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4 February 1965

*To be published in the proceedings of the Royal Society

A b s t r a c t

The relativistic structure of the 143, 364, 572, 4212, and 5940 $\tilde{U}(12)$ multiplets is exhibited in the conventional wave formulation of quantum fields. Expressions for free Lagrangians, free propagators and interaction Lagrangians are given for S-matrix calculations. These last multiplets (4212, 5940) include spin 2 particles and experimental evidence for them has been recently accumulating.

1. INTRODUCTION

A covariant theory of strong-interaction symmetries incorporating a $\widetilde{U}(12)$ structure was developed in Part I of this paper (Salam, Delbourgo, and Strathdee, 1965). The chief feature of the formalism was to exhibit essentially the <u>free-particle</u> structure of the (intrinsically broken) $\widetilde{U}(12)$ irreducible multiplets corresponding to ($\underline{143}$) mesons and ($\underline{364}$) baryons.

Now the <u>free-particle</u> structure of the multiplets is all that is required if one wishes to generate the simplest S-matrix elements (vertex functions). We were thus able to give expressions for the meson-baryon and meson-meson form factors. For more sophisticated computations, however, it is desirable to develop a $\widetilde{U}(12)$ invariant field theory of the particles we are dealing with and this will be attempted in the present paper. We wish to stress that there is no assumption here that the particles we are concerned with (the baryons N and the meson M) correspond to any "fundamental fields".

The contents of the paper are the following: In Section 2 we write the free-particle structure of the $\underline{143}$, $\underline{364}$, and $\underline{572}$ multiplets, and the free Lagrangians from which the corresponding Bargmann-Wigner (1948) equations can be derived. We also give in Section 2 the free propagators. In Section 3 we write the interaction Lagrangians for $\bar{N}NM$ and MMM coupling. These were essentially given in Paper I, except that we now also include interactions of the meson singlet. This formal apparatus should be sufficient for most higher order calculations. Section 4 gives the detailed relativistic decomposition of the $\underline{4212}$ and $\underline{5940}$ meson multiplets.

2. THE STRUCTURE OF FREE-PARTICLE MULTIPLETS

A. The multi-spinor formalism of Paper I is essential to exhibit the $\widetilde{U}(12)$ symmetry. Also this is the simplest formalism for writing invariant interaction Lagrangians. It is, however, somewhat unfamiliar and we shall attempt in this paper to write all our expressions in the conventional wave formulation of fields.

B. Three of the lowest $\widetilde{U}(12)$ multiplets which find application are the $\underline{143}$, $\underline{364}$, and $\underline{572}$. (These correspond to the $\underline{36}$, $\underline{56}$, and $\underline{70}$ of $U(6)$ theory.) In terms of $\widetilde{U}(4) \times SU(3)$ their contents are as follows:

$$\underline{143} = \Big(\underbrace{\phi}_{8} \;;\; \underbrace{\phi_5 \,,\, \phi_{\mu 5}}_{1+8} \;;\; \underbrace{\phi_\mu \,,\, \phi_{[\mu\nu]}}_{1+8} \Big) \qquad (2.1)$$

$$\underline{364} = \Big(\underbrace{V}_{1} \;;\; \underbrace{\psi \,,\, \psi_\mu}_{8} \;;\; \underbrace{\psi_\mu \,,\, \psi_{[\mu\nu]}}_{10} \Big) \qquad (2.2)$$

$$\underline{572} = \Big(\underbrace{V}_{8} \;;\; \underbrace{\psi \,,\, \psi_\mu}_{1+8} \;;\; \underbrace{\psi_\mu \,,\, \psi_{[\mu\nu]}}_{8} \;;\; \underbrace{\psi \,,\, \psi_\mu}_{10} \Big) \qquad (2.3)$$

where the $SU(3)$ dimensionality is shown underneath each set of quantities that comprise a $\widetilde{U}(4)$ multiplet. For homogeneous Lorentz transformations (which are contained in $\widetilde{U}(4)$) ϕ is scalar; ϕ_μ = 4-vector, $\phi_{[\mu\nu]}$ = antisym. tensor, ψ = Dirac-spinor, ψ_μ = Rarita-Schwinger spinor... When equations of motion are applied, the $\widetilde{U}(4)$ symmetry is broken and one finds a relation among pairs [1] $(\phi_5, \phi_{\mu 5})$, (ψ, ψ_μ), $(\phi_\mu, \phi_{[\mu\nu]})$, and $(\psi_\mu, \psi_{[\mu\nu]})$. In detail, as a consequence of Bargmann-Wigner equations[2] one may write the multiplets in the form:[3]

