ON NONCOMPACT GROUPS

II. REPRESENTATIONS
OF THE 2+1 LORENTZ GROUP

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Abstract

A simple algebraic method based on multispinors with a complex number of indices is used to obtain the linear (and unitary) representations of noncompact groups. The method is illustrated in the case of the 2+1 Lorentz group. All linear representations of this group, their various realizations in Hilbert space as well as the matrix elements of finite transformations have been found. The problem of reduction of the direct product is also briefly discussed.
I. INTRODUCTION

Recently physicists have become more interested in the theory of noncompact groups. The early work on this subject was pioneered by Wigner (1939) in his study of the unitary representations of the inhomogeneous Lorentz group and its various subgroups. Later a systematic study of the representations of simple noncompact Lie groups was initiated by Mackey (1955) and by the Russian school. 1)

Instances where the unitary representations of noncompact groups have been worked out completely are remarkably few. They include the 2+1 Lorentz group (Bargmann, 1947, Gel'fand, Graev and Vilenkin, 1962), the homogeneous Lorentz group (Naimark, 1964), some work on the 3+2 and 4+1 de Sitter groups (Thomas, 1947, Newton, 1949, 1950, Ehrman, 1957, Phillips, 1962) and on some non-compact forms of unitary groups (Graev, 1958). Considering the large number of semi-simple noncompact Lie groups, 2) the study of noncompact groups seems to be only beginning, at least for physicists. Essentially the difficulties arise from the fact that the interesting (unitary) representations of noncompact groups are all infinite dimensional and seem to require a formidable mathematical apparatus. The purpose of this paper is to show that simpler algebraic techniques are sufficient to obtain the representations of noncompact groups. These methods seem to us to be sufficiently general that we hope this paper may serve as an introduction to the theory of representations of noncompact groups, although we shall be concerned with a simple example.

The example is the 2+1 Lorentz group which is not only the smallest of all non-trivial simple real noncompact Lie groups, but is also fundamental in that it appears as a subgroup of all others. It is also interesting in its own right for physical applications.
In other reports we show that these methods are by no means confined to the 2+1 Lorentz group.

In Section II we list some properties of the 2+1 Lorentz group, the essential spinor group, and the covering group. These were all given by Bargmann (1947), but are included here for completeness.

In Section III we write down the spinor representation of the associated Lie algebra, and introduce the principal technique: the construction of multi-spinors with non-integral, in general complex, numbers of indices. In this way all linear representations of the algebra are found. In Section IV those representations that are equivalent to unitary representations are selected and their unitary forms are given.

In Section V we show how a number of equivalent realizations of the underlying Hilbert space may be obtained. In particular, some light is thrown on the relationship between the three main series of representations. In particular we study the contraction of some of the representations into unitary, irreducible representations of the two-dimensional Poincaré group.

In Section VI some problems of the reduction of product representations into sums of irreducible representations are discussed. This problem was solved by Pukánszky (1961) in a very special case.

In Section VII, finally, we rederive the matrix elements of finite transformations found already by Bargmann, using algebraic rather than analytic methods.

II. THE GROUP AND THE LIE ALGEBRA

The 2+1 Lorentz group is the group \(^3\) of transformations in a real three-dimensional vector space that leaves invariant the indefinite form

\[
x^2 = x_1^2 + x_2^2 - x_3^2 = g^{\mu\nu} x_\mu x_\nu
\]  \hspace{1cm} (2.1)

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This group has three real parameters and is generated by a Lie algebra spanned by three independent generators. The generators and their commutation relations are

\[
\begin{align*}
L_{12} & , L_{13} , L_{23} \quad & L_{\mu \nu} + L_{\nu \mu} = 0 \\
[L_{\mu \nu} , L_{\alpha \lambda}] & = -i \, g_{\mu \nu} L_{\alpha \lambda} \\
g_{11} = g_{22} = 1 , & \quad g_{33} = -1
\end{align*}
\]

(2.2)

Another basis for the Lie algebra is particularly useful, namely

\[
\begin{align*}
M^\pm & = \sqrt{2} \left( i \, L_{13} \pm L_{23} \right) , \quad L_{12} \\
[L_{12} , M^\pm] & = \pm M^\pm , \quad [M^+ , M^-] = L_{12}
\end{align*}
\]

(2.3)

We have defined \( M^\pm \) and \( L_{12} \) in such a way that their commutation relations are exactly the same for the 2+1 group as for the three-dimensional rotation group. The difference between the two groups lies in the range of the parameters. Thus, if

\[
L = \varepsilon_{12} L_{12} + \varepsilon_+ M^+ + \varepsilon_- M^-
\]

(2.4)

is an element of the algebra, then \( \varepsilon_+ = \varepsilon_- \) for the rotation group and \( \varepsilon_+ = -\varepsilon_- \) for the 2+1 Lorentz group; \( \varepsilon_{12} \) is real in either case. This means that in a unitary representation

\[
L_{12}^\dagger = L_{12} \quad , \quad (M^+)^\dagger = M^-
\]

(2.5)

for the rotation group, but

\[
L_{12}^\dagger = -L_{12} \quad , \quad (M^+)^\dagger = -M^-
\]

(2.6)

for the 2+1 Lorentz group.
The 2+1 Lorentz group is locally isomorphic to the group of real unimodular 2-by-2 matrices, and also to the group of all 2-by-2 matrices of the form

\[ g = g(\alpha, \beta) = \begin{pmatrix} \alpha & \beta \\ \beta & \bar{\alpha} \end{pmatrix} \]

(2.7)

\[ |\alpha|^2 - |\beta|^2 = 1 \]

This latter group is called the spinor group. More precisely, (2.7) is a two-valued representation of the 2+1 Lorentz group. In quantum mechanics one is not interested only in representations in the narrowest sense of the word; i.e., one-valued representations, but also in multivalued representations. Consequently, we shall use the word representation for either one-valued or multivalued representations. In that case it is not important, from the point of view of representations, to distinguish between the 2+1 Lorentz group, the associated spinor group, or their covering group. However, it is important to know how many times the 2+1 Lorentz group is covered by its covering group; for this will determine the multivaluedness of the representations. This is determined by noting that (2.7) may be factorized as follows (Bargmann, 1947):

\[ g = g(\omega, \gamma) = \begin{pmatrix} e^{i\omega} & 0 \\ 0 & e^{-i\omega} \end{pmatrix} (1 - |\gamma|^2)^{-\frac{1}{2}} \begin{pmatrix} 1 & \gamma \\ \gamma^* & 1 \end{pmatrix} \]

(2.8)

Here \( 0 \leq \omega < 2\pi \) and \( \gamma = \gamma_1 + i\gamma_2 \) is a complex number. Thus the topology of the spinor group is the product of the circle and the plane; it follows that the topology of the covering group is the product of the line and the plane. The covering is thus infinitely many-fold and infinitely many-valued representations must be expected.

