - \sigma(\bar{K}^0p) = \sigma(\pi^+p) - \sigma(\pi^-p) \qquad (5.16)$$

which by charge symmetry gives

$$\sigma(K^+n) - \sigma(K^-n) = \frac{1}{2}\left[\sigma(K^+p) - \sigma(K^-p)\right]$$

in excellent agreement with experiment at moderate to high energies[75]. The πp data give less encouraging comparisons with Kp data.

(3) Considering reactions to which only one amplitude contributes, say $\mathcal{C}$, zero polarization is predicted for the outgoing baryon[73]. e.g. $K^- p \rightarrow \Xi^- K^+$. This is badly contradicted by the experiments since the outgoing Ξ has polarization of about 0.8. On the other hand when M_1 amplitudes are considered the polarization predictions no longer obtain.

(4) The U(6,6) predictions for the annihilation channel are described later together with annihilation into three and four mesons.

(ii) <u>BBBB Scattering</u>

Once again M_o contains four independent amplitudes,

$$\begin{aligned} M_o = \; & \mathcal{A}\; \bar{\Psi}^{\{ABC\}}(p_3)\, \Psi_{\{ABC\}}(p_1)\, \bar{\Psi}^{\{DEF\}}(p_4)\, \Psi_{\{DEF\}}(p_2) \\ & - \tilde{\mathcal{A}}\; \bar{\Psi}^{\{ABC\}}(p_3)\, \Psi_{\{ABC\}}(p_2)\, \bar{\Psi}^{\{DEF\}}(p_4)\, \Psi_{\{DEF\}}(p_1) \\ & + \mathcal{B}\; \bar{\Psi}^{\{ABC\}}(p_3)\, \Psi_{\{ABD\}}(p_1)\, \Psi^{\{DEF\}}(p_4)\, \Psi_{\{CEF\}}(p_2) \\ & - \tilde{\mathcal{B}}\; \bar{\Psi}^{\{ABC\}}(p_3)\, \Psi_{\{ABD\}}(p_2)\, \Psi^{\{DEF\}}(p_4)\, \Psi_{\{CEF\}}(p_1) \end{aligned} \tag{5.17}$$

where $$\mathcal{A}(s,t,u) = \tilde{\mathcal{A}}(s,u,t) \;,\quad \mathcal{B}(s,t,u) = \tilde{\mathcal{B}}(s,u,t)$$
$$s = (p_1+p_2)^2 \;,\quad u = (p_1-p_4)^2 \;,\quad t = (p_1-p_3)^2$$

by the generalized Pauli principle. The following conclusions ensue[76] from M_o :

(1) Scattering lengths in the triplet and singlet states a_T, a_S are related as follows:

$$a_S^{np} = a_T^{np} \;,\quad a_S^{\Lambda p} = a_T^{\Lambda p} \;,$$
$$8\, a_S^{\Lambda p} = a_T^{\Sigma^+ p} + 7\, a_T^{\Lambda p} \;,\quad 6\, a_S^{\Lambda p} = a_S^{\Sigma^- p} + 5\, a_S^{np} \tag{5.18}$$

Experimentally the very first conclusion $a_S^{np} = a_T^{np}$ is so badly violated that other comparisons are rendered meaningless. Thus, experimentally,

$$a_S^{np} = -23.4 \; f \quad \text{and} \quad a_T^{np} = 5.4 \; f \; . \tag{5.19}$$

This "catastrophic" result was inherent in the original Wigner theory and cannot be blamed on SU(3) symmetry breaking nor on the neglect of M_1 irregular amplitude. Most likely the discrepancy comes about because of the great sensitivity of these threshold statements to the positions of the real (and virtual) deuteron poles.

(2) Apart from SU(3) results such as $\sigma(\Xi^0 p) = \sigma(\Sigma^- p)$ U(6, 6) has the following extra consequences for total cross sections,

$$\begin{aligned} \sigma(\Sigma^+ p) - 3\sigma(\Sigma^- p) &= 4\sigma(np) - 6\sigma(\Lambda p) \\ 3\sigma(\Sigma^+ p) - 8\sigma(\Sigma^- p) + 4\sigma(\Xi^- p) &= 3\sigma(np) - 4\sigma(\Lambda p) \end{aligned} \tag{5.20}$$

At low energies the first relation is badly violated; however in this region $\sigma(np)$ shows large variations and Coulomb interference effects may be quite strong.

(3) The U(6, 6) invariance hypothesis provides a single constraint on the 5 independent $I\mathcal{L}_4$ amplitudes for every SU(3) channel, that predicts among other things, that the correlation parameter[78] $C_{pq} = 0$. This conclusion remains[11] even with M_1 terms relaxing the symmetry to $U(3) \otimes U(3)$, whereas the pp scattering experiments at 400 MeV show $C_{pq} \approx 0.4$.

(4) In the annihilation channel U(6, 6) seems to work better in that it correctly predicts the dominance of the elastic channel reaction $p\bar{p} \to p\bar{p}$ over all other inelastic reactions $p\bar{p} \to B\bar{B}$, near threshold i.e. it automatically

provides a reasonable description of absorptive effects. Making simple approximations near threshold one obtains[76]

$$\sigma(n\bar{n}) : \sigma(\Lambda\bar{\Lambda}) : \sigma(\Sigma^0\bar{\Lambda}) : \sigma(\Sigma^+\bar{\Sigma}^+) \approx 130 : 80 : 3 : 4$$

$$\sigma(\Sigma^+\bar{\Sigma}^+) : \sigma(\Sigma^-\bar{\Sigma}^-) : \sigma(\Xi\bar{\Xi}^-) \approx 4 : 25 : 1 \qquad (5.22)$$

while the experimental 3 GeV/c lab momentum annihilations give

$$\begin{array}{ll} \sigma(p\bar{p}) = 21 \text{ mb} & \sigma(n\bar{n}) = 0 + 1.3 \text{ mb} \\ \sigma(\Lambda\bar{\Lambda}) = 117 \pm 18\ \mu b & \sigma(\Sigma^0\bar{\Lambda}) = 51 \pm 8\ \mu b \\ \sigma(\Sigma^+\bar{\Sigma}^+) = 36 \pm 16\ \mu b & \sigma(\Sigma^-\bar{\Sigma}^-) = 10 \pm 4\ \mu b \\ & \sigma(\Xi\bar{\Xi}^-) = 2 \pm 1\ \mu b \end{array} \qquad (5.23)$$

Excepting the over-estimate of $\sigma(\Lambda\bar{\Lambda})$ there is fair qualitative agreement.