$$\underline{143} = \left(\begin{array}{c} 0 \\ \hline i m \phi_5 \\ p_\mu \phi_5 \\ \hline i m \phi_\mu \\ p_\mu \phi_\nu - p_\nu \phi_\mu \end{array} \right) \begin{array}{l} \text{(trivial)} \\ \left.\begin{array}{c} \\ \\ \end{array}\right\} 0^- \\ \left.\begin{array}{c} \\ \\ \end{array}\right\} 1^- \end{array}, \quad \underline{364} = \left(\begin{array}{c} 0 \\ \hline i m \psi \\ p_\mu \psi \\ \hline i m \psi_\mu \\ p_\mu \psi_\nu - p_\nu \psi_\mu \end{array} \right) \begin{array}{l} \text{(trivial)} \\ \left.\begin{array}{c} \\ \\ \end{array}\right\} \tfrac{1}{2}^+ \\ \left.\begin{array}{c} \\ \\ \end{array}\right\} \tfrac{3}{2}^+ \end{array} \quad \underline{572} = \left(\begin{array}{c} 0 \\ \hline i m \psi \\ p_\mu \psi \\ \hline i m \psi_\mu \\ p_\mu \psi_\nu - p_\nu \psi_\mu \\ \hline i m \psi \\ p_\mu \psi \end{array} \right) \begin{array}{l} \text{(trivial)} \\ \left.\begin{array}{c} \\ \\ \end{array}\right\} \tfrac{1}{2}^+ \\ \left.\begin{array}{c} \\ \\ \end{array}\right\} \tfrac{3}{2}^+ \\ \left.\begin{array}{c} \\ \\ \end{array}\right\} \tfrac{1}{2}^+ \end{array}$$

(2.4)

Note the uniform structure of 4-gradients and 4-curls which occurs for all multiplets considered. The presence of the "derivative" components $p_\mu \phi$, $p_\mu \psi$, etc. is necessitated by the requirement of $\tilde{U}(12)$ invariance[4].

C. The free meson Lagrangian is the following:

$$\mathcal{L}(\underline{143}) = \tfrac{1}{2} m \left[(\partial_\mu \phi^i_{\mu 5}) \phi^i_5 - \phi^i_{\mu 5} (\partial_\mu \phi^i_5) \right] + \tfrac{1}{2} m^2 \left(\phi^{i\,2}_5 + \phi^{i\,2}_{\mu 5} - \phi^{i\,2} \right)$$

$$+ \tfrac{1}{2} m \phi^i_{\mu\nu} (\partial_\mu \phi^i_\nu - \partial_\nu \phi^i_\mu) - \tfrac{1}{4} m^2 \phi^i_{\mu\nu} \phi^i_{\mu\nu} - \tfrac{1}{2} m^2 \phi^i_\mu \phi^i_\mu \tag{2.5}$$

The propagators are most easily obtained by the functional differentiation method of Glashow (1958) where a source term of the type $\frac{1}{2} m^2 j^A_B \bar{\Phi}^B_A$ is introduced to compute the appropriate Green's functions. Leaving out the unitary spin factor δ_{ij}, these read:

$$(\phi_5, \phi_5)_+ = \frac{1}{p^2 - m^2}$$

$$(\phi_{\mu 5}, \phi_{\nu 5})_+ = \frac{p_\mu p_\nu}{m^2 (p^2 - m^2)} - \frac{g_{\mu\nu}}{m^2}$$

$$(\phi_{\mu 5}, \phi_5)_+ = \frac{-i p_\mu}{m (p^2 - m^2)}$$

$$(\phi_\mu, \phi_\nu)_+ = \frac{-g_{\mu\nu} + p_\mu p_\nu / m^2}{p^2 - m^2}$$

$$(\phi_{\lambda\mu}, \phi_\nu)_+ = \frac{i (p_\lambda g_{\mu\nu} - p_\mu g_{\lambda\nu})}{m (p^2 - m^2)}$$

$$(\phi_{\kappa\lambda}, \phi_{\mu\nu})_+ = \frac{p_\kappa p_\mu g_{\lambda\nu} - p_\kappa p_\nu g_{\lambda\mu} + p_\lambda p_\nu g_{\kappa\mu} - p_\lambda p_\mu g_{\kappa\nu}}{m^2 (p^2 - m^2)} -$$

$$- \frac{1}{m^2} (g_{\kappa\mu} g_{\lambda\nu} - g_{\kappa\nu} g_{\lambda\mu}) \tag{2.6}$$

