SOME UNITARY REPRESENTATIONS
OF A NON-COMPACT FORM OF SU₄

CHRISTIAN FRONSDAL

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PIAZZA OBERDAN
TRIESTE
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Christian Fronsdal

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It has been suggested that certain non-compact groups, among them $SU_{12}$, may be relevant for the theory of elementary particles. In that case it would be of interest to study their unitary representations. As a beginning we study the noncompact subgroup $SU_4$ of $SU_{12}$. We find that among several noncompact real forms of $SU_4$ only that which is isomorphic to the rotation group $R_{4,2}$ is of interest. For this group we determine all the unitary irreducible representations for which the energy operator has a positive definite spectrum. Then we study the relationship between these representations and those of the Poincaré group.
INTRODUCTION

Recently it has appeared that certain non-compact groups may be important for describing the interactions of elementary particles, beyond the usual application of the Poincaré group. Here we study some unitary irreducible representations of one of the smallest of these groups.

The "classical" discussions of symmetry properties of elementary particles dealt with groups of the form $P \otimes SU_3$ where $P$ is the Poincaré group and $SU_3$ was sometimes replaced by other compact groups. A desire to incorporate these groups in a larger one was widespread and many attempts were made. The theory of Gürsey and Radicati was an outgrowth of these attempts, but did not constitute a complete theory because part of $P$ was ignored. In fact, only the compact subgroup $SU_2$ of $P$ was considered.

The group $SU_2 \otimes SU_3$ was enlarged to $SU_6$ by considering the set of all unitary unimodular matrices in the space $H_2 \otimes H_3$, where $H_2$ is a two-dimensional basis for $SU_2$ and $H_3$ is a three-dimensional basis for $SU_3$. Several authors have discussed the application of this method to the full Poincaré group. Thus, take a low dimensional representation of $P$ in some Hilbert space $H(P)$ and consider linear transformations in $H(P) \otimes H_3$.

The essential complication which arises in working with the full group $P$ is that this group is not compact. Therefore the finite-dimensional space $H(P)$ does not admit any unitary representations, and the basis vectors in $H(P) \otimes H_3$ (the "quarks") cannot be interpreted as real physical particles, and the same applies to (ordinary) quark compounds. Let us consider the simplest non-trivial $H(P)$, namely the four-dimensional space $H_4$ that admits a non-unitary representation of the Poincaré group. This leads to $H_4 \otimes H_3 = H_{12}$, and hence to some subgroup $G_{12}$ of the set of $12 \times 12$ matrices. This group $G_{12}$ may contain, an addition to $P$ and $SU_3$, two kinds of matrices: 1) matrices operating on $H_4$ only, and 2) matrices that do not have this property. Here we shall consider only that subgroup $G_4$ of $G_{12}$ that acts exclusively on $H_4$. 

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and for $G_4$ we shall choose the group $\tilde{SU}_4$, which is one of the real non-compact forms of $SU_4$.

First we discuss several ways in which $\mathcal{P}$ may be embedded in $\tilde{SU}_4$, and then we classify those unitary irreducible representations that may be interpreted as particles.

**EMBEDDING OF $\mathcal{P}$ IN $SU_4$**

We exploit the isomorphism between $SU_4$ and $R_6$ (the group of rotations in a real six-dimensional space), and consider the non-compact forms $R_{5,1}$, $R_{4,2}$, $R_{3,3}$ that are obtained by introducing a non-definite metric with 1, 2 or 3 minus-signs. These groups have 15 elements

$$L_{\alpha\beta} = -L_{\beta\alpha}; \quad \alpha, \beta = 1, 2, 3, 0, 5, 6,$$

satisfying the commutation relations

$$[L_{\alpha\beta}, L_{\gamma\nu}] = -\epsilon_{\alpha\beta\gamma\nu} L_{\rho\rho}; \quad \alpha, \beta, \gamma = 1, ..., 6$$

where in $R_{5,1}$, $R_{4,2}$, and $R_{3,3}$ the signs of $g_{55}$ and $g_{66}$ are $- - + +$ and $- +$, respectively.

There are (at least) two essentially different ways of interpreting this group.

