IC/65/29

INTERNATIONAL ATOMIC ENERGY AGENCY

INTERNATIONAL CENTRE FOR THEORETICAL PHYSICS

GENERALIZATIONS OF THE POINCARÉ GROUP

P. BUDINI

AND

C. FRONSDAL

1965

PIAZZA OBERDAN

TRIESTE

IC/65/29

INTERNATIONAL ATOMIC ENERGY AGENCY

INTERNATIONAL CENTRE FOR THEORETICAL PHYSICS

GENERALIZATIONS OF THE POINCARE GROUP *

P. Budini

and

C. Fronsdal

Trieste

March 1965

* To be submitted to "Physics Letters"

GENERALIZATIONS OF THE POINCARE GROUP

I. INTRODUCTION

The recent successful application of the symmetry group SU_6 has opened the floodgates of speculation. The original formulation of SU_6 symmetry[1)] suggested that it was incompatible with relativity, and many attempts to formulate a "relativistic version" were made.[2)] Except for the suggestions of WYLD,[3)] and MAHANTHAPPA and SUDARSHAN,[4)] all these generalizations are plagued by difficulties of interpretation of the most serious kind. Here we investigate relativistic "generalizations" of internal symmetry groups. Our main conclusion is that an algebra $\mathcal{A}$ that includes that of the Poincaré group must be a semi-direct product with $\mathcal{P} = \mathcal{A}/\mathcal{S}$. The assumptions under which this result is derived are : (1) Relativistic covariance, and, (2) that the mass spectrum is not continuous. We have studied several choices of $\mathcal{S}$, giving relativistic generalizations of Wigner's supermultiplet theory,[5)] as well as Gürsey and Radicati's SU_6 theory.

II. GENERAL CONSIDERATIONS

It is our aim to determine every real Lie algebra that satisfies certain conditions that are necessary for a physical interpretation. Let $\mathcal{P}$ be the algebra of the Poincaré group, and let the 10 basis elements of $\mathcal{P}$ be chosen as follows

$$\mathcal{P} = \{ L_{ij} \, , \, P_\mu \, , \, L_{i0} \}$$

$$i, j = 1, 2, 3 \; ; \; \mu = 0, 1, 2, 3$$

Let $\mathcal{P}_{LT}$ be the largest subalgebra of $\mathcal{P}$ that commutes with P_0 and let $\mathcal{P}_L$ be the homogeneous part of $\mathcal{P}_{LT}$. Then the structure of $\mathcal{P}$ is

$$\mathcal{P} = \{ \mathcal{P}_{LT} , L_{oi} \} \tag{2.1}$$

$$\mathcal{P}_{LT} = \mathcal{P}_{L} \ \boxdot \ \{ P_\mu \} \tag{2.2}$$

$$\mathcal{P}_{L} = \{ L_{ij} \} \tag{2.3}$$

The semi-direct sum will always be written with the invariant subalgebra last.

Let $\mathcal{A}$ be an algebra that contains $\mathcal{P}$ as a subalgebra, and let $\mathcal{A}_{LT}$ be the largest subalgebra of $\mathcal{P}$ that commutes with P_0. Then we shall show that the physical interpretation requires the following structure for $\mathcal{A}$:

$$\mathcal{A} = \{ \mathcal{A}_{LT} , L_{oi} \} \tag{2.4}$$

$$\mathcal{A}_{LT} = \mathcal{A}_{L} \ \boxdot \ \{ P_\mu \} \tag{2.5}$$

$$\mathcal{A}_{L} = \{ L_{ij} \} \ \boxdot \ \mathcal{S} \tag{2.6}$$

$$\mathcal{A} = \mathcal{P} \ \boxdot \ \mathcal{S} \tag{2.7}$$

From (2.4) and (2.5) there follows that $\{P_\mu\}$ is an invariant subalgebra of $\mathcal{A}$. If $\{P_\mu\}$ is an invariant subalgebra of $\mathcal{A}$, and if in addition the mass operator $P_\mu P^\mu$ is an invariant of $\mathcal{A}$, then (2.6) and (2.7) follow. This important result was obtained by MICHEL.[6] Here we shall assume only that the spectrum of $P_\mu P^\mu$ is not continuous, and not, a priori, that $\{ P_\mu \}$ is an invariant subalgebra.