D. For the $\underline{364}$ multiplet, let N, N_μ stand for the spin $\frac{1}{2}$ particles and D_μ , $D_{\mu\nu}$ for the spin 3/2 particles. Then neglecting unitary spin,

$$\begin{aligned}\mathcal{L}\,(\underline{364}) = \;& \bar{N}\,(i\gamma\partial - m)\,N + \bar{N}_\mu \partial_\mu N - \bar{N}\,\partial_\mu N_\mu - m\bar{V}V \\ & - m\bar{N}_\mu N_\mu - (\partial_\mu \bar{N})(\partial_\mu N)/m + \bar{D}_\mu\,(i\gamma\partial - m)\,D_\mu \\ & - \frac{i}{3}\bar{D}_\mu\,(\gamma_\mu\partial_\nu + \partial_\mu\gamma_\nu)\,D_\nu + \frac{1}{3}\bar{D}_\mu\gamma_\mu\,(i\gamma\partial + m)\,\gamma_\nu D_\nu \\ & + \bar{D}_\mu\,(\partial^2 g_{\mu\nu} - \partial_\mu\partial_\nu)\,D_\nu/m - \frac{1}{2}m\,\bar{D}_{\mu\nu}D_{\mu\nu} \\ & + \frac{1}{2}\bar{D}_{\mu\nu}\,(\partial_\mu D_\nu - \partial_\nu D_\mu) + \frac{1}{2}(\bar{D}_\mu\partial_\nu - \bar{D}_\nu\partial_\mu)\,D_{\mu\nu}\end{aligned} \tag{2.7}$$

This Lagrangian provides propagators in close similarity with those of the mesons:

$$\begin{aligned}(N,\bar{N})_+ &= \frac{1}{\gamma p - m} \\ (N_\mu,\bar{N}_\nu)_+ &= \frac{p_\mu p_\nu}{m^2\,(\gamma p - m)} - \frac{g_{\mu\nu}}{m} \\ (N_\mu,\bar{N})_+ &= \frac{-i p_\mu}{m\,(\gamma p - m)} \\ (D_\mu,\bar{D}_\nu)_+ &= \frac{\nabla_{\mu\nu}(p)}{p^2 - m^2} \\ (D_{\lambda\mu},\bar{D}_\nu)_+ &= \frac{i\,(p_\mu\nabla_{\lambda\nu} - p_\lambda\nabla_{\mu\nu})}{m\,(p^2 - m^2)} \\ (D_{\kappa\lambda},\bar{D}_{\mu\nu})_+ &= \frac{p_\mu p_\lambda\nabla_{\nu\kappa} - p_\nu p_\lambda\nabla_{\mu\kappa} + p_\nu p_\kappa\nabla_{\mu\lambda} - p_\mu p_\kappa\nabla_{\nu\lambda}}{m^2\,(p^2 - m^2)} \\ &\quad - \frac{1}{m}\,(g_{\kappa\mu}g_{\lambda\nu} - g_{\kappa\nu}g_{\lambda\mu})\end{aligned} \tag{2.8}$$

where

$$\nabla_{\mu\nu}(p) = (\gamma p + m)\left[-g_{\mu\nu} + \frac{1}{3}\gamma_\mu\gamma_\nu + \frac{1}{3m}(\gamma_\mu p_\nu - \gamma_\nu p_\mu) + \frac{2}{3m^2}p_\mu p_\nu\right]$$

$$- \frac{1}{3m^2}(p^2 - m^2)\left[\gamma_\mu p_\nu - \gamma_\nu p_\mu + (\gamma p - m)\gamma_\mu\gamma_\nu\right] \tag{2.9}$$

Note the (uniform) appearance of "contact" terms involving $g_{\mu\nu}$ and $g_{\kappa\mu} g_{\lambda\nu}$, etc. in the propagators. These extra terms are characteristic of all linearized field theories. For gauge theories of mesons it is easy to understand their significance; these terms take care of the so-called C-parts (two-meson, two-photon contact interactions). For spin $\frac{1}{2}$ and 3/2 particles their appearance (for the first time in the present theory) presents a completely new feature.

3. INTERACTION LAGRANGIANS

A. The three-point interactions of the <u>143</u> and <u>364</u> multiplets are unique in $\tilde{U}(12)$. Thus

$$\mathcal{L}(\phi\phi\phi) = g f^{ijk}\begin{bmatrix} 3\phi^i_\mu \phi^j_\nu \phi^k_{\mu\nu} + 3\phi^i_{\mu 5}\phi^j_{\mu\nu}\phi^k_{\nu 5} + \\ + 6\phi^i_5\phi^j_\mu\phi^k_{\mu 5} + \phi^i_{\mu\nu}\phi^j_{\nu\lambda}\phi^k_{\lambda\mu} \end{bmatrix} +$$

$$+ g d^{ijk}\begin{bmatrix} \phi^i\phi^j\phi^k + 3\phi^i\phi^j_\mu\phi^k_\mu - 3\phi^i\phi^j_5\phi^k_5 - \\ - 3\phi^i\phi^j_{\mu 5}\phi^k_{\mu 5} + \frac{3}{2}\phi^i\phi^j_{\mu\nu}\phi^k_{\mu\nu} + \\ + \frac{3}{2}\epsilon_{\kappa\lambda\mu\nu}\left(\phi^i_\kappa\phi^j_{\lambda\mu}\phi^k_{\nu 5} + \frac{1}{2}\phi^i_{\kappa\lambda}\phi^j_{\mu\nu}\phi^k_5\right) \end{bmatrix} \tag{3.1}$$