**Conformal interpretation:** If $\tilde{SU}_4$ is interpreted as the conformal group in four dimensions, then the generators are identified as follows:

**Lorentz transformations:**

$$L_{\mu\nu}; \quad \mu, \nu = 1, 2, 3, 0$$

**Translations:**

$$p_{\mu} = \sqrt{E} (L_{\mu\nu} - L_{\nu\mu})$$

**Accelerations:**

$$a_{\mu} = \sqrt{E} (L_{\mu\nu} + L_{\nu\mu})$$

**Dilatation:**

$$d = L_{56}$$
In this case we have to take the variety $R_{4,2}$, then the commutation relations are

$$[L_{\mu \nu}, L_{\sigma \lambda}] = -i g_{\mu \lambda} L_{\nu \sigma}, [L_{\mu \nu}, P_\mu] = -i g_{\mu \nu} P_\nu$$  (7)

$$[P_\mu, A_\nu] = -i L_{\mu \nu} - i g_{\mu \nu} d, [L_{\mu \nu}, A_\mu] = -i g_{\mu \nu} A_\nu$$  (8)

$$[d, P_\mu] = i P_\mu, [d, A_\mu] = -i A_\mu, [P_\mu, P_\nu] = [A_\mu, A_\nu] = 0$$  (9)

The first line shows that the Poincaré group $P$ is a subgroup of $R_{4,2}$.

In this interpretation the energy $P_0$ does not have a discrete spectrum, and the eigenvalues always range from $-\infty$ to $+\infty$. This makes a physical interpretation with a positive definite metric impossible, and we shall not discuss this alternative further.

De Sitter interpretation. The subgroup that leaves the sixth co-ordinate invariant is $R_{4,1}$ or $R_{3,2}$. These groups are called the 4+1 De Sitter group and the 3+2 De Sitter group, respectively. From the De Sitter groups it is possible to descend to the Poincaré group by "contraction", though the latter group is not a subgroup of the De Sitter groups. In this interpretation the generators are not identified as in (3) – (6), but instead as follows,

Lorentz transformations: $L_{\mu \nu}; \mu, \nu = 1, 2, 3, 0$  (10)

Translations: $P_\mu = \psi \xi_{\mu \nu}, \psi > 0$  (11)

These operators form the algebra of the De Sitter group and satisfy the commutation relations

$$[L_{\mu \nu}, L_{\sigma \lambda}] = -i g_{\mu \lambda} L_{\nu \sigma}$$  (12)

$$[L_{\mu \nu}, P_\mu] = -i g_{\mu \nu} P_\nu, [P_\mu, P_\nu] = -i \rho L_{\mu \nu} g_{s s}$$  (13)
The other 5 operators $\mathbf{L}_\mu$ and $\mathbf{L}_\nu$ are not given a physical interpretation. The remaining commutation relations are

$$[\mathbf{L}_\mu, \mathbf{L}_\nu] = -i g_{\mu\nu} \mathbf{L}_\rho, \quad [\mathbf{L}_\mu, \mathbf{L}_\nu] = -i \sqrt{g} g^{\mu\nu} \mathbf{P}_\mu$$  \hspace{1cm} (14)

$$[\mathbf{P}_\mu, \mathbf{L}_\nu] = -i \sqrt{g} g^{\nu\sigma} \mathbf{L}_\sigma, \quad [\mathbf{L}_\mu, \mathbf{L}_\nu] = -i g_{\mu\nu} \mathbf{L}_\rho$$  \hspace{1cm} (15)

$$[\mathbf{L}_\mu, \mathbf{L}_\nu] = 0, \quad [\mathbf{P}_\mu, \mathbf{L}_\nu] = i g_{\nu\sigma} \sqrt{g} \mathbf{L}_\sigma$$  \hspace{1cm} (16)

The Poincaré group is obtained as the limit $\rho \to 0$ of the De Sitter subgroup, both in the $R_{4,1}$ and the $R_{3,2}$ cases. Since we do not wish to have $\mathbf{P}_0 > 0$ as $\rho \to 0$, it is necessary, as is seen from (14), that $g_{\mu\nu} L_{\rho\sigma}$ remain finite. This gives rise to several possibilities to which we shall return later.