(iii) MMMM Scattering

This four point function has not received much attention so far because of its obvious unphysicality. GRIFFITHS and WELLING[70] have used trilinear and quadrilinear M_o couplings to study $X^0 \rightarrow \eta\pi\pi$, $K^* \rightarrow K\pi\pi$, $\omega \rightarrow 3\pi$ etc. These authors use the discrepancy between $\Gamma = 5.4$ MeV obtained by SAKITA and WALI[5] for $\omega \rightarrow 3\pi$ and the experimental value $\Gamma = 9.4$ MeV, as a measure of the quadrilinear coupling $Tr(\bar{\Phi}\Phi\bar{\Phi}\Phi)$. Using this information together with an estimate of the $\pi\pi$ coupling constant λ they are able to compute the strength of the second quadrilinear coupling $Tr(\bar{\Phi}\Phi)Tr(\bar{\Phi}\Phi)$. Finally they arrive at the theoretical estimates

$$\Gamma(X^0 \rightarrow \eta\pi\pi) \approx 1 \text{ MeV} , \quad \Gamma(K^* \rightarrow K\pi\pi) \approx 5.5 \text{ KeV}.$$

The first agrees with the value obtained by comparison with the c.m. decay $X^0 \rightarrow \pi\pi\gamma$ and the second is consistent with $\Gamma(K^* \rightarrow K\pi\pi) < 100$ KeV.

(iv) $p\bar{p} \rightarrow$ Mesons, at rest

(1) Annihilation into 2 mesons.

Under U(6, 6) this is forbidden[79]. Apart from the mode $p\bar{p} \rightarrow \rho\pi$ which accounts for about 4 % of all meson annihilations, there is good qualitative agreement of this prediction with experiment. Introducing M_1 kineton couplings to account for the small cross sections one recovers the HARARI-LIPKIN result[80] (neglecting $K\pi$ mass differences)

$$\sigma(\pi^+\pi^-) : \sigma(K^+K^-) : \sigma(K^0\bar{K}^0) = 1 : 4 : 1$$

derived from $SU(6)_W$. If these irregular amplitudes are computed on the basis of a baryon exchange model,the $K\pi$ mass differences, in kinematic factors improve the ratio to

$$\sigma(\pi^+\pi^-) : \sigma(K^+K^-) : \sigma(K^0\bar{K}^0) = 5 : 1 : 1/4$$

Experimentally,

$$\sigma(\pi^+\pi^-) : \sigma(K^+K^-) : \sigma(K^0\bar{K}^0) = 3 : 1 : 0.4 \quad .$$

(2) Annihilation into 3 mesons.

Calculations have not been carried out which take derivative couplings into account and all the conclusions reported below refer to the U(6, 6) invariant M_o amplitudes only[79, 81]. The major results concern the suppression of strange particle pairs $K\bar{K}$, and the absence of production of the physical φ Also one has $M(p\bar{p} \rightarrow \pi\pi\rho) = M(p\bar{p} \rightarrow \pi\pi\omega)$ together with many other well-defined quantitative predictions[82] for the allowed channels on account of the uniqueness of M_o coupling:

$$M_0 = \bar{\Phi}^{\{ABC\}}(-p)\,\Phi_{\{DEF\}}(p)\,\Phi_A^D(-k_1)\,\Phi_B^E(-k_2)\,\Phi_C^F(-k_3) \qquad (5.24)$$

All these predictions are in qualitative agreement with the experiments if one admits an initial state interaction that enhances the $^3S_1$ relative to the $^1S_0$ state.

(3) Annihilation into 4 mesons

Here 12 M_o couplings survive at rest, but for the case of strange particle production only one of these is significant[83]. Typical predictions then are that $p\bar{p} \rightarrow K^0 \bar{K}^0 \pi^+ \pi^-$ is forbidden at rest, and that doubly charged K modes are preferred over singly charged and neutral K modes. These are hard to confirm.

If one neglects the momentum dependence of the form factors in M_o the symmetric coupling of the 4 mesons just remains. In the $^3S_1$ state one finds

$$\frac{\sigma(p\bar{p} \rightarrow \rho\, 3\pi)}{\sigma(p\bar{p} \rightarrow \omega\, 3\pi)} \approx \frac{25}{3}$$

to be compared with the experimental ratio of 8.

6. THE INHOMOGENEOUS GROUP

(A) The reduction of U(6, 6) multiplets under the maximal compact subgroup U(6) ⊗ U(6) which was found necessary in Section 3 in order to make physical associations, may be viewed in a different light. In the same way that finite dimensional representations of the Lorentz group are associated with unitary representations of the Poincaré group we can make the non-unitary representations of U(6, 6) correspond to unitary representations of another group IU(6, 6). The new group is to be obtained by adjoining to U(6, 6) a group of translations whose generators, P_A^B, constitute a multiplet of U(6, 6). This will provide an elegant, though for physical applications highly frustrating and tantalizing view point.

As will be seen from the following treatment, if the group of translations is chosen so that the little group (for the physically relevant representations) coincides with the maximal compact subgroup U(6) ⊗ U(6), then the multispinor $\phi_{A_1 A_2 \cdots}^{B_1 \cdots}(p)$ belongs to a direct sum of irreducible unitary representations of IU(6, 6) in the same sense as it belongs to a direct sum of irreducible representations of U(6) ⊗ U(6). The momenta p_A^B are subject to the constraints

$$p_A^B \, p_B^C = m^2 \delta_A^C \quad \text{and} \quad (\gamma_0)_A^B \, p_B^A > 0 .$$

Thus it is useful to investigate some of the main features of the representations of IU(6, 6). But since these are all present in the simpler group IU(2, 2) we shall confine our attention to that in the following

(B) INHOMOGENEOUS SU(2, 2)

Since the finite dimensional representations of the homogeneous group are not unitary and therefore are unsuitable in themselves to characterize physical states, we can introduce some translations as is done in passing from the Lorentz group to the Poincaré group. There is an arbitrariness in defining the set of translation operators. For example the isomorphism of SU(2,2) with rotations in a 6-dimensional pseudo-euclidean space could lead one to introduce naturally six translations. We wish however to use a structure like the Dirac equation to generate the representations of the inhomogeneous group and since the γ^R's form a 15-fold it seems natural to generalize to 15 translations, p^R , such that $\gamma^R p^R \psi$ transforms like ψ. By inhomogeneous SU(2,2) we shall mean therefore the semi-direct product of SU(2,2) with T_{15}, the group of translations in 15 dimensions. For the generators we take J_α^β and P_α^β which satisfy the commutation rules

$$\begin{aligned} &[P_\alpha^\beta , P_\gamma^\delta] = 0 , \\ &[J_\alpha^\beta , P_\gamma^\delta] = \delta_\alpha^\delta P_\gamma^\beta - \delta_\gamma^\beta P_\alpha^\delta , \\ &[J_\alpha^\beta , J_\gamma^\delta] = \delta_\alpha^\delta J_\gamma^\beta - \delta_\gamma^\beta J_\alpha^\delta , \end{aligned} \tag{6.1}$$

and the reality conditions

$$(\gamma_0 J^\dagger \gamma_0)_\alpha^\beta = J_\alpha^\beta ,$$

$$(\gamma_0 P^\dagger \gamma_0)_\alpha^\beta = P_\alpha^\beta . \qquad (6.2)$$

Among the sixteen J_α^β there are of course only fifteen independent ones and the same is true of P_α^β,

$$J_\alpha^\alpha = 0 \quad , \quad P_\alpha^\alpha = 0 . \qquad (6.3)$$

Some of the unitary representations of this group can be obtained by the usual methods · if one begins by requiring that the values taken by P_α^β in an irreducible representation shall consist of the set of points obtainable by homogeneous operations of the group from the fixed point

$$\hat{P}_\alpha^\beta = m\,(\gamma_0)_\alpha^\beta \qquad (6.4)$$

where m is a positive number. That is we require that every physical state can be brought to rest. The group of transformations which leave $\hat{P}$ invariant constitutes the so-called <u>little group</u>.