First we show that (2.4) is necessary. Let $\mathcal{D}$ be a particle-like representation of $\mathcal{A}$ (i.e. one in which the spectrum of P_0 is bounded below by m, say) and let $\mathcal{H}$ be the Hilbert space in which the operators of $\mathcal{D}$ act. Let $\mathcal{H}_L$ be the subspace of $\mathcal{H}$ on which P_0 has the eigenvalue m, and let $\mathcal{D}_L$ be the representation of $\mathcal{A}_L$ induced in $\mathcal{H}_L$. The basis vectors of $\mathcal{H}_L$ are, for an appropriate choice of $\mathcal{D}$, the states of a single particle at rest; we may call them $|\alpha\rangle$, $\alpha = 1, 2, \cdots$ where α stands for discrete quantum numbers like spin, charge and strangeness. Let a_i, $i = 1, 2, \cdots$, be a maximal set of basis elements in $\mathcal{A}$ that are linearly independent modulo $\mathcal{A}_{LT}$. Let $U(\epsilon)$ be a unitary operator $1 + \sum \epsilon_i a_i$ where the ϵ_i are arbitrarily small real numbers. Then $U(\epsilon)|\alpha\rangle = |\alpha, \epsilon\rangle$ is not in $\mathcal{H}_L$. Thus, the elements of $\mathcal{A}$ that are not in $\mathcal{A}_{LT}$ may be used, in addition to the label α, as labels to denote those vectors of $\mathcal{H}$ that are close to $\mathcal{H}_L$. Let us decompose this part of $\mathcal{H}$ into a direct sum of subspaces $\mathcal{H}_\alpha$, where $\mathcal{H}_\alpha$ consists of all vectors $|\alpha, \epsilon\rangle$ with fixed α. Then $\mathcal{H}_\alpha$ contains all the states of small velocity of a particle with well-defined internal quantum numbers. Now we come to our main point, namely: the dimension of $\mathcal{H}_\alpha$ must be 3. It is at least 3, because the three components of momentum are independent of each other. It is not more than 3 because the principle of relativity requires that, if a particle is found in a well-defined state by an observer at rest relative to it, then its state must likewise be well defined as seen by an observer moving slowly relatively to it. It follows that the three operators L_{0i}, when adjoined to $\mathcal{A}_{LT}$, complete $\mathcal{A}$, and we have proved (2.4).

Let $a \in \mathcal{A}_{LT}$ and consider the commutator

$$[a, L_{0i}] = -C(a)_i^{\,j} L_{0j} + b \qquad (2.8)$$

where $b \in \mathcal{A}_{L T}$. Calculating the commutator of both sides of (2.8) with P_0 we obtain

$$[a, P_i] = - C(a)_i{}^j P_j \tag{2.9}$$

which proves (2.5).

In (2.9) let $a \in \mathcal{A}_L$ and let us now introduce the assumption that the spectrum of $P_\mu P^\mu$ is not continuous. Then the matrices $C(a)_i{}^j$ are antisymmetric, and form a faithful representation of the subalgebra $\{L_{ij}\}$ of $\mathcal{A}_L$. Therefore $\mathcal{A}_L$ must have an invariant subalgebra S, say, such that S commutes with $\{P_i\}$ and $\{L_{ij}\}$ is the factor algebra $\mathcal{A}_L / S$. Thus we have proved (2.6); (2.7) follows immediately by the observation that the matrices $C(a)_i{}^j$ in (2.8) and in (2.9) are the same. Note that S commutes with $\{P_\mu\}$.

Let s_A, $A = 1, 2, \cdots$ be a basis in the algebra S, then

$$[s_A, L_{\mu\nu}] = C_{A,\mu\nu}^{B} \, s_B \tag{2.10}$$

where the matrices $C_{A,\mu\nu}^{B}$ form areal, finite-dimensional representation of the Lorentz algebra $\{L_{\mu\nu}\}$. Such a representation is a direct sum of tensor representations. The index A may be replaced by an aggregate of indices $(\lambda_1 \cdots \lambda_n, a)$ where all except the last one are four-vector indices, such that (2.10) takes the form of a set of equations

$$[s_{\lambda_1 \cdots \lambda_n, a}, L_{\mu\nu}] = i\,(g_{\lambda_1 \mu} \delta_\nu^{\lambda_1'} - g_{\lambda_1 \nu} \delta_\mu^{\lambda_1'}) s_{\lambda_1' \cdots \lambda_n, a}$$

$$+ \cdots + i\,(g_{\lambda_n \mu} \delta_\nu^{\lambda_n'} - g_{\lambda_n \nu} \delta_\mu^{\lambda_n'}) s_{\lambda_1 \cdots \lambda_n', a} \tag{2.11}$$

with $n = 0, 1, \cdots$. The range of the index a will, in general, depend on n.