The two-valued representation (2.7) is called the fundamental irreducible representation \(^4\) of the 2+1 Lorentz group. The form of the generators in this representation is given by
A parametrization of the finite elements of the spinor group \( (2.7) \) that relates more directly to the generators is

\[
\mathbf{g} = \mathbf{g}(\mathbf{\Theta}) = e^{\mathbf{\Theta} \cdot \mathbf{L}} = e^{\cos \frac{\mathbf{\Theta} \cdot \mathbf{L}}{2} + \frac{i}{2} \mathbf{\Theta} \cdot \mathbf{L} \sin \frac{\mathbf{\Theta} \cdot \mathbf{L}}{2}}
\]  

(2.10)

Here \( \mathbf{\Theta} = \pm \left[ \Theta_1^2 + \Theta_2^2 + \Theta_3^2 \right]^{\frac{1}{2}} \), which may be real or imaginary.

The parametrization \( (2.7) \) and the parametrization corresponding to 2-by-2 real unimodular matrices \( \mathbf{g} \), are related to each other by

\[
\tilde{\mathbf{g}} = \mathbf{T} \mathbf{g} ^{-1}, \quad \mathbf{T} = \sqrt{\frac{1}{2}} \begin{pmatrix} 1 & -i \\ i & 1 \end{pmatrix}
\]

(2.11)

We note a geometrical interpretation of \( \mathbf{g} \) and \( \tilde{\mathbf{g}} \). The transformations \( \mathbf{g}(\alpha , \beta) \), Eq. \( (2.7) \), correspond to conformal transformations of the interior of the unit circle into itself given by

\[
\mathbf{z}' = \frac{\alpha \mathbf{z} + \beta}{\beta \mathbf{z} + \alpha}
\]

(2.12)

and to the transformations \( \tilde{\mathbf{g}}(\alpha , \beta) \) correspond the conformal transformations of the upper half of the complex plane onto itself (and also the projective transformations of the real line).

III. DERIVATION OF THE LINEAR REPRESENTATIONS

Let \( (\mathbf{f}_1, \mathbf{f}_2) \) be a spinor, i.e., a basis for the fundamental representation \( (2.7) \), and consider the linear vector space spanned by the "monomials"

\[
|a, b\rangle = \mathbf{f}_1^a \mathbf{f}_2^b
\]

(3.1)
If \(a, b\) vary over a set of positive integers \(0, 1, 2, \ldots, n\), then this vector space is the space of symmetric multispinors with \(n\) indices, and the transformations of \(\mathbf{F}\) induce on (3.1) an irreducible representation of the algebra. This technique has been used by Wigner (1959) and by Van der Waerden (1932) to derive the irreducible unitary representations of the rotation group. Here we shall use a different range of values of the exponents in order to discuss the representations of the 2+1 Lorentz group.\(^5\)

We remarked that the commutation relations, when written in terms of the operators \(L_{12}, M^+, \text{ and } M^-\), are precisely the same for the two algebras. Therefore it will be convenient to start with arbitrary complex values of \(a\) and \(b\), obtain linear representations valid for both algebras, and then show that the unitarity conditions for the rotation group require \(a\) and \(b\) to be positive integers, but give entirely different conditions on \(a\) and \(b\) in the case of the 2+1 Lorentz group.

When the two-dimensional representation (2.9) is expressed in terms of \(L_{12}, M^+, \text{ and } M^-\) it reads

\[
\begin{aligned}
L_{12} &= \frac{1}{2} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} = \frac{1}{2} \left( \mathbf{1} \mathbf{F}_1 \frac{\partial}{\partial \mathbf{F}_1} - \mathbf{1} \mathbf{F}_2 \frac{\partial}{\partial \mathbf{F}_2} \right) \\
M^+ &= \sqrt{2} \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} = \sqrt{2} \mathbf{1} \mathbf{F}_1 \frac{\partial}{\partial \mathbf{F}_1} \\
M^- &= \sqrt{2} \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} = \sqrt{2} \mathbf{1} \mathbf{F}_2 \frac{\partial}{\partial \mathbf{F}_2}
\end{aligned}
\]  

(3.2)

The last forms are particularly convenient for deriving the transformations induced on (3.1). We find immediately

\[
\begin{aligned}
L_{12} |a, b\rangle &= \frac{1}{2} (a - b) |a, b\rangle \\
M^+ |a, b\rangle &= \frac{b}{\sqrt{2}} |a + 1, b - 1\rangle \\
M^- |a, b\rangle &= \frac{a}{\sqrt{2}} |a - 1, b + 1\rangle
\end{aligned}
\]  

(3.3)
Notice that $L_{12}$ is diagonal; this is not accidental. One sees from (3.3), or directly from the commutation relations (2.3), that $M^+(M^-)$ raises (lowers) the eigenvalue of $L_{12}$ by one unit. If $L_{13}$ had been taken diagonal, and if raising and lowering operators were constructed for $L_{13}$, then we should have found that the eigenvalue would be raised or lowered by $i$, and this is of course not possible for unitary representations. However, closer examination reveals that this conclusion rests on the assumption that the eigenvectors of $L_{13}$ are normalizable. Hence we conclude that the eigenvectors of the two generators, $L_{13}$ and $L_{23}$, are in fact not normalizable, consequently the spectra of $L_{13}$ and $L_{23}$ are continuous. The reason for this difference between $L_{13}$ and $L_{23}$, and $L_{12}$ is that only $L_{12}$ generates a compact subgroup of the 2+1 Lorentz group. Although $L_{13}$ or $L_{23}$ can in principle be diagonalized it would require a completely different mathematical apparatus.

To obtain irreducible representations we must find the invariant subspaces of the operators (3.3). First it is clear that every irreducible, invariant subspace is characterized by a unique value of

$$\frac{1}{2}(a + b) = 2\Phi$$

In other words, $\Phi$ is an invariant, and it must have a fixed value in an irreducible representation. It is therefore not surprising to find that $\Phi$ is related to the eigenvalue of the Casimir operator:

$$Q = 2M^-M^+ + L_{12}(L_{12} + 1)$$

But this is not the only invariant. For $\frac{1}{2}(a - b)$ can only change by multiples of $1$ within each irreducible subspace. Hence another invariant is the fractional part $\epsilon_0$ of $\frac{1}{2}(a - b)$:

$$\frac{1}{2}(a - b) = \epsilon_0 + m$$

When $\Phi$ and $\epsilon_0$ are fixed, then the basis vectors are labelled unambiguously by $m$. Furthermore, each value of $m$ corresponds to an eigenvector of $L_{12}$, and no two eigenvectors have the same eigenvalue. Therefore, the representation $\mathcal{D}(\Phi, \epsilon_0)$, character-
ized by fixed $\Phi$ and fixed $E_0$, is irreducible unless one of the coefficients in (3.3) vanishes. This happens if $a$ or $b$ or both are integers. Thus we have the following cases:

A. If $a$, $b$ are not integers, then $\mathcal{D}(\Phi, E_0)$, defined by (3.3), (3.4) and (3.6), is irreducible. Clearly it is no loss of generality in this case to impose

$$-\frac{1}{2} \leq \Re E_0 < \frac{1}{2} \quad (3.7)$$

Two representations $\mathcal{D}(\Phi, E_0)$ and $\mathcal{D}(\Phi', E_0')$ cannot be equivalent unless the Casimir operator is the same for both, and the spectra of $L_2$ are the same. Thus they are equivalent only if $E_0 = E_0'$ and either $\Phi' = \Phi$ or $\Phi' = -\Phi - 1$. It is easy to verify that $\mathcal{D}(\Phi, E_0)$ and $\mathcal{D}(-\Phi - 1, E_0)$ are in fact equivalent. Hence it is more convenient to label the representation by $Q$ rather than $\Phi$. Thus: If $a$ and $b$ are not integers, then (3.3), (3.4), (3.5), (3.6) and (3.7) define an irreducible representation $\mathcal{D}(Q, E_0)$, and these representations are all inequivalent.