For an infinitesimal transformation of SU(2,2) we have

$$\delta\,\hat{P}_\alpha^\beta = i\,\epsilon^R\,[\gamma^R, m\gamma_0]_\alpha^\beta \quad , \qquad (6.5)$$

and the matrices which commute with γ_0 may be taken in the form

$$\frac{1+\gamma_0}{2}\,\sigma_{ij} \quad , \quad \frac{1-\gamma_0}{2}\,\sigma_{ij} \quad , \quad \gamma_0 \quad ; \quad i,j = 1,2,3 \qquad (6.6)$$

indicating that the little group is SU(2) ⊗ SU(2) ⊗ U(1). The remaining eight matrices

$$\gamma_5 \ , \quad \gamma_5\sigma_{ij} \ , \quad \gamma_0\gamma_5 \ , \quad \gamma_0\gamma_5\sigma_{ij} \tag{6.7}$$

carry $\hat{P}$ out into an 8-dimensional "mass-shell". This mass-shell can otherwise be specified as the set of points satisfying the equations

$$P_\alpha^\beta P_\beta^\gamma = m^2 \delta_\alpha^\gamma \quad \text{and} \quad \mathrm{tr}(\gamma_0 P) > 0 \ . \tag{6.8}$$

This becomes rather more clear in the notation appropriate to O(4.2). In the basis provided by the Dirac matrices $(\gamma_{IJ})_\alpha^\beta$, $I, J = 0,1,2,3,5,6$ where

$$\gamma_{IJ} = -\gamma_{JI} = \begin{pmatrix} \sigma_{\mu\nu} & i\gamma_\nu\gamma_5 & \gamma_\nu \\ -i\gamma_\mu\gamma_5 & 0 & \gamma_5 \\ -\gamma_\mu & -\gamma_5 & 0 \end{pmatrix} , \tag{6.9}$$

we can write

$$J_\alpha^\beta = \frac{1}{2} J_{IJ} (\gamma_{IJ})_\alpha^\beta \ ,$$
$$P_\alpha^\beta = \frac{1}{2} P_{IJ} (\gamma_{IJ})_\alpha^\beta \ , \tag{6.10}$$

thereby defining the hermitian generators J_{IJ}, P_{IJ} . Using the anticommutors

$$\{\gamma_{IJ}, \gamma_{KL}\} = 2\left(g_{IK}g_{JL} - g_{IL}g_{JK}\right) + \epsilon_{IJKLMN}\gamma_{MN} , \tag{6.11}$$

we find that the mass-shell equation (6.8) is equivalent to

$$\frac{1}{2} P_{IJ} P_{IJ} = m^2 , \tag{6.12}$$

$$\epsilon_{IJKLMN} P_{KL} P_{MN} = 0 , \tag{6.13}$$

or

$$P_{\mu 6} P_{\mu 6} - P_{\mu 5} P_{\mu 5} + \frac{1}{2} P_{\mu\nu} P_{\mu\nu} - P_{5\cdot} P_{56} = m^2 \tag{6.14}$$

$$\epsilon_{\lambda\mu\nu\rho} P_{\lambda\mu} P_{\nu\rho} = 0 \tag{6.15}$$

$$\epsilon_{\lambda\mu\nu\rho} P_{\lambda\mu} P_{\nu 5} = 0 \tag{6.16}$$

$$\epsilon_{\lambda\mu\nu\rho} P_{\lambda\mu} P_{\nu 6} = 0 \tag{6.17}$$

$$e_{\lambda\mu\nu\rho} \left(P_{\nu\rho} P_{56} + P_{\rho 5} P_{\nu 6} + P_{5\nu} P_{\rho 6} \right) = 0 \tag{6.18}$$

It is easy to verify that the last four of these (6.15),..,(6.18) are satisfied by taking

$$P_{\nu\rho} P_{56} + P_{\rho 5} P_{\nu 6} + P_{5\nu} P_{\rho 6} = 0 \tag{6.19}$$

which consist of six independent conditions. Thus (6.14) and (6.19) together reduce the 15 P_{IJ} to 8 independent ones. It can also be verified that $P_{06} \geqslant m$ on the sheet of this 8-surface which contains $\hat{P}$.

The states with momentum $\hat{P}$ must group themselves into multiplets $D(k, l, r)$ of the little group $SU(2) \otimes SU(2) \otimes U(1)$ where $k, l = 0, 1/2, 1, \ldots$ and $r = 0, \pm 1/2, \pm 1, \ldots$ denote the eigenvalues of the Casimir operators. These operators are most easily constructed with the help of a new generalized PAULI-LUBANSKI(10) operator W_α^β defined by

$$W_\alpha^\beta = \frac{1}{2} \left(P_\alpha^\gamma J_\gamma^\beta + J_\alpha^\gamma P_\gamma^\beta \right) , \tag{6.20}$$

which is translationally invariant for the representations of interest

$$[W_\alpha^\beta , P_\gamma^\delta] = 0 \quad \text{when} \quad (P^2)_\alpha^\beta = m^2 \delta_\alpha^\beta \tag{6.21}$$

For other varieties of representation where $(P^2)^\beta_\alpha$ is not proportional to δ^β_α the operators W^β_α would not commute with the translations - this circumstance of course does not arise for the Poincaré group* where $(P^2)^\beta_\alpha$ is automatically proportional to δ^β_α. Since W^β_α is evidently a tensor under homogeneous transformations the following must be invariants of the full group

---

* Footnote

It may be of interest to note that in the subspaces where all P_{IJ} vanish except $P_{\mu 6}$ the surviving components of W_{IJ} constitute a representation of the CALOGERO algebra[1]

$$[W_\mu , W_\nu] = i\, \epsilon_{\mu\nu\lambda\rho} P_\lambda W_\rho$$

$$[W_\lambda , W_{\mu\nu}] = \frac{i}{2} \left(g_{\lambda\mu} \epsilon_{\nu\kappa\rho\sigma} P_\kappa - g_{\lambda\nu} \epsilon_{\mu\kappa\rho\sigma} P_\kappa + \epsilon_{\mu\nu\rho\sigma} P_\lambda \right) W_{\rho\sigma}$$

$$[W_{\kappa\lambda} , W_{\mu\nu}] = \frac{i}{2} \left(\epsilon_{\kappa\lambda\nu\rho} W_\mu - \epsilon_{\kappa\lambda\mu\rho} W_\nu + \epsilon_{\mu\nu\kappa\rho} W_\lambda - \epsilon_{\mu\nu\lambda\rho} W_\kappa \right) P_\rho$$

where $W_\mu = W_{\mu 5}$ and $P_\mu = P_{\mu 6}$.