Among the alternatives that exist for choosing the signs of $g_{55}$ and $g_{46}$ we shall argue that $g_{55} = +1$ and that $g_{46} = -1$. For if $g_{55} = -1$, then the one parameter subgroup generated by $P_0$ is non-compact. We find an energy spectrum that is continuous and runs from $-\infty$ to $+\infty$. For this reason we shall henceforth take $g_{55}$ positive, so that the 10 parameter subalgebra (10), (11) generates the 3+2 De Sitter group. With this choice of $g_{55}$ the positive sign for $g_{46}$ would give an energy spectrum symmetric about zero, which is certainly not interesting. Thus we shall always take

$$g_{55} = g_{46} = +1$$  \hspace{1cm} (17)

$$g_{\mu} = g_{22} = g_{33} = g_{46} = -1$$  \hspace{1cm} (18)

Hence we are dealing with $R_{4,2}$. 

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UNITARY PARTICLE-LIKE REPRESENTATIONS.

We shall call a representation $\mathcal{D}$ particle-like if the energy spectrum is bounded below. Consider the Hilbert space $H_\mathcal{D}$ of all states belonging to the lowest energy eigenvalue $m$:

$$P_0 |\hat{\rho}, \nu> = m |\hat{\rho}, \nu>, \quad |\hat{\rho}, \nu> \in H_\mathcal{D} \quad (19)$$

where the index $\nu$ serves to label the states in $H_\mathcal{D}$.

We define the subgroup $\mathcal{G}_\mathcal{D}$ as that subgroup of $SU_4$ that leaves $H_\mathcal{D}$ invariant, or more precisely, $\mathcal{G}_\mathcal{D}$ is the subgroup that commutes with $P_0$. It is, clearly, the subgroup of rotations in $1, 2, 3, 6$ space. This group is the compact rotation group $R_4$. Any representation $\mathcal{D}$ of $SU_4$ induces a representation $\mathcal{D}_\mathcal{D}$ of $\mathcal{G}_\mathcal{D}$ on $H_\mathcal{D}$. By a simple adaptation of the well-known proof of the statement that the highest weight of an irreducible representation of a compact, simple Lie group is non-degenerate, one can prove that if $\mathcal{D}$ is irreducible then so is $\mathcal{D}_\mathcal{D}$. Finally, also by adapting a similar theorem about compact, simple Lie groups, one easily shows that $\mathcal{D}$ is fixed up to equivalence by $\mathcal{G}_\mathcal{D}$.

Hence we have the

**Theorem 1.** Every particle-like irreducible representation $\mathcal{D}$ of $SU_4$ (precisely: $R_4$) is determined up to equivalence by the lowest value $m$ of $P_0$ and by an irreducible representation $\mathcal{D}_\mathcal{D}$ of the subgroup $\mathcal{G}_\mathcal{D}$ ($\mathcal{G}_\mathcal{D}$ is $R_4$). Thus we may write without ambiguity $\mathcal{D} = \mathcal{D}(\mathcal{D}_\mathcal{D}, m)$.

Let us now discuss the unitarity of the representation $\mathcal{D}(\mathcal{D}_\mathcal{D}, m)$. First it is obvious that $\mathcal{D}_\mathcal{D}$ must be unitary. Next consider the following four subalgebras

$$\mathcal{A} = \{ L_{\lambda A}, L_{\pi A}, L_{\phi A} \}, A = 1, 2, 3, 6 \quad (20)$$

All four are non-compact 2+1 Lorentz groups. Thus for each of them,
the only representations that can occur are sums of irreducible representations of the type $\mathcal{D}(\Phi^*)$ (Bargmann's discrete series), with $
abla = -m\phi^{-1}$ \[m > 0\] These are unitary if and only if

and this is thus a necessary condition of unitarity; but it is not sufficient.

To obtain results stronger than (21) we have to study $\mathcal{D}_\Phi$. The group $G_\Phi$ is the ordinary compact 4-dimensional rotation group. This is isomorphic to a direct product of two compact 3-dimensional rotation groups.

\[G_\Phi = R_4 \cong R_3 \otimes R_3\] though neither factor is the group of rotations in ordinary physical 3-space. Thus

\[\mathcal{D}_\Phi = \mathcal{D}_\Phi (k_1, k_2)\]

where $2k_1$ and $2k_2$ are arbitrary non-negative integers. A typical weight diagram is shown in Fig 1; each dot stands for one (non-degenerate) eigenstate of $L_{12}$ and $L_{36}$.