B. If $a$ takes integer values, then one sees from (3.3) that no operator transforms a vector from the subspace $a \geq 0$ to the subspace $a < 0$, because $M^{-} |0, b\rangle = 0$. Therefore the subspace $a \geq 0$ is invariant, and the representation (3.3) is reducible even after $\Phi$ and $E_0$ have been fixed. It is not fully reducible, however. That is, the subspace $a < 0$ is not invariant, since $M^{+} |1, b\rangle \sim |0, b\rangle$ with a nonvanishing coefficient. In the subspace $a \geq 0$ we have an irreducible representation $\mathcal{D}^+(\Phi)$ if $\Phi$ is fixed and $2\Phi$ is not a positive integer or zero. The spectrum of $\frac{1}{2}(a - b)$ is

$$\frac{1}{2}(a - b) = E_0 + m, \quad m = 0, 1, 2, \ldots \left\{ E_0 = -\Phi \right\} \quad (3.8)$$

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C. In a similar way we find that if \( b \) (but not \( a \)) is integer, an irreducible representation \( \mathcal{D}^-(\Phi) \) is the subspace \( b \geq 0 \), with

\[
\frac{1}{2}(a-b) = E_0 + m, \quad m = 0, -1, -2, \ldots
\]

\[
E_0 = \Phi
\]

Of course, in these cases \( E_0 \) cannot be restricted to the domain (3.7).

D. Finally both \( a \) and \( b \) may take integral values. If \( a + b < 0 \), then \( b < 0 \) in the subspace \( a \geq 0 \), and the representation induced in this subspace is \( \mathcal{D}^+(\Phi) \). Thus this corresponds to special cases of \( \mathcal{D}^+(\Phi) \). In other words, for \( \mathcal{D}^+(\Phi) \), the only values of \( 2\Phi \) that are excluded are the non-negative integers. If \( a + b \geq 0 \), then the only invariant subspace is \( a \geq 0, b \geq 0 \), and on it we have the finite dimensional representation \( \mathcal{D}(\Phi) \):

\[
\frac{1}{2}(a-b) = -\Phi, -\Phi + 1, \ldots, \Phi
\]

To put it in another way, if \( a \) is bounded below and \( b \) is bounded above, then the highest value of \( a \) and the lowest value of \( b \) must both be equal to the same integer \( \Phi \) and one gets the \( (2\Phi + 1) \)-dimensional finite representation. For \( \Phi = 0 \) we have the identity representation. The results are summarized in Table I.

Our results may be compared with those of Bargmann on the one- and two-valued representations, and we find that restricting ourselves from the covering group to the spinor group by requiring \( E_0 \) to be integer or half integer—none of the representations have been missed. It is interesting that all representations can be constructed in the form of symmetrized products of the fundamental representations, even though the latter is not unitary, and the products have non-integer numbers of factors.

Since we have worked with the algebra rather than the finite
group elements we do not yet know if the representations may be extended to the whole group. This is certainly true of the unitary representations, however.

IV. THE UNITARY REPRESENTATIONS

If a representation found in the preceding section can be made unitary by a nonsingular transformation of the basis vectors, then it may be done with \( L_{12} \) remaining diagonal. Therefore, it may be done by simply introducing normalizers in (3.1):

\[
| \Phi, m \rangle = N_m | a, b \rangle = N_m \left( f_1^a f_2^b \right) = N_m \left( \frac{f_1}{f_2} \right)^{E_{0+m}}
\]

In this new representation the operators (3.3) are

\[
\begin{align*}
L_{12} | \Phi, m \rangle &= (E_0 + m) | \Phi, m \rangle \\
M^+ | \Phi, m \rangle &= \sqrt{m} (\Phi - E_0 - m) \frac{N_m}{N_{m+1}} | \Phi, m+1 \rangle \\
M^- | \Phi, m \rangle &= \sqrt{m} (\Phi + E_0 + m) \frac{N_m}{N_{m-1}} | \Phi, m-1 \rangle
\end{align*}
\]

We now define the inner product by

\[
< \Phi, m | \Phi, m' > = \delta_{m,m'}
\]

and impose the unitarity condition (2.5) for the rotation group, to obtain

\[
\begin{align*}
\text{Im} E_0 &= 0 \\
\left| \frac{N_{m+1}}{N_m} \right|^2 &= \frac{\Phi^* - E_0 - m}{m + E_0 + \Phi + 1}
\end{align*}
\]

(for rotation group)

Clearly these recursion relations can be solved only if the values
of \( m \) are bounded both above and below. This corresponds to the well-known result that every unitary, irreducible representation of the rotation group is finite dimensional.

Returning to the 2+1 Lorentz group, we use (2.6) to obtain

\[
\operatorname{Im} E_0 = 0
\]

\[
\left| \frac{N_{m+1}}{N_m} \right|^2 = \frac{m + E_0 + \frac{1}{2} - (\Phi + \frac{1}{2})^2}{m + E_0 + \frac{1}{2} + (\Phi + \frac{1}{2})^2}
\]

These relations can be solved only if the right-hand side is positive for all integer values of \( m \) (in the case of \( D(Q, E_0) \)), for all non-negative values of \( m \) (in the case of \( D^+(\Phi) \)), or for all negative values of \( m \) (in the case of \( D^-(\Phi) \)). This gives the following conditions

\[
\begin{align*}
D(Q, E_0) & : Q < E_0 (E_0 - 1) \\
D^+(\Phi) & : \Phi < 0 \\
D^-(\Phi) & : \Phi = 0
\end{align*}
\]

(4.5)

Note that in the case of \( D(Q, E_0) \) the allowed values of \( Q \) correspond to the following range of \( \Phi \):

\[
Q > -\frac{1}{4} : \quad \Phi \text{ real }, \quad |\Phi + \frac{1}{2}| < |E_0 + \frac{1}{2}|
\]

(4.6)

\[
Q < -\frac{1}{4} : \quad \Phi = -\frac{1}{2} + \lambda, \quad \lambda \text{ real}
\]

(4.7)

Another way of imposing the unitarity condition is to require that the eigenvalues of \( L_\mu \), \( Q \) are real and those of \( M^* M \) and \( M^T M \) real and negative definite.

In the case (4.7), (4.4) is consistent with \( N_m = 1 \). In
the other cases we may take, provided (4.5) is satisfied,

\[ N_m = \left[ \frac{(m+E_0 - 1 - \Phi)!}{(m+E_0 + \Phi)!} \right]^{1/2} \]  

(4.8)

where \( Z \) is defined, here and throughout this paper, as \( \Gamma(z+1) \).

With this choice (4.2) is unitary. These results have been entered in Table I and illustrated by Fig. 1.

The representations (4.7) and (4.6) are called respectively the principal series and the supplementary series of representations.