$$\begin{aligned} A_0 &= \tfrac{1}{4} P^\beta_\alpha P^\alpha_\beta , \\ A_1 &= \tfrac{1}{4} W^\alpha_\alpha , \\ A_2 &= \tfrac{1}{4} W^\beta_\alpha W^\alpha_\beta , \\ A_3 &= \tfrac{1}{4} W^\beta_\alpha W^\gamma_\beta P^\alpha_\gamma . \end{aligned} \tag{6.22}$$

To find their values it is sufficient to take the states

$$P^\beta_\alpha = \hat{P}^\beta_\alpha = m(\gamma_0)^\beta_\alpha \tag{6.23}$$

for which

$$W_\alpha^\beta = \frac{m}{2}\left((\gamma_0)_\alpha^\gamma J_\gamma^\beta + J_\alpha^\gamma (\gamma_0)_\gamma^\beta \right)$$

$$= \frac{m}{4} J_{IK} \{ \gamma_{06} , \gamma_{IK} \}_\alpha^\beta \tag{6.24}$$

Only those J_{IK} survive here which correspond to operations of the little group because

$$\{\gamma_{06}, \gamma_{IK}\} = \begin{cases} 2\gamma_{06}\gamma_{IK} & , \gamma_{IK} \in \text{little group} , \\ 0 & \text{otherwise} . \end{cases} \tag{6.25}$$

Hence on the $\hat{P}$ states we can write

$$W_\alpha^\beta = m(\gamma_0)_\alpha^\gamma \left[\frac{1+\gamma_0}{2} \underline{\sigma}\cdot\underline{K} + \frac{1-\gamma_0}{2} \underline{\sigma}\cdot\underline{L} + \gamma_0 \Gamma \right]_\gamma^\beta \tag{6.26}$$

where $\underline{K}$, $\underline{L}$ and Γ are the generators of $SU(2) \otimes SU(2) \otimes U(1)$

$$[K_i , K_j] = i\epsilon_{ijk} K_k ,$$

$$[L_i , L_j] = i\epsilon_{ijk} L_k ,$$

$$[K_i , L_j] = [K_i , \Gamma] = [L_i , \Gamma] = 0 . \tag{6.27}$$

In terms of the little group generators then

$$A_1 = m\Gamma ,$$

$$A_2 = \tfrac{1}{2} m^2 (\underline{K}^2 + \underline{L}^2 + 2\Gamma^2) ,$$

$$A_3 = \tfrac{1}{2} m^2 (\underline{K}^2 - \underline{L}^2) , \tag{6.28}$$

and for an irreducible representation $\mathcal{D}(m,k,\ell,\Gamma)$ of the inhomogeneous group the Casimir operators are given by

$$
\begin{aligned}
A_0 &= m^2 , \\
A_1 &= m\Gamma , \\
A_2 &= \frac{m^2}{2}\left(k(k+1) + \ell(\ell+1) + 2\Gamma^2\right), \\
A_3 &= \frac{m^3}{2}\left(k(k+1) - \ell(\ell+1)\right),
\end{aligned}
\tag{6.29}
$$

where $m > 0$, $k,\ell = 0, 1/2, 1, \ldots$ and $\Gamma = 0, \pm 1/2, \pm 1, \ldots$

Finally, the complete basis for one of these representations is obtained by operating on the rest states

$$|\hat{P}; k, \ell, \Gamma; k_3, \ell_3\rangle , \quad -k \le k_3 \le k , \quad -\ell \le \ell_3 \le \ell$$

with an 8-parameter family of unitary boosts $\mathcal{U}[L_P]$ where $(L_P)_\alpha^\beta$ is an SU(2,2) matrix that takes $\hat{P}_\alpha^\beta = m(\gamma_0)_\alpha^\beta$ into P_α^β on the mass-shell:

$$|P; k, \ell, \Gamma; k_3 \ell_3\rangle = \mathcal{U}[L_P]\, |\hat{P}; k, \ell, \Gamma; k_3, \ell_3\rangle \tag{6.30}$$

It is then a simple matter to show, in general terms, the effect on these states of a finite transformation, $\mathcal{U}[S]$ where S belongs to SU(2,2). Now

$$P = L_P \hat{P} L_P^{-1} \tag{6.31}$$

and suppose

$$P' = S P S^{-1} . \tag{6.32}$$

Let R(P,S) be defined by

$$S L_P = L_{P'} R(P, S) . \tag{6.33}$$

It then follows that

$$R \hat{P} R^{-1} = \hat{P} \tag{6.34}$$

so that R belongs to the little group. Operating in the representation we have

$$\mathcal{U}[S]\,|p\ldots\rangle = \mathcal{U}[S L_p]\,|\hat{p}\ldots\rangle$$

$$= \mathcal{U}[L_{p'}]\,\mathcal{U}[R]\,|\hat{p}\ldots\rangle\,,$$

but

$$\mathcal{U}[R]\,|\hat{p};k,\ell,\Gamma;k_s\ell_s\rangle = \sum_{k_s'\ell_s'} |\hat{p};k,\ell,\Gamma;k_s'\ell_s'\rangle\langle k_s'\ell_s'|D^{k\ell\Gamma}(R)|k_s\ell_s\rangle$$

so that

$$\mathcal{U}[S]\,|p;k,\ell,\Gamma;k_s\ell_s\rangle = \sum_{k_s'\ell_s'} |p';k,\ell,\Gamma;k_s'\ell_s'\rangle\langle k_s'\ell_s'|D^{k\ell\Gamma}(R)|k_s\ell_s\rangle \qquad (6.35)$$

where $D^{k\ell\Gamma}$ denotes one of the unitary finite-dimensional representations of the little group. This completes the general discussion of the representations of the inhomogeneous group. In the next part we shall examine the multi-spinor representations.

(C) Multispinor Representations

It is convenient for the introduction of multispinor representations to construct local free fields in terms of which the transformation laws of inhomogeneous U(2, 2) as well as time reversal and antiparticle conjugation are most clearly formulated. The crossing properties also are made unambiguous by this approach.

We begin with the quark fields of which there are two basic types which may be written $\psi_\alpha(x)$ and $\bar{\psi}_\alpha(x)$. Both of these transform under inhomogeneous U(2, 2) according to the same law,

$$\psi_\alpha(x) \to \psi_\alpha'(x') = S_\alpha^\beta\,\psi_\beta(x)$$

$$\bar{\psi}_\alpha(x) \to \bar{\psi}_\alpha'(x') = S_\alpha^\beta\,\bar{\psi}_\beta(x) \qquad (6.36)$$

where $S^\dagger \gamma_0 S = \gamma_0$ and $x' = S x S^{-1}$. The fifteen coordinates are here arranged as a 4x4 traceless matrix satisfying the reality condition $\gamma_0 x^\dagger \gamma_0 = x$. It should perhaps be emphasized that the two fields, ψ_α and $\bar{\psi}_\alpha$ are not adjoints of one another and that there is no bilinear invariant to be constructed from them. They are to be distinguished by their equations of motion, namely

$$(i\partial\!\!\!/ - m)_\alpha^\beta \psi_\beta(x) = 0 ,$$

and

$$(i\partial\!\!\!/ + m)_\alpha^\beta \bar{\psi}_\beta(x) = 0 . \tag{6.37}$$

where $(\partial\!\!\!/)_\alpha^\beta = \partial/\partial x_\beta^\alpha$. The corresponding adjoint fields $\bar{\psi}^\alpha$ and ψ^α are now defined in the usual way

$$\bar{\psi}^\alpha = (\psi_\beta)^\dagger (\gamma_0)_\beta^\alpha ,$$

$$\psi^\alpha = (\bar{\psi}_\beta)^\dagger (\gamma_0)_\beta^\alpha . \tag{6.38}$$