The structure constants of S itself, defined by

$$[s_{\lambda_1\cdots\lambda_\ell,a}\,,\,s_{\rho_1\cdots\rho_m,b}] = C^{\sigma_1\cdots\sigma_m,c}_{\lambda_1\cdots\lambda_\ell,a,\rho_1\cdots\rho_m,b}\, s_{\sigma_1\cdots\sigma_m,c} \qquad (2.12)$$

must of course satisfy the usual conditions that make S a Lie algebra. In addition, (2.11) and (2.12) are consistent if and only if (2.12) is Lorentz covariant.

In general S will include elements with no vector indices. These commute with $\mathcal{P}$ and form the algebra S_0 of the internal symmetry group.

III. EXAMPLES WITHOUT INTERNAL SYMMETRIES

To construct the smallest A that is not simply a direct sum of S_0 and $\mathcal{P}$ let some of the elements of S be labelled by a single vector index, i.e. $s_\mu \in S, \mu = 0,1,2,3$. Then the commutator $[s_\mu, s_\nu] \equiv s_{\mu\nu}$ is an antisymmetric tensor. If $s_{\mu\nu} = 0$ then we may take S to consist exclusively of $\{s_\mu\}$, thus $S_0 = 0$. In this example $A_L = \{L_{ij}\} \boxplus \{s_\mu\}$ is isomorphic to the direct sum $U_1 \oplus E_3$, where E_3 is the three-dimensional Euclidean group. If $s_{\mu\nu} \neq 0$, then it cannot be expressed linearly and covariantly in terms of the s_μ; hence $s_{\mu\nu}$ would be independent elements of S.

For another example, suppose that $s_{\mu\nu} = -s_{\nu\mu} \in S$. Then it is possible to write covariant commutation relations, for example

$$[s_{\mu\nu}\,,\,s_{\lambda\rho}] = -i\,(g_{\mu\lambda}s_{\nu\rho} - g_{\mu\rho}s_{\nu\lambda} - g_{\nu\lambda}s_{\mu\rho} + g_{\nu\rho}s_{\mu\lambda}) \qquad (3.1)$$

This algebra is of order six;[7] it is isomorphic to $SL(2,C)$. It has a two-dimensional representation

$$s_{ij} = \tfrac{1}{2}\sigma_k \quad , \quad s_{0i} = \tfrac{1}{2i}\sigma_i \qquad (3.2)$$

The algebra $\mathcal{A}_L$ is $\{L_{ij}\} \boxminus \{S_{\mu\nu}\}$. From the commutation relations (2.11) and (3.1) it follows that $\{S_{\mu\nu}\}$ commutes with $\{L_{ij}-S_{ij}\}$; therefore $\mathcal{A}_L$ is isomorphic to $SL(2,C) \oplus SU_2$, where the second term is $\{L_{ij}-S_{ij}\}$ and <u>not</u> $\{L_{ij}\}$. The unitary irreducible representations of $\mathcal{A}_L$ are given by a pair of unitary irreducible representations of the invariant subalgebras ; let us consider, briefly, those representations of $\mathcal{A}_L$ that are obtained by choosing the trivial representation for the second one.

The unitary irreducible representations of $SL(2,C)$ were given by NAIMARK.[8] They may be reduced according to its compact subalgebra, which is isomorphic to SU_2, and are then found to contain an infinite sum of irreducible representations. Each irreducible representation of the SU_2 subalgebra with "spin" larger some minimum value occurs precisely once. These representations may be associated with the rotational levels of nuclei for fixed isotopic spin.[9]

It is important to realize that, in the type of representation just considered, the operators L_{ij} and S_{ij} are equal only in the rest system, i.e. on $\mathcal{H}_L$. Because the commutation relations between L_{ij} and S_{ij} with accelerations and with momenta are entirely different, this equality does not hold in other reference systems. In fact, on states with momentum $\vec{p}$:

$$L_{ij} = L_{ij}^{(0)} - i\left(p_i \frac{\partial}{\partial p_j} - p_j \frac{\partial}{\partial p_i}\right)$$

where $L_{ij}^{(0)}$ are the spin operators in the rest system, while

$$S_{\mu\nu} = \Lambda_\mu{}^\lambda \Lambda_\nu{}^\rho S_{\lambda\rho}^{(0)}$$

where $\Lambda_\mu{}^\lambda$ is the 4 by 4 matrix of the Lorentz transformation that transforms p_μ to rest. We have $L_{ij}^{(0)} = S_{ij}^{(0)}$ but $L_{ij} \neq S_{ij}$ for states with $\vec{p} \neq 0$.