We have yet to construct a Hilbert space, but this is easily done. Contrary to the impression that may be gained by reading the literature it is neither necessary nor convenient to introduce a realization of the Hilbert space by means of Lebesgue-square integrable functions or variations thereof. A vector in the space spanned by the basis vectors (4.1) may be written

\[ \Psi = \sum C_m \Phi, m \]  

(4.9)

In order to construct a Hilbert space we have to introduce an inner product \( (\Psi, \Psi') \), that is linear in the postfactor and antilinear in the prefactor. Thus, because of (4.3)

\[ (\Psi, \Psi') = \sum_{m',m} C_{m'} \overline{C}_m \langle \Phi, m' | \Phi, m \rangle \]

\[ = \sum_m C_m \overline{C}_m \]  

(4.10)

Accordingly, (4.9) is normalizable if

\[ \sum_m |C_m|^2 < \infty \]  

(4.11)
and the set of all vectors (4.9) for which (4.11) holds form a Hilbert space with respect to the inner product (4.10). This is the standard example of a Hilbert space constructed from a denumerably infinite set of orthogonal basis vectors.

V. EQUIVALENT REPRESENTATIONS

In the literature one finds the representations of the principal series realized on the space of square integrable functions on the unit circle, those of the supplementary series on a similar space, while the representations $\mathfrak{D}^\phi (\Phi)$ are realized on the space of functions of one complex variable, analytic inside the unit circle. It may be of some interest to see precisely what is the equivalence transformation that connects these realizations with those discussed in the preceding sections.

Let us define a set of vectors by

$$|\phi> = \sum_m N_m e^{-im\phi} |\Phi, m>$$

and confine ourselves to the unitary representations. This set is a complete set of vectors in the sense that (5.1) may be "solved":

$$|\Phi, m> = \frac{1}{2\pi N_m} \int_0^{2\pi} e^{im\phi} d\phi |\phi>$$

Therefore an arbitrary normalizable vector (4.9) may be expanded as follows

$$\Psi = \frac{1}{2\pi} \int_0^{2\pi} \psi(\phi) d\phi |\phi>$$

where the "wave function" $\psi(\phi)$ is related to the (square summable) coefficients $C_m$ by

$$\psi(\phi) = \sum_m \frac{N_m}{N_{\Phi}} C_m e^{im\phi}$$

$$C_m = \frac{N_m}{2\pi} \int_0^{2\pi} e^{-im\phi} \psi(\phi) d\phi$$
The transformation properties of $\psi(\varphi)$ are easily calculated in the case of the representations of the principal or supplementary series with the result that

$$
L_z \psi(\varphi) = (E_\varphi - i \frac{\partial}{\partial \varphi}) \psi(\varphi)
$$

$$
M^+ \psi(\varphi) = \sqrt{2} (\mp E_\varphi + i \frac{\partial}{\partial \varphi}) e^{i \varphi} \psi(\varphi)
$$

$$
M^- \psi(\varphi) = \sqrt{2} (\mp E_\varphi - i \frac{\partial}{\partial \varphi}) e^{-i \varphi} \psi(\varphi)
$$

(5.6)

which agrees with the expressions given by Bargmann for the cases $E_\varphi = 0, 1/2$ (the spinor group).

In the case of the representations $D^+(\varphi)$ the results (5.6) cannot be valid, as we see from the fact that $e^{-i \varphi} \psi(\varphi)$ is not of the form (5.4) unless $C_\varphi = 0$. (Remember that, for $D^+(\varphi)$, $m = 0, 1, 2, \ldots$ and that $-\Phi + E_\varphi - 1 \neq 0$.) It is easy to obtain the correct formulae, however, and the result is that for $D^+(\varphi)$ only the expression for $M^-$ is changed to the following

$$
M^- \psi(\varphi) = \sqrt{2} (\mp E_\varphi - i \frac{\partial}{\partial \varphi}) e^{-i \varphi} \left[ \psi(\varphi) - \frac{C_\varphi}{N_\varphi} \right]
$$

(5.7)

The functions $\psi(\varphi)$ should not be confused with those of Bargmann's Hilbert space. Bargmann's functions are realizations of the vectors $|\varphi\rangle$. For $D^-(\varphi)$ there is a corresponding correction to $M^+$, only. These peculiarities may be the reason why the basis (5.1) has not been considered suitable for the representations $D^z(\varphi)$; more about this below.

The inner product may be calculated in a straightforward manner:

$$
\langle \psi, \psi' \rangle = (2\pi)^2 \int d\varphi \int d\varphi' \psi^*(\varphi) \psi'(\varphi) \langle \varphi | \varphi' \rangle
$$

(5.8)

$$
\langle \varphi | \varphi' \rangle = \sum_m |N_m|^2 e^{im(\varphi - \varphi')}
$$

(5.9)
We evaluate (5.9) in the several cases.

**Principal series.** Here \( N_m = 1 \), and for real \( \Phi \),

\[
\langle \Phi | \Phi' \rangle = 2\pi \delta (\Phi - \Phi') \tag{5.10}
\]

\[
(\psi^*, \psi') = (2\pi)^{-1} \int d\Phi \psi^*(\Phi)\psi'(\Phi) \tag{5.11}
\]

Therefore the Hilbert space is the space of functions that are Lebesgue-square integrable over the unit circle.

**Supplementary series.** First we note that

\[
N_m \to m^{-2\Phi - 1} \quad \text{as} \quad |m| \to \infty \tag{5.12}
\]

Since in the sum in (5.9) \( m \) goes from \(-\infty\) to \(+\infty\), it can converge only if \(-2\Phi - 1 < 0\), or \( \Phi > -\frac{1}{2} \). But this is not an essential restriction for the supplementary series since reflection of \( \Phi \) around \(-\frac{1}{2}\) gives an equivalent representation. Now let

\[
\chi = e^{im(\Phi^* - \Phi')} \tag{5.13}
\]

then it may be verified that (5.9) satisfies the following differential equation

\[
\left[ (1 - \chi) \frac{\partial^2}{\partial \chi^2} + \frac{1}{\lambda} (E_0 + \Phi) + \Phi - E_0 \right] \langle \Phi | \Phi' \rangle = 0 \tag{5.14}
\]

the solution of which is

\[
\langle \Phi | \Phi' \rangle \sim e^{-iE_0(\Phi^* - \Phi')} \left[ 1 - c_8 \cos(\Phi^* - \Phi') \right] \Phi \tag{5.15}
\]

When this is substituted into (5.8) we obtain a formula for
the inner product that again agrees with the results of Bargmann in the case \( E_0 = 1 \). In (5.15)

\[-\frac{1}{2} < \Phi < -1 \]

**Discrete series \( D^+(\Phi) \).** The important difference between the principal and supplementary series and the discrete series is that in the case of the latter the sum in (5.9) includes only positive values (or, for \( D^-(\Phi) \), only negative values) of \( m \). Therefore, for \( D^+(\Phi) \), (5.9) converges inside the unit circle in the \( x \)-plane; \( x \) was defined by (5.13). The sum is now easily calculated by direct summation, and the result is the same formula (5.15) as for the supplementary series. The convergence of (5.9) requires that \( \text{Im} \Phi < 0 \), or \(|x| < 1 \). One may either replace the unit circle by a circle of radius slightly less than unity, or one may continue \( \psi(\Phi) \) to the whole interior of the unit circle, to obtain a representation on the space of functions analytic there.

We now discuss another realization of \( D^+(\Phi) \) which is adapted to the following physical application.