The subgroups $H_A$ contain operators

\[M_A^\pm = \sqrt{2} (cL_\alpha \pm L_5) \quad A = 1, 2, 3, 6\] that raise or lower the eigenvalue of $L_\sigma$ by one unit. To obtain restrictions stronger than (21) we consider linear combinations that also raise or lower $L_\alpha$. This is true of the raising and lowering operators of the subgroups generated by

\[A_5: \left( \frac{1}{\sqrt{2}} (L_{\alpha_1} - L_2), \frac{1}{\sqrt{2}} (L_{\alpha_1} + L_2), \frac{1}{\sqrt{2}} (L_{\alpha_2} - L_1), \frac{1}{\sqrt{2}} (L_{\alpha_2} + L_1) \right)\]

\[A_6: \left( \frac{1}{\sqrt{2}} (L_{\alpha_1} + L_3), \frac{1}{\sqrt{2}} (L_{\alpha_1} - L_3), \frac{1}{\sqrt{2}} (L_{\alpha_2} + L_3), \frac{1}{\sqrt{2}} (L_{\alpha_2} - L_3) \right)\]

From the unitarity of the representations of these 2+1 Lorentz subgroups we must have positive spectra for $L_\sigma \pm L_\alpha$. 

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The symmetry of the problem tells us that also \( L_{\alpha} + L_{\beta} \) must be positive. In \( D_{\rho}(k_1, k_2) \) \( k_1 \) and \( k_2 \) are the highest eigenvalues of \( L_{\alpha} + L_{\beta} \); hence the highest value of \( L_{12} \) and of \( L_{36} \) is \( k_1 + k_2 \), and we have a new and stronger condition for unitarity:

\[
m > \sqrt{\rho} (k_1 + k_2)
\]

(25)

Naturally, for massive particles this is not a serious restriction when \( \rho \to 0 \). There is good reason to believe that this condition is sufficient. Summarizing, we have

**Theorem 2.** Every particle-like, irreducible representation of \( \tilde{S}U_q \) (i.e. \( R_{4,2} \)) is determined up to equivalence by three numbers \( m, k_1 \), and \( k_2 \). These may be defined as follows: \( m \) is the lowest eigenvalue of \( \rho = \sqrt{\rho} L_{\alpha} \), \( k_1 \) and \( k_2 \) are the highest values of \( \sqrt{\rho} (L_{\alpha} + L_{\beta}) \) that occur in the subspace in which \( \rho = m, 2k_1 \) and \( 2k_2 \) have to be non-negative integers. In addition the inequality \( m > \sqrt{\rho} (k_1 + k_2) \) is a necessary (and probably sufficient) condition for the representation \( D(m, k_1, k_2) \) to be unitary.

**INTERPRETATION.**

Let the label \( \vec{\phi} \) on the Hilbert space \( H_{\rho} \) of states of minimum energy be a set of five real numbers:

\[
\vec{\phi} = (\hat{\phi}_1, \hat{\phi}_2, \hat{\phi}_3, \hat{\phi}_4, \hat{\phi}_5) = (0, 0, 0, m, 0)
\]

(26)

and let us call \( |\vec{\phi}, \mu> \) the rest system states. About these states we know

\[
\rho_o |\vec{\phi}, \mu> = \hat{\rho}_o |\vec{\phi}, \mu>
\]

(27)

\[
L_{AB} |\vec{\phi}, \mu> = (L_{AB}^{(o)})_{\sigma, \mu} |\vec{\phi}, \sigma>
\]

(28)

\[
L_{SA} |\vec{\phi}, \mu> = \lambda L_{OA} |\vec{\phi}, \mu> ; A, B = 1,2,3,6
\]

(29)
The matrices \( (L_{AB})^{(o)} \) form an irreducible unitary finite dimensional 
\( (2k+1) \times (2k+1) \) representation of \( R_4 \); the last equation says that the 
energy-lowering operators \( M_A \) (Eq. (23)) annihilate the states 
\( \hat{\rho}, \kappa > \) of lowest energy.