In conjunction with the free Lagrangian written earlier this gives the typical equations of motion,

$$(\partial^2 + m^2)\,\phi_5^i = 6g\left[f^{ijk}\phi_\mu^j\phi_{\mu 5}^k - d^{ijk}\phi^j\phi_5^k + \tfrac{1}{8}\epsilon_{\kappa\lambda\mu\nu}d^{ijk}\phi_{\kappa\lambda}^j\phi_{\mu\nu}^k\right] \quad (3.2)$$

$$m^2\phi^i = 3g\,d^{ijk}\left[\phi^j\phi^k + \phi_\mu^j\phi_\mu^k + \tfrac{1}{2}\phi_{\mu\nu}^j\phi_{\mu\nu}^k - \phi_{\mu 5}^j\phi_{\mu 5}^k - \phi_5^j\phi_5^k\right] \quad (3.3)$$

Note that the trivial component ϕ^i is no longer zero and the static Eq. (3.3) provides a definition of it in terms of the other fields[5].

B. Turning to the interaction of the mesons with the nucleon octet, we construct the (multispinor) invariant:

$$\mathcal{L}(\bar{D}D\phi) = \tfrac{3}{2}\bar{D}^{pqr}_{\lambda}(T^i)_p^{p'}\left[\phi^i + \gamma_5\phi_5^i + \gamma_\mu\phi_\mu^i + i\gamma_\mu\gamma_5\phi_{\mu 5}^i + \tfrac{1}{2}\sigma_{\mu\nu}\phi_{\mu\nu}^i\right]D_{\lambda,p'qr}$$
$$+ \tfrac{3}{4}\bar{D}^{pqr}_{\kappa\lambda}(T^i)_p^{p'}\left[\phi^i + \gamma_5\phi_5^i + \gamma_\mu\phi_\mu^i + i\gamma_\mu\gamma_5\phi_{\mu 5}^i + \tfrac{1}{2}\sigma_{\mu\nu}\phi_{\mu\nu}^i\right]D_{\kappa\lambda,p'qr}$$

$$\mathcal{L}(\bar{D}N\phi) = -\tfrac{i}{2}\bar{D}^{pqr}_{\lambda}(T^i)_p^{p'}\left[\phi^i\gamma_5 - \phi_5^i + \gamma_\mu\gamma_5\phi_\mu^i - i\gamma_\mu\phi_{\mu 5}^i + \tfrac{1}{2}\sigma_{\mu\nu}\gamma_5\phi_{\mu\nu}^i\right]\epsilon_{p'qs}N_r^s$$
$$+ \tfrac{1}{4}\bar{D}^{pqr}_{\kappa\lambda}(T^i)_p^{p'}\begin{bmatrix}\phi^i\gamma_5 - \phi_5^i + \gamma_\mu\gamma_5\phi_\mu^i - i\gamma_\mu\phi_{\mu 5}^i + \\ + \tfrac{1}{2}\sigma_{\mu\nu}\gamma_5\phi_{\mu\nu}^i\end{bmatrix}\sigma_{\kappa\lambda}\epsilon_{p'qs}N_r^s$$
$$+ \tfrac{1}{4}\epsilon_{\mu\nu\kappa\lambda}\bar{D}^{pqr}_{\mu\nu}(T^i)_p^{p'}\epsilon_{p'qs}\begin{bmatrix}\phi^i + \gamma_5\phi_5^i + \gamma_\rho\phi_\rho^i \\ + i\gamma_\rho\gamma_5\phi_{\rho 5}^i + \tfrac{1}{2}\sigma_{\rho\tau}\phi_{\rho\tau}^i\end{bmatrix}\gamma_\kappa N_{\lambda r}^s$$

$$\mathcal{L}(\bar{N}N\phi) = -\tfrac{1}{24}\left(\bar{N}^{[\beta\alpha]\gamma}\phi_\alpha^{\alpha'}(\underline{143})\,N_{[\alpha'\beta]\gamma}\right)_{12S+3D+5F}$$
$$+ \tfrac{1}{12}\left(\bar{N}^{[\beta\alpha]\gamma}\phi_\alpha^{\alpha'}(\underline{143})\,N_{[\alpha'\gamma]\beta}\right)_{3S+3D+2F} \quad (3.4)$$

where

$$
\begin{aligned}
(\bar{N}\phi N)_S &= \mathrm{Tr}\,(\bar{N}T^0 N)\,\phi^0 \\
(\bar{N}\phi N)_D &= \mathrm{Tr}\,(\bar{N}\{T^i, N\})\,\phi^i \qquad ; \ i = 1, \dots 8. \\
(\bar{N}\phi N)_F &= \mathrm{Tr}\,(\bar{N}[T^i, N])\,\phi^i \qquad ; \ i = 1, \dots 8.
\end{aligned}
\tag{3.5}
$$