The problem is that of formulating elementary particle physics in a space whose group of motions is the de Sitter group; i.e., the 3+2 Lorentz group. The difficulties of interpretation are precisely the same in the algebraically simpler case of a two-dimensional space-time and this leads us to the 2+1 Lorentz group. If we define

\[ P_\mu = \mathcal{F}^{-\frac{1}{2}} \mathcal{L}_\mu \quad ; \quad \mu = 1, 3 \]

and let \( \mathcal{F} \) tend to zero - this is a "contraction" (Inönü and Wigner, 1953) - then the algebra \( P_1, P_3, L_3 \) generates the two-dimensional Poincaré group. The operators \( P_1 \) and \( P_3 \) are interpreted as translations in time and space, respectively, and
in the limit $\rho \to 0$ their eigenvalues are the energy and the
momentum. For massive particles the energy must be bounded below;
hence we are led to the study of $D^+(\Phi)$.

We now give $D^+(\Phi)$ in the form that brings out the physical
content according to this application.

Let $\rho = (\rho_1, \rho_3)$ be a basis for a two-dimensional non-unitary
representation of the subgroup generated by $L_{31}$:

$$L_{31} \rho = \langle 0 \vert \rho$$

(5.17)

If $\hat{\rho}$ is the special "2-vector"

$$\hat{\rho} = (m, 0)$$

(5.18)

then there is one and only one "Lorentz transformation" generated
by $L_{31}$, such that

$$\alpha(\rho) \rho = \hat{\rho}$$

(5.19)

namely

$$\alpha(\rho) = e^{2i\theta L_{31}}, \quad 2\theta = \text{tan}^{-1} \frac{\rho_3}{\rho_1}$$

(5.20)

In $D^+(\Phi)$ let $\alpha(\rho)$ be represented by $D(\alpha(\rho))$, and define

$$|\rho\rangle = D(\alpha(\rho)) |\Phi, 0\rangle = e^{2i\theta L_{31}} |\Phi, 0\rangle$$

(5.21)

Then the label $\rho$ for the state $|\rho\rangle$ has the following properties:
1. For the state of lowest energy it is equal to $\hat{\rho} = (m, 0)$. 2. The
components of $\rho$ transform like a vector under the Lorentz transform-
ations of the two-dimensional space. 3. We shall see that, in
addition, as $\rho \to 0$, $\rho_1$ becomes the eigenvalues of $P_\mu$. All this
qualifies the label \( \rho \) to be referred to as the momentum of the state (5.21).

We calculate \( |\rho\rangle \) explicitly:

\[
|\rho\rangle = N_0 \left( \frac{5}{2} \cosh \Theta - \frac{5}{2} \sinh \Theta \right)^2 \Phi
\]

\[
= N_0 \sum_{k=0}^{\infty} \frac{(2E_0 + j)!}{k!(2E_0 - j)!} \left( \frac{5}{2} \cosh \Theta \right)^{-2E_0 - k} \left( \frac{5}{2} \sinh \Theta \right)^k
\]

\[
= \left( \frac{2m}{\rho_1 + m} \right) \sum_{k=0}^{\infty} \frac{N_0}{N_0} \left( \frac{\rho_3}{\rho_1 + m} \right)^k |\Phi, k\rangle
\]

where

\[
m = \sqrt{\rho_1^2 - \rho_3^2}
\]

We may note that

\[
|\rho\rangle = \left( \frac{2m}{\rho_1 + m} \right)^{E_0} N_0^{-1} \left( \rho_1 + m \right) \Phi, \quad \frac{\rho_3}{\rho_1 + m} = \frac{1}{i \sin \Theta}
\]

This allows us to obtain the properties of \( |\rho\rangle \) from those of \( |\Phi\rangle \), but we prefer a direct calculation. Thus the inner product is

\[
<\rho | q> = \sum_{k=0}^{\infty} \left( \frac{N_k}{N_0} \right)^2 \left( \cosh \Theta \cosh \Theta_q \right)^{-2E_0 - k} \left( \sinh \Theta \sinh \Theta_q \right)^k
\]

\[
= \left( \cosh \Theta \cosh \Theta_q - \sinh \Theta \sinh \Theta_q \right)^{-2E_0}
\]

\[
= \left( \frac{2m^2}{\rho_1 + m} \right)^{E_0}
\]

where \( \rho_4 = \rho_1 q_1 - \rho_3 q_3 \).
From (5.21) we find

\[ L_{31} | \phi > = -i \left( P_1 \frac{\partial}{\partial P_3} + P_3 \frac{\partial}{\partial P_1} \right) | \phi > \]  \hspace{1cm} (5.26)

To determine the other operators we combine

\[ M^- | \phi, 0 > = 0 \quad , \quad L_{12} | \phi, 0 > = E_0 | \phi, 0 > \]  \hspace{1cm} (5.27)

to

\[ L_{\mu \nu} | \phi > = \left[ E_0 (\frac{P_\mu}{m}) + (\frac{\partial}{\partial m}) L_{\nu \mu} \right] | \phi > \]  \hspace{1cm} (5.28)

Since this is covariant with respect to the "Lorentz" transformations generated by (5.26) we have

\[ L_{\mu \nu} | \phi > = \left[ E_0 (\frac{P_\mu}{m}) + (\frac{\partial}{\partial m}) L_{\nu \mu} \right] | \phi > \]  \hspace{1cm} (5.29)

Notice that (5.23) holds unconditionally. Writing this in terms of \( P_\mu \) we obtain

\[ P_\mu | \phi > = \left[ P_\mu + \frac{\sqrt{E}}{c m} P^\nu L_{\nu \mu} \right] | \phi > \], \( \mu = 1, 3 \)

provided the "mass" is related to \( E_0 \) and \( P \) by

\[ m = E_0 \sqrt{E} \]  \hspace{1cm} (5.30)

Taking the limit \( P \to 0 \) we keep \( m \) finite and thus obtain the desired result \( P_\mu \to P_\mu \).

The results (5.26) and (5.29) have an exact analogue in the case of a four-dimensional space time (Fronsdal, 1965), but here we
have the advantage of a simpler algebraic structure that allows us to reach a deeper understanding. The Hilbert space on which \( \mathcal{D}(\mathbb{F}) \) is realized may be taken to be discrete \((\mathbb{F}, \mathbb{H})\), or it may be taken as the set of functions \( \mathbb{F} \to \mathbb{F}(\mathbb{C}), \mathbb{C} = e^{i\varphi} \), holomorphic for \(|\mathbb{C}| < 1\), or it may be taken as the set of points on a circle \(|\mathbb{C}| = \mathbb{R} < 1\). In terms of \( \mathbb{F} \), the holomorphy domain \(|\mathbb{C}| < 1\) is the "future tube", \( \mathbb{Re} \mathbb{F} \in \mathbb{V}^+ \). Because of (5.23) only \( \mathbb{F}_3 \) is an independent variable. In terms of \( \mathbb{F}_3 \) the "future tube" is the entire complex plane, except for cuts from \( \pm i\mathbb{m} \) to infinity. In view of the application that we have in mind, however, none of these realizations are particularly "physical". We may attempt to stay near the real \( \mathbb{F}_3 \) axis and introduce wave functions \( \psi(\mathbb{F}) \) by writing

\[
\psi = \int_{-\infty}^{+\infty} \frac{d\mathbb{F}_3}{2\mathbb{F}_1} \psi(\mathbb{F}) |\mathbb{F}\rangle \tag{5.31}
\]

Then we run into the following peculiarities.