Let \( \rho_0 = (\rho_1, \rho_2, \rho_3, \rho_4, \rho_5) \) be a set of five real numbers with 
\( \rho_0 > 0 \) and \( \rho_0^2 - \rho_1^2 - \rho_2^2 - \rho_3^2 - \rho_4^2 = m^2 \). Then there exists a finite trans-
formation \( \alpha(\rho) \) generated by a linear combination of \( L_{AB} \), \( A = 1, 2, 3, 6 \), 
such that \( \alpha(\rho) \hat{\rho} = \rho \). Let us suppose that a unitary irreducible 
representation of the kind envisaged here does in fact exist, and let 
\( \mathcal{D}(\alpha(\rho)) \) be the representative of \( \alpha(\rho) \). Then we may define a set 
of states by

\[ |\rho, \kappa > \equiv \mathcal{D}(\alpha(\rho)) |\hat{\rho}, \kappa > \]

[Note: \( |\rho, \kappa > \) is a finite number of states. The result of applying 
the operator \( \mathcal{D}(\alpha(\rho)) \) to each of them is exactly the same number of 
states, and they may therefore be labelled in the same way. This 
defines the meaning of \( \kappa \) in \( |\rho, \kappa > \).] This procedure is precisely the 
same as that which is used to find the representations of the Poincaré 
group, \(^{11}\) the main difference being that a state \( |\rho, \kappa > \) is not an eigen-
state of the translation operators. We can now calculate the operators 
\( L_{AB} \) and \( L_{AB} \), \( A = 1, 2, 3, 6 \) in a straightforward way, and obtain,

on \( |\rho, \kappa > \), \( L_{AB} = L_{AB}^{(o)} - i (\frac{\partial}{\partial \rho} \rho_A - \frac{\partial}{\partial \rho} \rho_B) \)

(30)

\[ L_{AB} = \frac{\partial^B}{\partial \rho^A} L_{AB}^{(o)} - i (\frac{\partial}{\partial \rho} \rho_A - \frac{\partial}{\partial \rho} \rho_B) \]

(31)

Now it turns out that the remaining operators \( L_{AB} \), for \( A = 1, 2, 3, 6 \) 
do not give any new states. For (27) and (29) may be combined to read

on \( |\rho, \kappa > \), \( L_{AB} = \rho^\kappa \hat{\rho} A + \frac{i}{\kappa} \bar{\rho}^B L_{AB} \), \( A = 1, 2, 3, 0, 6 \)

(32)

This equation is covariant with respect to the subgroup 
generated by (30), (31), hence
These representations are unitary in spite of appearances.\(^\text{12}\)

Thus the physical interpretation is as follows:

1) In the rest system there are a finite number of states, labelled by \(k_1 = -k_1, -k_1 + 1, \ldots, k_1\) and \(h = 0, 1, \ldots, k_2\). This corresponds to a single value of the ordinary spin if \(k_1\) or \(k_2\) is zero, otherwise to several spin values.

2) When Lorentz transformations are applied to the rest system states, we obtain states labelled by, in addition to the spin labels, a four-vector \(\rho, \mu = 0, 1, 2, 3\) with \(\rho^2 - \rho^2 - \mu^2 = m^2\). In the limit \(\rho \to 0\), \(\rho^2\) is the eigenvalue of \(\rho^2\), \(\mu = 0, 1, 2, 3\). This set of states, \(|\rho, \mu\rangle\), form a basis for all the operators \(\Lambda_{\rho, \mu}\).

3) When the operators \(\Lambda_{\rho, \mu}\) are applied to the states \(|\rho, \mu\rangle\), then a new continuous quantum number \(\rho^2\) is needed to label the states. This may be interpreted as a change in the mass, since \(\rho^2 - \rho^2 = m^2 + \rho^2\).

We are happy about the simultaneous inclusion of several values of the spin. But the quantum number \(\rho^2\) still needs to be interpreted. Here we shall take rather drastic steps to ensure a physical interpretation. First we contract \(\rho \to 0\) in several different ways. Then we resume the discussion of the interpretation.

CONTRACTIONS.