Expanding out in terms of N and N_μ,

$$
\begin{aligned}
\tfrac{1}{4}\bar{N}^{[\beta\alpha]\gamma}\phi_\alpha^{\alpha'}N_{[\alpha'\beta]\gamma} = {} & 2\bar{N}\,(\phi + \phi_5\gamma_5)N + \\
& + i\bar{N}_\mu(\phi\gamma_\mu - \phi_\mu - i\phi_{\mu5}\gamma_5 + \gamma_\mu\gamma_5\phi_5)N + \\
& + i\bar{N}\,(-\phi\gamma_\mu + \phi_\mu - i\phi_{\mu5}\gamma_5 - \gamma_5\gamma_\mu\phi_5)N_\mu + \\
& + \bar{N}_\mu\begin{bmatrix}(g_{\mu\nu} + \gamma_\mu\gamma_\nu)\phi + i\phi_{\mu\nu} + \\ + i\phi_{\mu5}\gamma_\nu\gamma_5 + i\gamma_\mu\gamma_5\phi_{\nu5}\end{bmatrix}N_\nu
\end{aligned}
\tag{3.6}
$$

$$
\begin{aligned}
\tfrac{1}{4}\bar{N}^{[\beta\alpha]\gamma}\phi_\alpha^{\alpha'}N_{[\alpha'\beta]\gamma} = {} & \bar{N}\,(\phi + \tfrac{1}{2}\sigma_{\mu\nu}\phi_{\mu\nu} + \phi_5\gamma_5)N \\
& + \tfrac{i}{2}\bar{N}_\mu\left(\gamma_\mu\phi - \phi_\nu\gamma_\nu\gamma_\mu - \tfrac{1}{2}\phi_{\nu\lambda}\gamma_\mu\sigma_{\nu\lambda} + i\phi_{\nu5}\gamma_\nu\gamma_5\gamma_\mu + \gamma_\mu\gamma_5\phi_5\right)N \\
& - \tfrac{i}{2}\bar{N}\left(\gamma_\mu\phi - \gamma_\mu\gamma_\nu\phi_\nu - \tfrac{1}{2}\phi_{\nu\lambda}\sigma_{\nu\lambda}\gamma_\mu + i\phi_{\nu5}\gamma_\mu\gamma_5\gamma_\nu + \gamma_5\gamma_\mu\phi_5\right)N_\mu \\
& + \tfrac{1}{2}\bar{N}_\mu\begin{bmatrix}(g_{\mu\nu} + \gamma_\mu\gamma_\nu)\phi + \phi_\lambda(\gamma_\mu\gamma_\lambda\gamma_\nu - g_{\mu\nu}\gamma_\lambda) + \\ + \phi_5(2\gamma_\mu\gamma_5\gamma_\nu + g_{\mu\nu}\gamma_5) + \\ + \tfrac{1}{2}\phi_{\kappa\lambda}(\gamma_\mu\sigma_{\kappa\lambda}\gamma_\nu + g_{\mu\nu}\sigma_{\kappa\lambda} + 4ig_{\mu\lambda}g_{\kappa\nu}) + \\ + \phi_{\kappa5}(i\gamma_\mu\gamma_5 g_{\kappa\nu} + i\gamma_\nu\gamma_5 g_{\mu\kappa} - i\gamma_\kappa\gamma_5 g_{\mu\nu})\end{bmatrix}N_\nu
\end{aligned}
\tag{3.7}
$$

With the propagators (2.6) and (2.8) and the interaction Lagrangians (3.1), (3.6), and (3.7) the formal apparatus for writing Dyson S-matrix elements for meson-baryon interactions is complete. The lowest S-matrix expression for the form factors (vertex function on the mass shell) for example is obtained in the conventional manner by substituting for N_{μ}, $\phi_{\mu 5}$, and $\phi_{\mu\nu}$, $D_{\mu\nu}$ the free equations of motion,

$$i m N_{\lambda} = p_{\lambda} N \quad , \quad i\mu \phi_{\lambda 5} = p_{\lambda} \phi_{5} \quad ,$$

$$i\mu \phi_{\kappa\lambda} = p_{\kappa}\phi_{\lambda} - p_{\lambda}\phi_{\kappa} \, , \quad i m D_{\kappa\lambda} = p_{\kappa} D_{\lambda} - p_{\lambda} D_{\kappa}$$

This gave the expressions (5.15) and (5.16) of Paper I.