Both of these must transform according to the same law

$$\bar{\psi}^\alpha(x) \rightarrow \bar{\psi}'^\alpha(x') = \bar{\psi}^\beta(x)\,(S^{-1})_\beta^\alpha ,$$

$$\psi^\alpha(x) \rightarrow \psi'^\alpha(x') = \psi^\beta(x)\,(S^{-1})_\beta^\alpha . \tag{6.39}$$

They are distinguished by their equations of motion

$$\bar{\psi}^\beta(x)\,(i\overleftarrow{\partial\!\!\!/} + m)_\beta^\alpha = 0 ,$$

$$\psi^\beta(x)\,(i\overleftarrow{\partial\!\!\!/} - m)_\beta^\alpha = 0 . \tag{6.40}$$

These four independent fields, $\psi_\alpha, \bar{\psi}^\alpha, \bar{\psi}_\alpha, \psi^\alpha$ represent the various quarks out of which all the representations are to be constructed. They are, of course, no more than a part of the mathematical machinery and it is not necessary for us to regard the higher multiplets as bound systems of quarks. For practical work it is necessary to use the Fourier components of the fields. We define these for the free quarks by taking the plane wave expansions

$$\psi_\alpha(x) = \int d\Sigma(P)\left[u^c_\alpha(P)\,a_c(P)\,e^{-iPx} + u^{\hat{c}}_\alpha(-P)\,b^{\hat{c}+}(P)\,e^{iPx}\right]$$

$$\bar{\psi}_\alpha(x) = \int d\Sigma(P)\left[\bar{u}^{\hat{c}}_\alpha(P)\;a_{\hat{c}}(P)\,e^{-iPx} + \bar{u}^c_\alpha(-P)\,b^{c+}(P)\,e^{iPx}\right]$$

(6.41)

and the usual corresponding expressions for $\bar{\psi}^\alpha$ and ψ^α. These momentum space integrals extend over the positive sheet of the 8-dimensional mass-shell

$$(\not{P}^2)^\beta_\alpha = m^2\delta^\beta_\alpha \quad , \quad tr(\not{P}\gamma_0) > 0 \qquad (6.8)$$

on which $d\Sigma(P)$ is the invariant measure. The annihilation operators are $a_c(P)$, $a_{\hat{c}}(P)$ $(c=1,2)$ for quarks and $b^c(P), b^{\hat{c}}(P)$ for antiquarks. The positive and negative energy spinors satisfy the appropriate Dirac equation,

$$(\not{P}-m)^\beta_\alpha\; u^c_\beta(P) = 0$$

$$(\not{P}+m)^\beta_\alpha\; u^{\hat{c}}_\beta(-P) = 0$$

$$(\not{P}+m)^\beta_\alpha\; \bar{u}^{\hat{c}}_\beta(P) = 0$$

$$(\not{P}-m)^\beta_\alpha\; \bar{u}^c_\beta(-P) = 0$$

(6.42)

The reason for associating a_c with $b^{\hat{c}}$ in ψ_α and $a_{\hat{c}}$ with b^c in $\bar{\psi}_\alpha$ becomes clear when their transformation properties are exhibited. This can be done rather easily by applying the methods of Weinberg.

First let us choose a set of positive and negative energy spinors. Using $(L_P)_\alpha^\beta$ for the boost and $\mathcal{U}[L_P]$ for the corresponding Hilbert space operator we may take

$$\begin{aligned} u_\alpha^a(P) &= \langle 0|\, \psi_\alpha(0)\, |P, a\rangle \\ &= \langle 0|\, \psi_\alpha(0)\, \mathcal{U}[L_P]\, |\hat{P}, a\rangle \\ &= \langle 0|\, (L_P)_\alpha^\beta\, \psi_\beta(0)\, |\hat{P}, a\rangle \\ &= (L_P)_\alpha^\beta\ u_\beta^a(\hat{P}) \end{aligned}$$

where $|\hat{P}, a\rangle$ denotes a rest-state. In the notation of §5(b), $\hat{P} = m\gamma_0$ and so for $u_\alpha^a(\hat{P})$ we may take the first two columns of γ_0 (in the Pauli representation). Thus we have

$$u_\alpha^a(P) = (L_P)_\alpha^a , \quad a = 1, 2. \tag{6.43}$$

Similarly, we may define

$$\begin{aligned} u_\alpha^{\hat{a}}(-P) &= \langle P, \hat{a}|\, \psi_\alpha(0)\, |0\rangle \\ &= (L_P^{-1})_\alpha^\beta\ u_\beta^{\hat{a}}(-\hat{P}) \\ &= (L_P^{-1})_\alpha^{a+2} , \qquad a = 1, 2. \end{aligned} \tag{6.44}$$

The remaining spinors follow in the same way,

$$\bar{u}^{\hat{a}}_{\alpha}(p) = (L_p)^{a+2}_{\alpha} , \qquad a = 1,2$$

$$\bar{u}^{\hat{a}}_{\alpha}(-p) = (L_p^{-1})^{a}_{\alpha} , \qquad \text{"} \quad . \tag{6.45}$$

Having chosen this basis and given the transformation law for the fields, namely

$$\mathcal{U}[S]\, \psi_\alpha(x')\, \mathcal{U}^{-1}[S] = S^{\beta}_{\alpha}\, \psi_\beta(x) ,$$

$$\mathcal{U}[S]\, \bar{\psi}_\alpha(x')\, \mathcal{U}^{-1}[S] = S^{\beta}_{\alpha}\, \bar{\psi}_\beta(x) , \tag{6.46}$$

where U[S] is a unitary operator representing S in the Hilbert space, we can deduce the behaviour of the annihilation and creation operators. For example,

$$\mathcal{U}[S]\, (L_{p'})^{c}_{\alpha}\, a_c(p')\, \mathcal{U}^{-1}[S] = S^{\beta}_{\alpha}\, (L_p)^{c}_{\beta}\, a_c(p)$$

or

$$\mathcal{U}[S]\, a_c(p')\, \mathcal{U}^{-1}[S] = (L^{-1}_{p'}\, S\, L_p)^{c'}_{c}\, a_{c'}(p)$$