The transformation properties of \( \psi(\mathbb{F}) \) have the same unusual feature as that found earlier for \( \psi'(\mathbb{F}) \). A simple determination of \( \mathbb{M}^\pm \) and \( \mathbb{L}_1 \) on the basis of (5.31) gives the wrong answer, unless we remember that \( \mathbb{M}^{-} \) is a singular operator. A careful calculation gives

\[
\mathbb{L}_1 \psi(\mathbb{F}) = \left[ \frac{E_0 - 1}{\mathbb{m}} \mathbb{F}_3 - \frac{\mathbb{F}_3}{\mathbb{m}} \left( \mathbb{F}_1 \mathbb{F}_3 \mathbb{F}_3 + \mathbb{F}_3 \mathbb{F}_3 \mathbb{F}_3 \right) \right] \psi(\mathbb{F}) \tag{5.32}
\]

\[
\mathbb{M}^+ \psi(\mathbb{F}) = \sqrt{\mathbb{E}} \left[ \frac{E_0 - 1}{\mathbb{m}} \mathbb{F}_3 + \frac{\mathbb{F}_3}{\mathbb{m}} \left( \mathbb{F}_1 \mathbb{F}_3 \mathbb{F}_3 + \mathbb{F}_3 \mathbb{F}_3 \mathbb{F}_3 \right) \right] \psi(\mathbb{F}) \tag{5.33}
\]

\[
\mathbb{M}^- \psi(\mathbb{F}) = \sqrt{\mathbb{E}} \left[ \frac{E_0 - 1}{\mathbb{m}} \mathbb{F}_3 - \frac{\mathbb{F}_3}{\mathbb{m}} \left( \mathbb{F}_1 \mathbb{F}_3 \mathbb{F}_3 + \mathbb{F}_3 \mathbb{F}_3 \mathbb{F}_3 \right) \right] \psi(\mathbb{F}) \tag{5.34}
\]

where \( \psi' \) is \( \psi \) with the "vacuum part" substracted out. It may be defined by

\[
|\psi\rangle - |\hat{\psi}, 0\rangle \langle \hat{\psi}, 0|\psi\rangle = \int \frac{d\mathbb{F}_3}{2\mathbb{F}_1} \psi'(\mathbb{F}) \tag{5.35}
\]
and is given explicitly by

\[
\psi^*(\rho) = \psi^*(\rho) - \frac{i m}{\hbar} \frac{(\rho_1 + m)}{\rho_3} \int \frac{d\rho_3}{2\rho_1} \frac{\epsilon_0}{(\rho_1 + m)} \psi^*(\rho)
\]  

(5.36)

This formula leads us to another important point. The integration in (5.36) is a closed curve that includes the origin. In general, in order to construct an eigenstate of $\mathcal{L}_z$ from $|\rho\rangle$, such contour integrations are necessary. If we compare (5.31) with the expansion (4.9), then we find

\[
\psi^*(\rho) \sim \frac{1}{\rho_3} (\rho_1 + m) \sum_{\kappa=0}^{\infty} N_{\kappa} \left( \frac{\rho_1 + m}{\rho_3} \right)^{1/2} C_{\kappa}
\]  

(5.37)

which shows that $\psi(\rho)$ is singular at the origin. For states for which the sum (5.37) is finite, $\psi^*$ has only poles at the origin. If we insist on keeping $\rho$ real, then these functions are derivatives of $\delta$-functions; i.e., highly singular distributions. Perhaps the most convenient choice is to write all integrations in the following way

\[
\psi^* = \lim_{\epsilon \to 0^+} \left[ \int_{-\infty - i\epsilon}^{-\infty + i\epsilon} + \int_{+\infty - i\epsilon}^{+\infty + i\epsilon} \right] \frac{d\rho_3}{2\rho_1} \psi(\rho) |\rho\rangle
\]  

(5.38)

Finally we note that the inner product is, because of (5.25),

\[
\langle \psi, \psi^* \rangle = \int \frac{d\rho_1}{2\rho_1} \int \frac{d\varphi_3}{2\varphi_1} \psi^*(\rho) \left( \frac{2m^2}{\rho^2 + m^2} \right) \epsilon_0 \psi(\rho)
\]  

(5.39)
VI. ON THE REDUCTION OF PRODUCT REPRESENTATION

The reduction of a product of two irreducible representations into a sum of irreducible representations is perhaps the hardest aspect of the theory of noncompact groups. Nevertheless we shall show that methods that have been developed for compact groups may be applied, although a great deal of extra complication arises when one attempts to obtain complete results.

Let \( \mathcal{D}_1 \) and \( \mathcal{D}_2 \) be two unitary, irreducible representations, and let them be realized by means of multispinors:

\[
\mathcal{D}_1 : \quad \eta_1^c \eta_2^d, \quad \frac{1}{2} (c + d) = \Phi_1 \tag{6.1}
\]

\[
\mathcal{D}_2 : \quad \chi_1^e \chi_2^f, \quad \frac{1}{2} (e + f) = \Phi_2 \tag{6.2}
\]

The product \( \mathcal{D}_1 \otimes \mathcal{D}_2 \) is a reducible representation, and it may be reduced to a sum of unitary, irreducible representations. The problem is three-fold: 1. To determine which unitary, irreducible representations \( \mathcal{D} \) occur in the decomposition of \( \mathcal{D}_1 \otimes \mathcal{D}_2 \), 2. To determine the basis vectors for each \( \mathcal{D} \) that occurs, in terms of the basis vectors of the product, and 3. To write each basis vector of the product as a sum over basis vectors that belong to irreducible components. The first two parts of the problem may be reformulated as follows: 1. To find all unitary, irreducible representations \( \mathcal{D} \) that have the property that an invariant can be formed from the basis vectors of \( \mathcal{D}' \), \( \mathcal{D}_1 \) and \( \mathcal{D}_2 \), 6) where \( \mathcal{D}' \) is the representation contragredient to \( \mathcal{D} \), 2. To write down all these invariants in terms of the basis vectors of the three representations. We shall solve the problem in this form, and begin with a quick review of the simpler case of the compact rotation group.

Let \( \mathcal{D}' \) be realized in terms of multispinors:

\[
\mathcal{D}' : \quad \xi_1^a \xi_2^b, \quad \frac{1}{2} (a + b) = \Phi \tag{6.3}
\]
Following Regge (1965) we note that the only invariants that can be formed from the three spinors $\xi$, $\eta$, and $\chi$ are

$$(\xi \eta), \ (\xi \chi), \ (\eta \chi) = \eta_1 \chi_z - \eta_2 \chi_1$$

From these we attempt to construct an invariant coupling between $D'$, $D_1$, and $D_2$; i.e., an invariant of the form

$$I = \sum C_{abcdef} \xi_1^a \xi_2^b \eta_1^c \eta_2^d \chi_1^e \chi_2^f$$

where the ranges of the exponents are restricted by (6.1), (6.2), and (6.3), and in addition, $a + c + e = b + d + f$. If (6.5) is written in terms of the three invariants (6.4) it must have the form