IV. EXAMPLE WITH ISOTOPIC SPIN

Let $S_0 = \{ I_a \}$, where I_a, $a = 1, 2, 3$, are the isotopic spin operators, and let us construct a relativistic generalization of Wigner's supermultiplet theory. Both S_0 and the algebra $\{s_{\mu\nu}\}$ considered in the preceding section have two-dimensional representations, given by (3.2) and by $I_a = \frac{1}{2} \tau_a$. In these representations we may calculate anticommutators as well as commutators; it is particularly important that those of $\{s_{\mu\nu}\}$ may be expressed covariantly:

$$[s_{\mu\nu}, s_{\lambda\rho}]_+ = \tfrac{1}{2}(g_{\mu\lambda} g_{\nu\rho} - g_{\mu\rho} g_{\nu\lambda}) + \tfrac{i}{2} \epsilon_{\mu\nu\lambda\rho} \qquad (4.1)$$

Wigner's supermultiplet theory[5] is based on the algebra SU_4 given by the matrices

$$I_a = \tfrac{1}{2}(1 \otimes \tau_a), \; s_{ij} = \tfrac{1}{2}(\sigma_k \otimes 1), \; s_{ij,a} = \tfrac{1}{2}(\sigma_k \otimes \tau_a) \qquad (4.2)$$

The relativistic theory is constructed in the same way, and gives the algebra S:

$$I_a = \tfrac{1}{2}(1 \otimes \tau_a) \quad , \quad I'_a = \tfrac{1}{2i}(1 \otimes \tau_a)$$

$$s_{ij} = \tfrac{1}{2}(\sigma_k \otimes 1) \quad , \quad s_{0i} = \tfrac{1}{2i}(\sigma_i \otimes 1)$$

$$s_{ij,a} = \tfrac{1}{2}(\sigma_k \otimes \tau_a) \quad , \quad s_{0i,a} = \tfrac{1}{2i}(\sigma_i \otimes \tau_a) \qquad (4.3)$$

Because (3.1) and (4.1) are covariant we have covariant commutation relations for S; given by (3.1) and:

$$[I_a, I_b] = i I_c \; , \; [I_a, I'_b] = i I'_c \; , \; [I'_a, I'_b] = -i I_c \qquad (4.4)$$

$$[I_a, s_{\mu\nu}] = [I'_a, s_{\mu\nu}] = 0 \qquad (4.5)$$

$$[I_a, s_{\mu\nu,b}] = i s_{\mu\nu,c} \quad , [I'_a, s_{\mu\nu,b}] = -i \tilde{s}_{\mu\nu,c} \tag{4.6}$$

$$[s_{\mu\nu}, s_{\lambda\rho,a}] = -i(g_{\mu\lambda} s_{\nu\rho,a} - g_{\mu\rho} s_{\nu\lambda,a} - g_{\nu\lambda} s_{\mu\rho,a} + g_{\nu\rho} s_{\mu\lambda,a}) \tag{4.7}$$

$$[s_{\mu\nu,a}, s_{\lambda\rho,b}] = i(g_{\mu\lambda} g_{\nu\rho} - g_{\mu\rho} g_{\nu\lambda}) I_c - i \epsilon_{\mu\nu\lambda\rho} I'_c - i\delta_{ab}(g_{\mu\lambda} s_{\nu\rho} - g_{\mu\rho} s_{\nu\lambda} - g_{\nu\lambda} s_{\mu\rho} + g_{\nu\rho} s_{\mu\lambda}) \tag{4.8}$$

where $\tilde{s}_{\mu\nu,c} = g_{\mu\lambda} g_{\nu\rho} \epsilon^{\lambda\rho\sigma\tau} s_{\sigma\tau,c}$ and i, j, k and a, b, c are cyclic permutations of 1,2,3. These are the commutation relations of $SL(4,C)$.

The algebra $\mathcal{A}_L$ is $\{L_{ij}\} \boxplus SL(4,C)$. As in the previous example $\{L_{ij} - s_{ij}\}$ commutes with $SL(4,C)$, so $\mathcal{A}_L$ is isomorphic to $SL(4,C) \oplus SU_2$. Again we may take the trivial representation for SU_2, and construct unitary irreducible representations of $SL(4,C)$. These are sums of unitary, irreducible representation of SU_4; thus one representation of $SL(4,C)$ is an infinite set of Wigner supermultiplets. We repeat the warning that it is only on $\mathcal{H}_L$ that $L_{ij} = s_{ij}$.