It is interesting to note that whereas the usual gauge generation of electro-magnetic interactions by changing $\gamma p \rightarrow \gamma (p - eA)$ makes the Bargmann-Wigner set inconsistent, there apparently is no contradiction if this is done for the Lagrangians (2.7), (3.1), etc. This somewhat strange circumstance calls for further investigation.

4. HIGHER MESON REPRESENTATIONS IN $\tilde{U}(12)$

To accomodate spin 2 one must proceed to the higher representations $\underline{4212}$ and $\underline{5940}$ ($\underline{189}$ and $\underline{405}$ in SU(6)). Their decomposition in terms of SU(3) x $\tilde{U}(4)$ is

$$\begin{aligned}\underline{4212} = \; & (8, 84) + (1, 84) + (10, 45^*) + (10^*, 45) + (8, 45) + (8, 45^*) \\ & + (27, 20) + (8, 20) + (1, 20) + (27, 15) + (10, 15) + (10^*, 15) \\ & + 3\,(8, 15) + (1, 15) + (27, 1) + (8, 1) + (1, 1)\end{aligned}$$

$$\begin{aligned}\underline{5940} = \; & (27, 84) + (8, 84) + (1, 84) + (10^*, 45^*) + (10, 45) \\ & + (8, 45^*) + (8, 45) + (8, 20) + (1, 20) + (27, 15) + (10, 15) \\ & + (10^*, 15) + 3(8, 15) + (1, 15) + (27, 1) + (8, 1) + (1, 1)\end{aligned}$$

The detailed content is

$$\underline{4212} = (\underbrace{\phi, A, S}_{1+8} ; \underbrace{T, U}_{8+8+10+10^*} ; \underbrace{\eta, Y, X}_{1+8+27}) \tag{4.1}$$

$$\underline{5940} = (\underbrace{\phi, A, S}_{1+8+27} ; \underbrace{T, U}_{8+8+10+10^*} ; \eta, \underbrace{Y, X}_{1+8}) \tag{4.2}$$

where

$$(\phi, A, S) = (\phi ; A_{\lambda\mu}, A_{\lambda[\mu\nu]}, A_{[\kappa\lambda]\mu}, A_{[\kappa\lambda][\mu\nu]} ; S_{\lambda\mu}, S_{\lambda[\mu\nu]}, S_{[\kappa\lambda]\mu}, S_{[\kappa\lambda][\mu\nu]})$$

$$(T, U) = (T_\mu, T_{5\mu}, T_{\lambda 5\mu}, T_{[\mu\nu]}, T_{5[\mu\nu]}, T_{\lambda 5[\mu\nu]} ; U_5, U_{\mu 5}, U_\mu, U_{[\mu\nu]})$$

$$(\eta, Y, X) = (\eta ; Y_5, Y_\mu, Y_{\mu 5}, Y_{[\mu\nu]} ; X, X_5, X_{55}, X_{\mu 5}, X_{5\mu 5}, X_{\mu 5\nu 5}) \tag{4.3}$$

The correct $\tilde{U}(4)$ multiplet structure is assured by the constraints

$$A_{\lambda\mu} = -A_{\mu\lambda} , \quad A_{[\kappa\lambda][\mu\nu]} = A_{[\mu\nu][\kappa\lambda]}$$

$$A_{[\mu\lambda][\nu\lambda]} = -A_{[\nu\lambda][\mu\lambda]} , \quad A_{\lambda[\mu\nu]} = A_{[\nu\lambda]\mu}$$

$$S_{\lambda\mu} = S_{\mu\lambda} , \quad S_{\mu\mu} = 0 , \quad S_{[\kappa\lambda][\mu\nu]} = -S_{[\mu\nu][\kappa\lambda]}$$

$$S_{[\mu\lambda][\nu\lambda]} = S_{[\nu\lambda][\mu\lambda]} , \quad S_{\lambda[\mu\nu]} = -S_{[\mu\nu]\lambda}$$

$$T_{\mu 5\mu} = 0 , \quad T_{\mu 5\nu} - \frac{1}{2} T_{5[\mu\nu]} + \frac{1}{4} \epsilon_{\mu\nu\kappa\lambda} T_{[\kappa\lambda]} = 0$$

$$T_\mu + \frac{1}{2} \epsilon_{\mu\nu\kappa\lambda} T_{\nu 5[\kappa\lambda]} = 0 , \quad T_{\lambda 5[\lambda\mu]} + T_{5\mu} = 0$$

$$X = X_{55} + X_{\mu 5\mu 5} , \quad X_{\mu 5\nu 5} = X_{\nu 5\mu 5} \tag{4.4}$$