Denoting by R^{β}_{α} the matrix $(L^{-1}_{p'}\, S\, L_p)^{\beta}_{\alpha}$ we find in turn

$$\mathcal{U}\, a_c(p')\, \mathcal{U}^{-1} = R^{c'}_{c}\, a_{c'}(p)$$

$$\mathcal{U}\, b^{\hat{c}\dagger}(p')\, \mathcal{U}^{-1} = R^{c'+2}_{c+2}\, b^{\hat{c}'\dagger}(p)$$

$$\mathcal{U}\, a_{\hat{c}}(p')\, \mathcal{U}^{-1} = R^{c'+2}_{c+2}\, a_{\hat{c}'}(p)$$

$$\mathcal{U}\, b^{c\dagger}(p')\, \mathcal{U}^{-1} = R^{c'}_{c}\, b^{c'\dagger}(p) \tag{6.47}$$

It is simple to verify that in the Pauli representation the matrix R_α^β assumes the form

$$R = \begin{pmatrix} \mathcal{D}(\frac{1}{2}, 0, -\frac{1}{2}) & 0 \\ 0 & \mathcal{D}(0, \frac{1}{2}, \frac{1}{2}) \end{pmatrix} \tag{6.48}$$

where $\mathcal{D}(k, \ell, r)$ denotes a representation of the little group $SU(2) \otimes SU(2) \otimes U(1)$. Noticing that $\mathcal{D}(k, \ell, r)^*$ is equivalent to $\mathcal{D}(k, \ell, -r)$ we are able finally to assign the particles to representations of the little group,

$$\begin{aligned} a_c^+ &\sim \mathcal{D}(\tfrac{1}{2}, 0, \tfrac{1}{2}) \\ b^{\hat{c}+} &\sim \mathcal{D}(0, \tfrac{1}{2}, \tfrac{1}{2}) \\ a_{\hat{c}}^+ &\sim \mathcal{D}(0, \tfrac{1}{2}, -\tfrac{1}{2}) \\ b^{c+} &\sim \mathcal{D}(\tfrac{1}{2}, 0, -\tfrac{1}{2}) . \end{aligned} \tag{6.49}$$

Notice that $a_c b^c$ is a little group scalar whereas $a_c b^{\hat{c}}$ is not. This is the reason for our choice of a rather elaborate notation.

Arbitrary representations of inhomogeneous SU(2, 2) can be constructed from products of the quarks. To get $\mathcal{D}(k, \ell, r)$ we may use, for example the tensors

$$\phi_{a_1 \cdots a_r \hat{b}_1 \cdots \hat{b}_s}^{c_1 \cdots c_t \hat{d}_1 \cdots \hat{d}_u} . \tag{6.50}$$

Symmetrizing separately in the sets $a_1 \cdots a_r, b_1 \cdots b_s$, $c_1 \cdots c_t$ and $d_1 \cdots d_u$ and then extracting traces between upper and lower indices of the hatted and unhatted varieties respectively, yields an irreducible set of components with

$$k = \tfrac{1}{2}(r+t) \ , \quad \ell = \tfrac{1}{2}(s+u) \ , \quad r = \tfrac{1}{2}(r+u-s-t) \, . \qquad (6.51)$$

Generalization of SU(2, 2) to U(2, 2) would give significance to the remaining quantum number (r+s-t-u).

The multispinor corresponding to $D(k, \ell, r)$ is obtained finally by multiplying the little group tensor (6.50) by the appropriate positive and negative energy spinors, namely

$$\begin{array}{ll} u^{a}_{\alpha}(P) & \text{for lower unhatted indices,} \\ u^{\gamma}_{c}(-P) & \text{for upper unhatted indices,} \\ \bar{u}^{\hat{b}}_{\beta}(P) & \text{for lower hatted indices} \\ \bar{u}^{\delta}_{\hat{d}}(-P) & \text{for upper hatted indices} \end{array} \qquad (6.52)$$

For example the rank-4 multispinor containing one quark of each type is given by

$$\phi^{\gamma\delta}_{\alpha\beta} = u^{a}_{\alpha}(P)\, \bar{u}^{\hat{b}}_{\beta}(P)\, \phi^{c\hat{d}}_{a\hat{b}}\, u^{\gamma}_{c}(-P)\, \bar{u}^{\delta}_{\hat{d}}(-P) \, . \qquad (6.53)$$

Stated otherwise, $\phi^{\gamma\delta}_{\alpha\beta}$ must be a solution of the following B. W. equations,

$$\begin{aligned} (\not{P} - m)^{\alpha'}_{\alpha}\, \phi^{\gamma\delta}_{\alpha'\beta}(P) &= 0 \\ (\not{P} + m)^{\beta'}_{\beta}\, \phi^{\gamma\delta}_{\alpha\beta'}(P) &= 0 \\ \phi^{\gamma'\delta}_{\alpha\beta}(P)\, (\not{P} - m)^{\gamma}_{\gamma'} &= 0 \\ \phi^{\gamma\delta'}_{\alpha\beta}(P)\, (\not{P} + m)^{\delta}_{\delta'} &= 0 \end{aligned} \qquad (6.54)$$

The notation for multispinors introduced here is a little inadequate in that it fails to distinguish whether a given lower index (say) corresponds to the quark ψ_α or to $\bar{\psi}_\alpha$. However, since in the applications so far use has been found only for representations involving one kind of quark ψ_α and its adjoint $\bar{\psi}^\alpha$, we shall not elaborate. For representations of interest then,

$$\phi^{\beta_1\cdots}_{\alpha_1\alpha_2\cdots}(P) = u^{a_1}_{\alpha_1}(P)\, u^{a_2}_{\alpha_2}(P)\cdots \phi^{\hat{b}_1\cdots}_{a_1 a_2\cdots}(P)\; \bar{u}^{\beta_1}_{\hat{b}_1}(-P)\cdots \qquad (6.55)$$

or

$$(\not{P}-m)^{\alpha_1'}_{\alpha_1}\, \phi^{\beta_1\cdots}_{\alpha_1'\alpha_2\cdots}(P) = 0, \qquad \ldots \text{ (on lower indices)},$$

$$(\not{P}+m)^{\beta_1}_{\beta_1'}\, \phi^{\beta_1'\cdots}_{\alpha_1\alpha_2\cdots}(P) = 0, \qquad \ldots \text{ (on upper indices)}.$$

$$(6.56)$$

The number of lower indices and their symmetry types determine the k-spins, similarly the upper indices determine the l-spins, and finally, the total number of indices fixes the Γ value.

We proceed now to examine some particular transformations. Our method is to define them initially for the quark fields : the generalization to higher representations then being largely automatic.