V. EXAMPLE WITH UNITARY SYMMETRY

Let $S_0 = \{\lambda_a\}$ where λ_a, $a = 1, \cdots, 8$ are the unitary symmetry operators, and let us construct a relativistic generalization of Gürsey and Radicati's supermultiplet theory.[1] In (4.3) replace $\frac{1}{2}\tau_a$, $a = 1,2,3$, by λ_a, $a = 1, \cdots, 8$, taking for the latter one of the three-dimensional representations. Then the matrices in the left-hand column satisfy the commutation relations of SU_6, and those in the right-hand column complete this to S', which is isomorphic to $SL(6,C)$. The commutation relations are given by (3.1), (4.7) and

$$[\lambda_a, \lambda_b] = i f_{ab}^{c} \lambda_c, \ [\lambda_a, \lambda'_b] = i f_{ab}^{c} \lambda'_c, \ [\lambda'_a, \lambda'_b] = -i f_{ab}^{c} \lambda_c$$

$$[\lambda_a, S_{\mu\nu}] = [\lambda'_a, S_{\mu\nu}] = 0$$

$$[\lambda_a, S_{\mu\nu,b}] = i f_{ab}^{c} S_{\mu\nu,c}, \ [\lambda'_a, S_{\mu\nu,b}] = -i f_{ab}^{c} \tilde{S}_{\mu\nu,c}$$

$$\begin{aligned}[S_{\mu\nu,a}, S_{\lambda\rho,b}] = {} & \tfrac{i}{4}(g_{\mu\lambda} g_{\nu\rho} - g_{\mu\rho} g_{\nu\lambda}) f_{ab}^{c} \lambda_c - \tfrac{i}{4} \epsilon_{\mu\nu\lambda\rho} f_{ab}^{c} \lambda'_c \\ & - \tfrac{i}{2} d_{ab}^{c} (g_{\mu\lambda} S_{\nu\rho,c} - g_{\mu\rho} S_{\nu\lambda,c} - g_{\nu\lambda} S_{\mu\rho,c} + g_{\nu\rho} S_{\mu\lambda,c}) \\ & - i \tfrac{2}{3} \delta_{ab} (g_{\mu\lambda} S_{\nu\rho} - g_{\mu\rho} S_{\nu\lambda} - g_{\nu\lambda} S_{\mu\rho} + g_{\nu\rho} S_{\mu\lambda})\end{aligned}$$

The algebra $\mathcal{A}_L$ is $\{L_{ij}\} \boxplus SL(6,C)$ and is isomorphic to $SL(6,C) \oplus SU_2$ where the SU_2 generators are $L_{ij} - S_{ij}$ as in the other example. Taking for SU_2 the trivial representation, we may represent elementary particles by unitary irreducible representations of $SL(6,C)$, which are infinite sums of representations of SU_6. The full relativistic group is given by (2.7) where S is isomorphic to $SL(6,C)$. The translations commute with S and the commutation relations between S and the generators of the homogeneous Lorentz group are given by (2.11).

REFERENCES

1. F.GÜRSEY and L.A.RADICATI, Phys. Rev. Letters 13,173 (1964).

2. R.DELBOURGO, A.SALAM and J.STRATHDEE, Proc. Royal Soc.
T.FULTON and J.WESS, Preprints (Vienna);
W.RÜHL, Preprints (CERN);
K.BARDACKI, J.M.CORNWALL, P.G.O.FREUND and B.W.LEE, Phys.Letters 13, 698 (1964) and 14, 48 (1965).

3. H. W. WYLD, Preprint.

4. K. T. MAHANTHAPPA and E.C.G. SUDARSHAN, Preprint.

5. E.P.WIGNER, Phys. Rev. 51, 105 (1937)
P. FRANZINI and L.A.RADICATI, Phys. Letters 6, 322 (1963).

6. L. MICHEL, Preprint.

7. We may attempt to reduce this to three by taking $S_{\mu\nu}$ to be self-dual, but because of the indefiniteness of the metric we must introduce an imaginary factor: $\tilde{S}_{\mu\nu} = i S_{\mu\nu}$. Then $iS_{\mu\nu}$ are new elements of the real Lie algebra, and we still have six elements.

8. M. A. NAIMARK, "Les Représentations Linéaires du Groupe de Lorentz", Dunod, Paris 1962. This book discusses SL(2,C). For SL(m,C) see M.A.NAIMARK, Ann. Math. Soc. Translations, Series 2, 9, 155 and 9, 195 (1958).

9. A. O. BARUT and A. BÖHM, "Dynamical Groups and Mass Formulae" I.C.T.P. Preprint IC/65/18.