$$I \sim (\xi \eta)^\alpha (\xi \chi)^\beta (\eta \chi)^\gamma$$

Comparing the values of the exponents we find

$$\alpha + \beta + \gamma = \xi + \chi$$

The special feature that simplifies the problem in the case of the compact rotation group is that $a, b, c, d, e, f$ must all be non-negative integers. There follows that the same is true of $\alpha, \beta, \gamma$ and thus that the possible values of $\Phi$ differ by multiples of unity and lie between the limits

$$|\Phi_1 - \Phi_2| \leq \Phi \leq \Phi_1 + \Phi_2$$

Since every unitary, irreducible representation of the rotation group is given up to equivalence by the number $\Phi$, this is already the complete answer to the first problem.
To solve the second problem we merely expand (6.6):

$$\Gamma = \sum_{j,k,l} \frac{(-)^l\alpha!\beta!\gamma!}{(\frac{\alpha}{2}+j)!(\frac{\beta}{2}-j)!(\frac{\gamma}{2}+k)!(\frac{\gamma}{2}-k)!(\frac{\alpha}{2}+\ell)!(\frac{\gamma}{2}-\ell)!} \times \left[ (-\frac{r_1^2}{r_2^2})^{j-k}\left[ (\gamma_1 \gamma_2)^{j+k} \left( \frac{\gamma_1}{\gamma_2} \right)^{-j} \right] \left[ (\gamma_1 \gamma_2)^{k}\left( \frac{\gamma_1}{\gamma_2} \right)^{\ell-k} \right] \right] \tag{6.9}$$

Here $j$ runs from $-\frac{\alpha}{2}$ to $+\frac{\alpha}{2}$ in integer steps, and $k,\ell$ similarly.

If we introduce the normalization proper to the rotation group

$$|\Phi, m_1 > = N_{m_1}^{(1)} (\gamma_1 \gamma_2)^{k} \left( \frac{\gamma_1}{\gamma_2} \right)^{m_1} \tag{6.10}$$

$$N_{m_1}^{(1)} = \left[ (\Phi_r - m_1)! (\Phi_r + m_1)! \right]^{-\frac{1}{2}} \tag{6.11}$$

and write $|\Phi_1, m_1; \Phi_2, m_2 >$ for the product basis, and define the contragredient representation

$$<\Phi, m | = N_{m}^{-l} (-\frac{r_1^2}{r_2^2})^{k} \left( \frac{r_2}{r_1} \right)^{m} \tag{6.12}$$

then (6.9) takes the usual form

$$\Gamma = \sum_{m_1, m_2} C_{m_1, m_2} <\Phi, m_1 + m_2 | \Phi_1, m_1; \Phi_2, m_2 > \tag{6.13}$$

$$C_{m_1, m_2} = \sum_{j=-\frac{\alpha}{2}, \ldots, \frac{\alpha}{2}} \sum_{k=-\frac{\beta}{2}, \ldots, \frac{\beta}{2}} \sum_{\ell=-\frac{\gamma}{2}, \ldots, \frac{\gamma}{2}} \frac{(-)^l\alpha!\beta!\gamma! N_{m_1, m_2} / N_{m_1}^{(1)} N_{m_2}^{(1)}}{(\frac{\alpha}{2}-j)! \ldots \ldots (\frac{\gamma}{2}-\ell)!} \tag{6.14}$$

The constants (6.14) are, except for a normalization factor, the Clebsch-Gordan coefficients.
Now let us turn back to the 2+1 Lorentz group. Because the transformation properties of spinors are the same as for the rotation group, the spinor invariants (6.4) are the same, and the problem is to compare (6.5) and (6.6). The new feature is that, since we shall discuss only unitary representations, not all three of the exponents $\alpha$, $\beta$, and $\nu$ are integers.

First we dispose of certain trivial cases. It is clear that, if $\mathcal{D}_2$ is the identity representation, then $\mathcal{D}_1$ and $\mathcal{D}$ must be equivalent, and

$$I = (\gamma)(\Phi) - (\gamma_2),(\Psi_1)(\eta_1,\eta_2)\sum_{\alpha} (\frac{-\xi^2}{\xi_2})$$

$$= \sum_{\Phi, m_1, \Phi, m_2} \langle \Phi, m_1 | \Phi, m_2 > \delta_{m_1, m_2}$$

This is a formal expression that serves to define the invariant form $\delta_{m_1, m_2}$. The reduction of $\mathcal{D}^+(\Phi_1) \otimes \mathcal{D}^+(\Phi_2)$ and of $\mathcal{D}^-(\Phi_1) \otimes \mathcal{D}^-(\Phi_2)$ are quite analogous to the case of the rotation group. It is enough to illustrate by working out the former case. Then (6.5) and (6.6) contain only non-negative integral powers of $\eta_1$ and of $\nu_1$. This means that the first two factors in (6.6) must be expanded in powers of their second terms, and that $\nu$ must be a non-negative integer. Thus the expansion (6.9) still holds, with

$$j = -\frac{\Phi}{2}, -\frac{\Phi}{2} + 1, \cdots$$

$$k = \frac{\beta}{2}, \frac{\beta}{2} - 1, \cdots$$

$$\nu = -\frac{\gamma}{2}, \cdots, \frac{\gamma}{2}$$

With these new ranges of the parameters (6.13) and (6.14) also continue to hold, except that the normalizers $N_m$ of the rotation group must be replaced by the normalizers (4.8) appropriate to the 2+1 Lorentz group.

It is easy to extend the discussion of the invariants to the other products; e.g., to the product of two representations of the principal series. However, even when a formal invariant exists
between, say, $\mathcal{D}_1$, $\mathcal{D}_2$ and $\mathcal{D}'$ it is not always true that it can be written in terms of normalizable vectors in the Hilbert space of the product representation $\mathcal{D}_1 \otimes \mathcal{D}_2$. We hope to complete our work on this problem in a future report.

VII. MATRIX ELEMENTS FOR FINITE TRANSFORMATIONS

One advantage of the representation theory based on the multispinors (3.1) is that it allows us in a straightforward manner to evaluate the matrix elements of finite transformations. These are the quantities corresponding to the well-known $\mathcal{P}^j$-functions in the case of the rotation group and are important quantities for applications.

Let us denote a representation of an element of the spinor group (2.7) by $S(g)$. Noting the inverse transformation

$$g^{-1} = \begin{pmatrix} \bar{\alpha} & -\bar{\beta} \\ -\beta & \alpha \end{pmatrix}$$

we have

$$S(g) | \Phi, m \rangle = N_{m} \left( \bar{\alpha} \xi_{1} - \bar{\beta} \xi_{2} \right)^{a} \left( -\bar{\beta} \xi_{1} + \alpha \xi_{2} \right)^{b}.$$  \hspace{1cm} (7.1)

Because the binomial expansions can be defined even for complex powers $a$ and $b$ we can expand the right hand of (7.1) and obtain

$$S(g) | \Phi, m \rangle = N_{m} \sum_{x} \sum_{x'} (-1)^{x} \frac{a^{x}}{x!} \frac{b^{x'}}{x'!} \frac{(a - x)! (b - x')!}{(a - x)! (b - x')!} \left( \bar{\alpha} \xi_{1} - \bar{\beta} \xi_{2} \right)^{a} \left( -\bar{\beta} \xi_{1} + \alpha \xi_{2} \right)^{b}.$$

$$\times \xi_{1}^{a + b - x - x'} \xi_{2}^{x + x'}.$$  \hspace{1cm} (7.2)