Finally the following 144 relations guarantee that the $\tilde{U}(12)$ representation is irreducible,

$$\phi + \eta = 0 \ \binom{\text{singlet}}{\&\ \text{octet}}, \quad Z + 2U + Y = 0 \text{ (octet)}, \quad Y + Z = 0 \text{ (singlet)}$$

where

$$\begin{aligned}
Z_5 &= -\tfrac{1}{2}\, \epsilon_{\kappa\lambda\mu\nu}\, A_{[\kappa\lambda][\mu\nu]} \\
Z_\mu &= 2i \left[A_{\lambda[\lambda\mu]} - A_{[\mu\lambda]\lambda} \right] \\
Z_{\mu 5} &= \epsilon_{\mu\nu\kappa\lambda} \left(A_{\kappa[\lambda\nu]} + A_{[\kappa\lambda]\nu} \right) \\
Z_{\mu\nu} &= -2i \left(A_{\mu\nu} + A_{[\mu\kappa][\nu\kappa]} \right)
\end{aligned} \tag{4.5}$$

At this point we apply the Bargmann-Wigner equations and obtain

$$\partial_\mu A_{\mu\nu} = \partial_\mu S_{\mu\nu} = 0$$

$$im\, S_{[\kappa\lambda]\mu} = p_\kappa S_{\lambda\mu} - p_\lambda S_{\kappa\mu}, \quad p_\kappa S_{[\kappa\lambda]\mu} = -im\, S_{\lambda\mu}$$

$$im\, S_{\lambda[\mu\nu]} = p_\mu S_{\lambda\nu} - p_\nu S_{\lambda\mu}, \quad p_\mu S_{\lambda[\mu\nu]} = -im\, S_{\lambda\nu}$$

$$-m^2 S_{[\kappa\lambda][\mu\nu]} = p_\mu p_\kappa S_{\lambda\nu} - p_\mu p_\lambda S_{\kappa\nu} + p_\nu p_\lambda S_{\kappa\mu} - p_\nu p_\kappa S_{\lambda\mu}$$

with similar equations for $A_{[\kappa\lambda]\mu}$, $A_{\lambda[\mu\nu]}$ and $A_{[\kappa\lambda][\mu\nu]}$.

$$p_\mu U_5 = im\, U_{\mu 5}, \quad p_\mu U_{\mu 5} = -im\, U_5$$

$$im\, U_{[\mu\nu]} = p_\mu U_\nu - p_\nu U_\mu, \quad p_\mu U_{[\mu\nu]} = -im\, U_\nu$$

$$T = X = Y = 0.$$

This gives the complete relativistic $\tilde{U}(12)$ structure of[6] $\underline{4212}$ and $\underline{5490}$. Notice that the description of spin 2 exactly parallels the Fierz treatment of integer-spin particles except that we are forced by $\tilde{U}(12)$ invariance to include all Fierz (1938) alternative formulations of spin 2 theories within one multiplet. Observe a new feature of the formalism; one of the spin 1 structures appearing in $\underline{5940}$ (and $\underline{4212}$) contains 15 components instead of the 10 which describe the spin 1 particle in the $\underline{143}$. This "abnormal" behaviour should not be surprising since the multiplets concerned correspond to different values of the $\tilde{U}(4)$ Casimir operators. It, however, does introduce a new (perhaps weakly respected) selection rule in the theory.

FOOTNOTES

1. The unpaired quantities ϕ and V (the trivial components) equal zero as a consequence of the equations.

2. In the multispinor notation used in Paper I these states are expressed by

$$\bar{\Phi}_A^B(\underline{143}) = \bar{\Phi}_{\alpha r}^{\beta s} = \left[(\gamma p + m)(\gamma_5 \phi_5^i + \gamma_\mu \phi_\mu^i)\right]_\alpha^\beta (\tau^i)_r^s$$

$$\tilde{\bar{\Psi}}_{ABC}(\underline{364}) = \bar{\Psi}_{\{\alpha p \beta q \gamma r\}} = \sqrt{\tfrac{3}{2}}\left[(\gamma p + m)\gamma_\mu C\right]_{\alpha\beta} \psi_{\gamma\mu, pqr} + \frac{1}{\sqrt{6}}\left\{\left[(\gamma p + m)\gamma_5 C\right]_{\alpha\beta} \epsilon_{pqs} \psi_{\gamma, r}^{s} + \text{cyc.}\right\}$$