Included among the transformations of the inhomogeneous U(2, 2) there is the usual parity operation

$$\psi_\alpha(x) \rightarrow (\gamma_0)^\beta_\alpha\, \psi_\beta(\gamma_0 x \gamma_0),$$

$$\bar{\psi}_\alpha(x) \rightarrow (\gamma_0)^\beta_\alpha\, \bar{\psi}_\beta(\gamma_0 x \gamma_0). \qquad (6.57)$$

Since this does not go beyond the operations which have already been used in classifying the irreducible representations we do not expect it to give anything essentially new. In this respect our group differs from the Poincaré group. The behaviour of the creation and annihilation operators under this operation is easy to deduce since, for suitably chosen boosts one has

$$\gamma_0 L_p = L_{\gamma_0 p \gamma_0} \gamma_0 \tag{6.58}$$

and therefore

$$a_c(p) \to a_c(\gamma_0 p \gamma_0) \quad , \quad b^{\hat{c}}(p) \to -b^{\hat{c}}(\gamma_0 p \gamma_0) \tag{6.59}$$

while

$$a_{\hat{c}}(p) \to -a_{\hat{c}}(\gamma_0 p \gamma_0) \quad , \quad b^c(p) \to b^c(\gamma_0 p \gamma_0) \tag{6.60}$$

That is, the k-spinors a_c, b^c are even while the l-spinors $a_{\hat{c}}, b^{\hat{c}}$ are odd. For an irreducible representation (insofar as it is permissible to look on it as an S-wave quark system) we obtain $(-)^{2\ell}$ for the intrinsic parity.

More interesting would be the operation

$$\begin{aligned} \psi_\alpha(x) &\to (\gamma_0)^\beta_\alpha \, \psi_\beta(\gamma_0 x \gamma_0) \\ \bar{\psi}_\alpha(x) &\to -(\gamma_0)^\beta_\alpha \, \psi_\beta(\gamma_0 x \gamma_0) \end{aligned} \tag{6.61}$$

which does not belong to inhomogeneous U(2, 2). Under this parity operation $\psi^\alpha \psi_\alpha$ is a pseudoscalar while $\bar{\psi}^\alpha \psi_\alpha$ is a scalar. For an arbitrary representation the parity is given simply by the number of antiquarks involved.

Another interesting transformation which does not belong to the group is that of antiparticle conjugation. Let $\mathcal{C}$ be a Hilbert space operator defined by

$$\mathcal{C}\,\psi_\alpha(x)\,\mathcal{C}^{-1} = C_{\alpha\beta}\;\bar{\psi}^\beta(x^c)$$

$$\mathcal{C}\,\bar{\psi}^\alpha(x)\,\mathcal{C}^{-1} = (C^{-1})^{\alpha\beta}\,\psi_\beta(x^c) \qquad (6.62)$$

with similar relations for $\bar{\psi}_\alpha$ and ψ^α. $C_{\alpha\beta}$ denotes the usual transposing matrix defined by $C^{-1}\gamma_\mu C = -\gamma_\mu^T$ and x^c is defined by $C^{-1}x\,C = -(x^c)^T$. It is necessary to include the inhomogeneous part $x \to x^c$ in this transformation in order to preserve the form of the Dirac equation. In the limit where all x's vanish except the usual $x_\mu = (1/4)\,tr(\gamma_\mu x)$ we have $x^c = x$ and our operator $\mathcal{C}$ simulates the usual charge conjugation.

The precise behaviour of the momentum space operators depends, of course, on the choice of positive and negative energy spinors used in the plane wave expansions. However, since these spinors constitute a complete set of solutions we can always write

$$C_{\alpha\beta}\,\bar{u}^\beta_b(p^c) = u^{\hat{a}}_\alpha(-p)\,\mathcal{L}_{\hat{a}b}(p), \qquad C_{\alpha\beta}\,\bar{u}^\beta_{\hat{b}}(-p^c) = u^a_\alpha(p)\,\mathcal{L}_{a\hat{b}}(p)$$

where $\mathcal{L}_{\hat{a}b}(p) = \mathcal{L}_{b\hat{a}}(p^c)$ denotes a unitary 2x2 matrix. The form of this matrix is unimportant and it is usual to choose the spinor basis such that it becomes the lowering matrix ϵ_{ab}. In general we have

$$\mathcal{C}\;a_b(p^c)\;\mathcal{C}^{-1} = \epsilon_{ba}\;b^{\hat{a}}(p)\;,$$

$$\mathcal{C}\;b^{\hat{a}}(p)\;\mathcal{C}^{-1} = -\epsilon_{ab}\;a_b(p^c)\;. \qquad (6.63)$$

with similar expressions in $a_{\hat{b}}$, b^a. The higher representations transform in corresponding fashion. For example, the quark-antiquark

$$\mathcal{C}\;\psi^\beta_\alpha(x)\;\mathcal{C}^{-1} = \omega\;(C^{-1})^{\beta\beta'}\,C_{\alpha\alpha'}\;\phi^{\alpha'}_{\beta'}(x^c) \qquad (6.64)$$

where $\omega = \pm 1$ is an undetermined parity. In momentum space this reads

$$\mathcal{C}\, \phi_a^{\hat{b}}(P^c)\, \mathcal{C}^{-1} = -\omega\, \epsilon_{aa'}\, \epsilon_{bb'}\, \phi_{b'}^{\hat{a}'}(P) \; . \tag{6.65}$$

One of the most useful features of the multispinor formalism is the simple method it supplies for analyzing matrix elements. We sketch this briefly.

Each particle in the initial and final states is represented by an appropriate multispinor, $\phi_{\alpha\ldots}^{\beta\ldots}(\pm P)$ of the type discussed above. The matrix element is then expressed as a suitably contracted product of these with the so-called M-function $M_{\alpha_1\ldots}^{\beta_1\ldots}(P_1\ldots)$ which must then have the invariance property

$$S_{\alpha_1}^{\alpha_1'} \cdots M_{\alpha_1'\ldots}^{\beta_1'\ldots}(P_1\cdots)\,(S^{-1})_{\beta_1'}^{\beta_1} \cdots = M_{\alpha_1\ldots}^{\beta_1\ldots}(P_1'\cdots) \tag{6.66}$$

where $P_1' = S P_1 S^{-1}$. This assures that the matrix element

$$\prod\left(\phi_{\beta_j\ldots}^{\alpha_j\ldots}(P)\right)\, M_{\alpha_1\ldots}^{\beta_1\ldots}(P_1\cdots)$$

has the correct transformation behaviour under the inhomogeneous U(2, 2). An M-function satisfying such an invariance requirement can be expanded in a set of scalar amplitudes which depend only on invariant combinations of the P's with coefficients made up from products of δ_α^β and the various P_α^β . For example the matrix element of a 15-current $J_\alpha^\beta(0)$ between quark states is given by

$$\langle P' |\, J_\alpha^\beta(0)\, | P \rangle = \bar{u}^\gamma(P')\, M_{\gamma\alpha}^{\beta\delta}(P', P)\, u_\delta(P) \tag{6.67}$$

where $M_{\gamma\alpha}^{\beta\delta}$ must have the general form

$$M^{\beta\delta}_{\gamma\alpha}(P',P) = \left(\delta^{\beta}_{\gamma}\delta^{\delta}_{\alpha} - \frac{1}{4}\delta^{\beta}_{\alpha}\delta^{\delta}_{\gamma}\right)F_1 + \left(P^{\beta}_{\gamma}\delta^{\delta}_{\alpha} - \frac{m}{4}\delta^{\beta}_{\alpha}\delta^{\delta}_{\gamma}\right)F_2 + \ldots \text{ etc.} \qquad (6.68)$$

where the F_j are functions of $tr(PP')$, $tr((PP')^2)$, ... etc. Account must be taken of the trace condition $J^{\alpha}_{\alpha} = 0$ and the usual simplifications resulting from use of the Dirac equations.