Let us now put in (7.2)
\[ a + b - x - x' = a' \]
\[ x' + x' = b' \]

and

\[ \tilde{m}' = \frac{1}{2} (a' - b') = \frac{1}{2} (a + b) - (x + x') = \Phi' - (x + x'). \]

We then obtain

\[ S(g) |\Phi, m\rangle = \sum_{m'} S_{mm'}^{\Phi'}(g) |\Phi, m'\rangle, \]  \hspace{1cm} (7.3) \]

where the matrix elements are given by

\[ S_{mm'}^{\Phi'}(g) = \frac{N_m}{N_{m'}} \sum_{x} (-1)^x \frac{a - x}{\beta} \frac{b - \Phi + m' + x}{\Phi - m' - x} \frac{a! b!}{x! (\Phi - m' - x)! (a - x)! (b - \Phi + m' + x)!}. \]  \hspace{1cm} (7.4) \]

or, with

\[ a = \xi + m + E_0, \quad b = \Phi - m - E_0 \]
\[ \tilde{m}' = m' + E_0 \]

\[ S_{mm'}^{\Phi'}(g) = \frac{N_m}{N_{m'}} \frac{\Phi + m + E_0)! (\Phi - m - E_0)!}{(m' - m)!} \frac{1}{\beta} \frac{\Phi + m + E_0}{\Phi - m' - E_0} \frac{m' - m \cdot m}{(-\beta)} \]
\[ \times \sum_{x=0}^{\infty} (-1)^x \frac{(\phi a)^x}{(\Phi + m + E_0)!} \frac{(-\beta \bar{\beta})^x}{(\Phi - m' - E_0 - x)!} \frac{1}{x! (m' - m + x)!}. \]  \hspace{1cm} (7.4')
In the last sum we recognize the expansion of the hypergeometric function \( F(a, b, c; z) \) of argument

\[
z = (\alpha \beta)^{-\gamma} (-\alpha \beta)^{\gamma} = \frac{-\alpha \beta}{1+\beta \alpha}
\]

because of (2.7). Transforming the argument to \( \frac{z}{1+z} \) we obtain

\[
S_{m}^{m'}(g) = \frac{N_{m}}{N_{m'}} \left( \frac{\alpha}{\alpha + m + E_0} \frac{\beta + m + E_0}{(-\beta)^{m'} m'} \right) \left( m' - m \right)
\]

(7.5)

In writing these formulae in terms of hypergeometric functions \( F(a, b, c; z) \) care must be taken that \( c \) does not take negative integer values, for only the ratio \( F(a, b, c; z)/\Gamma(c) \) is an entire analytic function of \( a, b, c \) if \( z \) is fixed and \( |z| < 1 \) (Erdelyi, 1953).

It is interesting that all matrix elements can be expressed as a linear combination of \( F(\pm \phi, -\phi; 1; -\beta \phi) \) and one of the following functions contiguous to \( F \):

\[
F(-\phi, -\phi; 1; -\beta \phi); F(-\phi, -\phi; 1; -\beta \phi);
\]

with coefficients which are rational functions of \( \phi \) and \(-\beta \phi\).

This follows from the corresponding properties of the hypergeometric functions because \( m, m' \) are integers.

If we replace in (7.5) \( \beta \) by \(-\beta\) we obtain the matrix elements of the rotation group (Wigner, 1959). For \( \phi \) integer and \( m' - \phi \) or \(-m - \phi \) equal to 0, -1, -2, ......., the hypergeometric function reduces to a polynomial.

The matrix elements (7.5) agree with those of Bargmann (1942) found via the solutions of differential equations.
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Fig. 1 Range of parameters of the unitary representations. The Casimir operator is $Q = \hat{Q}(\Phi + i)$, and $\Phi$ is plotted against the eigenvalues of $L_{12}$. Thick solid lines indicate the lowest (highest) values of $L_{12}$ in the discrete representations $\mathcal{D}^+(\Phi)$ ($\mathcal{D}^-(\Phi)$). The values of $E_0$, the fractional part of the eigenvalues of $L_{12}$, for the supplementary series of representations are in the shaded triangle, not including the boundary; the reflection of the triangle around $\Phi = -\frac{1}{2}$ gives only equivalent representations. For principal series $\Phi = -\frac{1}{2} + i\lambda$, a plane perpendicular to the plane of drawing at $\Phi = -\frac{1}{2}$. For comparison the range of unitary irreducible representations of the rotation group is also indicated by black dots. The origin is the identity representation.
Table I. Summary of linear representations.
FOOTNOTES

1. See for example the review by Berezin, Gel'fand, Graev and Naimark (19 ).

2. See for example the first paper in this series (Barut and Rączka, 1965).

3. Like the 3+1 Lorentz group, this group consists of four disconnected parts. Here we study only that which is continuously connected to the identity.

4. A semi-simple Lie group of rank $\ell$ has $\ell$ inequivalent fundamental irreducible representations.

5. To apply this method to a semi-simple Lie group of rank $\ell$ one has to study monomials in $\sum_i d_i$ variables where $d_i$ is the dimension of the $i$'th fundamental irreducible representation.

6. This problem is not completely equivalent to that of deciding if $\hat{\mathbf{A}}$ is contained in $\hat{\mathbf{A}}_1 \otimes \hat{\mathbf{A}}_2$; see below.
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\textbf{Table I}

<table>
<thead>
<tr>
<th>Name</th>
<th>Invariants</th>
<th>Spectrum</th>
<th>Unitary</th>
</tr>
</thead>
<tbody>
<tr>
<td>$D_{0} (Q, E_{0})$</td>
<td>$-\frac{1}{2} &lt; \Re E_{0} \leq \frac{1}{2}$</td>
<td>$L_{12} - E_{0} = 0$</td>
<td>$\Im E_{0} = 0$ $\bar{\phi} = -\frac{1}{2} + i \lambda$</td>
</tr>
<tr>
<td>$D_{s} (Q, E_{0})$</td>
<td>$0, \pm 1, \pm 2, \ldots$</td>
<td>$0, \pm 1, \pm 2, \ldots$</td>
<td>$\Im E_{0} = \Im \bar{\phi} = 0$ $\lambda = \frac{1}{2} + i \lambda$</td>
</tr>
<tr>
<td>$D^{+} (\phi)$</td>
<td>$\phi + E_{0} = 0$ $2\phi \neq 0, 1 + 2, \ldots$</td>
<td>$L_{12} - E_{0} = 0$</td>
<td>$\Im E_{0} = 0$ $\phi &lt; 0$</td>
</tr>
<tr>
<td>$D^{-} (\phi)$</td>
<td>$\bar{\phi} - E_{0} = 0$ $2\bar{\phi} \neq 0, 1 + 2, \ldots$</td>
<td>$L_{12} - E_{0} = 0$</td>
<td>$\Im E_{0} = 0$ $\bar{\phi} &lt; 0$</td>
</tr>
<tr>
<td>$D (\phi)$</td>
<td>$E_{0} = 0$ $2\phi = 0, 1, 2, \ldots$</td>
<td>$L_{12} = \frac{1}{2} - \phi, -\phi + 1, \ldots$</td>
<td>$\phi = 0$</td>
</tr>
</tbody>
</table>
Fig. 1