$$\tilde{\bar{\Psi}}_{ABC}(\underline{572}) = \bar{\Psi}_{[\alpha p \beta q]\gamma r} = \tfrac{1}{2}\left[(\gamma p + m)\gamma_\mu C\right]_{\alpha\beta} \epsilon_{pqs} \psi_{\gamma\mu, r}^{s} + \frac{1}{\sqrt{6}}\left\{\left[(\gamma p + m)\gamma_5 C\right]_{\beta\gamma} \epsilon_{qrs} \psi_{\alpha, p}^{s} + \left[(\gamma p + m)\gamma_5 C\right]_{\alpha\gamma} \epsilon_{rps} \psi_{\beta, q}^{s}\right\}$$
$$+ \tfrac{1}{6}\epsilon_{pqr}\left\{\left[(\gamma p + m)\gamma_5 C\right]_{\alpha\gamma} \psi_\beta + \left[(\gamma p + m)\gamma_5 C\right]_{\beta\gamma} \psi_\alpha\right\} + \frac{1}{\sqrt{2}}\left[(\gamma p + m)\gamma_5 C\right]_{\alpha\beta} \psi_{\gamma, \{pqr\}}$$

Our normalization of, for instance, the 364 multiplet differs from the SU(6) normalization of Sakita (1964) by an overall factor of $\sqrt{3}$ (to give charge e for the proton). An extra factor of $1/\sqrt{2}$ enters in the decomposition of the symmetric rank 3 multispinor into a spin 3/2 Rarita-Schwinger field and again produces the correct charges $, 2e(N^{*++})^\dagger N^{*++}$, etc.

3. The parity assignment of the multiplets is unique in $\tilde{U}(12)$ theory and is the subject of a separate discussion by A. Salam, J. Strathdee, J.M. Charap, and P.T. Matthews. (to be published in Phys. Letters)

4. Whenever we use the phrase "$\tilde{U}(12)$ invariance" we use it in the context of its intrinsic breaking through the inhomogeneous Lorentz Group.

4. (Continued)

It is worth mentioning that (2.5) can be recast in the form

$$\mathcal{L} = -\frac{1}{2}m^2 \bar{\Phi}_A^B \bar{\Phi}_B^A + \frac{1}{2}m\left[\phi_{\mu\nu}^i(\partial_\mu \phi_\nu^i - \partial_\nu \phi_\mu^i) + (\partial_\mu \phi_{\mu 5}^i)\phi_5^i - \phi_{\mu 5}^i(\partial_\mu \phi_5^i)\right]$$

The form explicitly shows that the kinetic energy terms are $\tilde{U}(12)$ non-invariant while the mass terms are.

5. Using (3.3) one may eliminate ϕ^i from the other equations (or effectively from the Lagrangian). As mentioned in Paper I, one can treat the trivial components ϕ^i as spurious. Consistently with C- and P-conservation, if one looks for an unsymmetrical solution, for example with only $\phi^8 \neq 0$, one obtains a natural mechanism for SU(3)-breaking. Thus an approximate symmetry-breaking solution of Eq. (3.3) is $\phi^8 \approx m/g\sqrt{3}$.

6. The multispinor forms for the non-trivial components η, U, ϕ, A and S appear more simply as

$$\Phi^{[\alpha\beta]}_{[\gamma\delta]} = [(\gamma p + m)\gamma_5 C]_{\gamma\delta}\,[C^{-1}\gamma_5(\gamma p - m)]^{\alpha\beta}\,\eta(p)$$

$$\Phi^{\{\alpha\beta\}}_{[\gamma\delta]} = [(\gamma p + m)\gamma_5 C]_{\gamma\delta}\,[C^{-1}\gamma_\mu(\gamma p - m)]^{\alpha\beta}\,U_\mu(p)$$

$$\Phi^{\{\alpha\beta\}}_{\{\gamma\delta\}} = [(\gamma p + m)\gamma_\mu C]_{\gamma\delta}\,[C^{-1}\gamma_\nu(\gamma p - m)]^{\alpha\beta}\,\phi_{\mu\nu}(p)$$

$$(p^2 - m^2)\,\eta = (p^2 - m^2)\,U_\mu = (p^2 - m^2)\,\phi_{\mu\nu} = 0$$

with

$$p_\mu U_\mu(p) = p_\mu \phi_{\mu\nu}(p) = 0$$

and

$$\phi = \phi_{\lambda\lambda}\;;\quad A_{\mu\nu} = \phi_{[\mu\nu]}\;;\quad S_{\mu\nu} = \phi_{\{\mu\nu\}}\;,\quad S_{\mu\mu} = 0\,.$$

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NOTE:

After this paper had been printed the authors received two papers from Drs. M. A. B. Bèg and A. Pais and Drs. B. Sakita and K. C. Wali, who have used a similar formalism. We are grateful to them for sending their preprints.