Further symmetries are included by imposing more conditions on the M-functions. Space reflection invariance is of course automatic for processes involving only one type of quark and so we shall consider the less trivial case of antiparticle conjugation invariance. The procedure may be exemplified through application to the matrix element of $J^{\beta}_{\alpha}(0)$ between quark states. Let us write - without regard for the finer points of 15-dimensional field theory -

$$M^{\beta\delta}_{\gamma\alpha}(P',P) = \int dx\,dy\; e^{iP'x - iPy} \langle 0|T\left(\psi_{\gamma}(x)\; J^{\beta}_{\alpha}(0)\,\bar{\psi}^{\delta}(y)\right)|0\rangle$$

Using the antiparticle conjugation operator $\mathcal{C}$ defined by

$$\mathcal{C}\,\psi_{\alpha}(x)\,\mathcal{C}^{-1} = C_{\alpha\beta}\,\bar{\psi}^{\beta}(x^c) \quad \text{and} \quad \mathcal{C}\,\bar{\psi}^{\alpha}(x)\,\mathcal{C}^{-1} = (C^{-1})^{\alpha\beta}\,\psi_{\beta}(x^c)$$

where $x^T = -C^{-1}x^c C$ and supposing that the current J^{β}_{α} satisfies a relation of the type

$$\mathcal{C}\; J^{\beta}_{\alpha}(0)\;\mathcal{C}^{-1} = \omega\;(C^{-1})^{\beta\beta'}\;J^{\alpha'}_{\beta'}(0)\;C_{\alpha'\alpha}$$

where $\omega = \pm 1$, then, if the vacuum is invariant under $\mathcal{C}$ we find

$$\langle 0|T\left(\psi_{\gamma}(x)\,J^{\beta}_{\alpha}(0)\,\bar{\psi}^{\delta}(y)\right)|0\rangle = \omega\,(C^{-1})^{\beta\beta'}(C^{-1})^{\delta\delta'}\langle 0|T\left(\psi_{\delta'}(y^c)\,J^{\alpha'}_{\beta'}(0)\,\bar{\psi}^{\gamma'}(x^c)\right)|0\rangle C_{\alpha'\alpha}C_{\gamma'\gamma}.$$

Hence, for the M-function $\mathcal{C}$-invariance implies the condition

$$M^{\beta\delta}_{\gamma\alpha}(P',P) = (C^{-1})^{\beta\beta'}(C^{-1})^{\delta\delta'}M^{\alpha'\gamma'}_{\delta'\beta'}(-P^c,-P'^c)\,C_{\alpha'\alpha}\,C_{\gamma'\gamma} \qquad (6.69)$$

which may in turn be translated into conditions on the invariant amplitudes F_i. (These conditions simplify a good deal when the multicomponent momenta are cut back to 4-vectors.)

Since this formal apparatus is to be applied ultimately in a 4-dimensional world it will have to undergo an amputation. The passage from 15 dimensions to 4 is not at this time clear. The most direct approach, however, is simply to restrict the Hilbert space of physical states to those with momentum vectors lying entirely within the 4-dimensional subspace. This Hilbert space of course lacks the full SU(2, 2) invariance. It can accommodate only Lorentz transformations (and other, unrelated, operations such as antiparticle conjugation).

The reduction of matrix elements into SU(2, 2) invariant components may have no relevance, except perhaps in some approximation, when the symmetry group is truncated. Certainly the number of invariant amplitudes permitted by Poincaré invariance is not generally the same as that permitted by inhomogeneous SU(2, 2). In the absence of any deep understanding of the dynamics this question of relevance can be dealt with, at this stage, only by reference to experiment.

7. THE OUTLOOK

We list here some problems which we believe deserve further investigation:

(1) We have made a first attempt to discover the possible origin of the $\tilde{U}(12)$ symmetry scheme within a simple quark Lagrangian model. The approximation that we used needs improving and the calculation should be repeated with the phenomenological super-multiplet $\tilde{U}(12)$ Lagrangians of section 4 to find conditions on coupling strength for the supermultiplets to persist. The possible dynamical appearance of kinetic super-multiplets[33] as bound state composites is another

(difficult) problem worth investigating. The whole subject is bound up with the fundamental experimental questions; Do triplets exist?

(2) One wishes to know why $\tilde{U}(12)$ zeroeth order predictions are good for the vertex function, i.e. what are the dynamical reasons for the suppression of possible unitarity connections?

(3) For the four-point function, no S-matrix calculations have as yet been performed with one particle exchange diagrams and $U(6)_W$ vertices as the (N/D) input. From the unexplained success of $U(6)_W$ predictions one may perhaps reasonably hope that a good fit may then be found for the scattering amplitude.

(4) It has commonly been assumed that SU(3) is a better symmetry than for example Wigner's SU(4). DYSON[86] has argued to the contrary by consideration of production cross-sections for N^*, Y^*, Ξ^* and Ω which go down by factors of 10 for each unit of strangeness, even allowing for different masses and by considering the superiority of the SU(4) mass formulae for vector mesons:-

$$4K^* = 3(\omega \text{ or } \phi) + \rho \quad \text{for } SU(3) \text{ vs. } \quad \rho + \phi = 2K^* \quad \text{for } SU(4).$$

In the relativistic theory the question would be rephrased in respect of superiority the reduction $\tilde{U}(12) \rightarrow \tilde{U}(8) \times \tilde{U}_Y(4)$ vs. $\tilde{U}(12) \rightarrow \tilde{U}(4) \times SU(3)$. Experimentally the breaking of supermultiplets according to j values is of the same order of magnitude as the SU(3) breaking at least for the baryons. We believe this is another point worth deeper study.

(5) A proper treatment of the bound state problem, starting perhaps from a quark Lagrangian; should throw some light on the possible existence of infinite numbers of levels corresponding to infinite dimensional representations[84] of U(6, 6) or higher groups, of which the known supermultiplets may be but one component. The situation here may be analogous to the consideration of the Bethe-Salpeter equation for the hydrogen atom[85].

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$$\bar{\Psi}^{\beta_1 \cdots}_{\alpha_1 \alpha_2 \cdots}(p) \rightarrow (\gamma_0)^{\gamma_1}_{\alpha_1} (\gamma_0)^{\gamma_2}_{\alpha_2} \cdots (\gamma_0)^{\beta_1}_{\delta_1} \cdots \bar{\Psi}^{\delta_1 \cdots}_{\gamma_1 \gamma_2 \cdots}(-p)$$

and belongs to the U(6, 6) transformations. Accordingly, the intrinsic parity of a U(6, 6) multiplet is $(-1)^n$ where n denotes the number of upper indices. See
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The authors find it amusing how expectations from different theories vary. The major successes of SU(3) symmetry lie in predicting multiplets and mass formulae and certain decay rates. Few S-matrix predictions of SU(3) have checked against experiment; one reason for the failure of such predictions is again the same unitarity difficulty (to preserve overall SU(3) invariance, one must neglect mass differences in intermediate and external states).

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