We study a family of different contractions. As remarked above, there are the following possibilities

\[
\rho_{\mu} = \rho^{\frac{1}{2}} \lambda_{\mu}\nu
\]

\[
\rho_{\nu} = \epsilon^{-1} \rho^{\frac{1}{2}} \lambda_{\nu}
\]

\[
\rho_{\mu}' = \epsilon \rho^{\frac{1}{2}} \lambda_{\mu}
\]
Here $t$ is a positive number that may go to zero or to infinity when $\rho \to 0$, while the operators on the left remain finite. The commutation relations are, first the $t$-independent set

\begin{align}
[\mathbf{L}_\mu, \mathbf{L}_\nu] &= -\imath g_{\mu\nu} \mathbf{L}_\lambda \quad , \quad [\mathbf{L}_\mu, P_\nu] = 0 \tag{37} \\
[\mathbf{L}_\mu, P_\nu] &= -\imath g_{\mu\nu} P_\nu \quad , \quad [\mathbf{L}_\mu, P'_\nu] = -\imath g_{\mu\nu} P'_\nu \tag{38} \\
[\mathbf{P}_\mu, P_\nu] &= -\imath \rho_\mu \mathbf{L}_\mu \quad , \quad [\mathbf{P}_\mu, P'_\nu] = -\imath \rho_\mu P'_\nu \tag{39}
\end{align}

and then the $t$-dependent relations

\begin{align}
[\mathbf{P}_\mu, P_\nu] &= -\imath t^{-2} \rho t P'_\nu \tag{40} \\
[\mathbf{P}_\mu, P'_\nu] &= i \rho t \mathbf{L}_\mu \tag{41} \\
[\mathbf{P}_\mu, P'_\nu] &= i \rho t \mathbf{g}_{\mu\nu} P_\nu \tag{42}
\end{align}

From (41) we see that $t^{2} \rho t$ must remain bounded, i.e. $t^{2} \rho t$ must tend to a finite value or zero.

A. First suppose that $t^{2} \rho t$ tends to 1. Then in addition to the commutators (37), (38) we have

\begin{align}
[\mathbf{P}_\mu, P_\nu] &= [\mathbf{P}_\mu, P'_\nu] = 0 \tag{43} \\
[\mathbf{P}_\mu, P'_\nu] &= i \mathbf{L}_\mu \quad , \quad [\mathbf{P}_\mu, P'_\nu] = i \mathbf{g}_{\mu\nu} P_\nu \tag{44}
\end{align}

and the group becomes the 4+1 dimensional Poincaré group.

B. Next suppose that $t^{2} \rho t$ tends to zero, while $t^{-2} \rho t$ tends to 1. Then

\begin{align}
[\mathbf{P}_\mu, P_\nu] &= -\imath \rho \mathbf{P}'_\nu \quad , \quad [\mathbf{P}_\mu, P'_\nu] = -\imath \rho \mathbf{P}_\nu \tag{45} \\
[\mathbf{P}_\mu, P'_\nu] &= \rho \mathbf{P}_\nu \quad , \quad [\mathbf{P}_\mu, P'_\nu] = \rho \mathbf{P}_\nu \tag{46}
\end{align}
Finally let $\epsilon_x^t \to 0$ to obtain

$$[P_\mu, P_\nu] = 0, \quad [P_\mu', P_\nu] = -i P_\mu$$  \hspace{1cm} (47)
$$[P_\mu', P_\nu'] = [P_\mu, P_\nu'] = [P_\mu, P_\nu] = 0$$  \hspace{1cm} (48)

These three possibilities are not in themselves exhaustive, but every other contraction may be done in several steps, starting with "A" or "C".

Now we insert (35) - (37) into the expressions (30), (31) and (33)

$$L_{ij} = L_{ij}^{(o)} - i\left(\phi_i \frac{\partial}{\partial \phi^a} - \phi_j \frac{\partial}{\partial \phi^b}\right)$$  \hspace{1cm} (49)

$$P'_\epsilon = \epsilon \rho^t \left[ L_{ij}^{(o)} - i\left(\phi_i \frac{\partial}{\partial \phi^a} - \phi_j \frac{\partial}{\partial \phi^b}\right)\right]$$  \hspace{1cm} (50)

$$P'_0 = \epsilon \rho^t \left[ \frac{\rho^t}{\rho_0 + m} L_{ij}^{(o)} - i\left(\phi_i \frac{\partial}{\partial \phi^a} - \phi_j \frac{\partial}{\partial \phi^b}\right)\right]$$  \hspace{1cm} (51)

$$L_{ij} = \frac{\rho^t}{\rho_0 + m} L_{ij}^{(o)} - i\left(\phi_i \frac{\partial}{\partial \phi^a} - \phi_j \frac{\partial}{\partial \phi^b}\right) + \frac{m^6}{\rho_0^6} L_{ij}^{(o)}$$  \hspace{1cm} (52)

$$P'_\mu = \rho'_\mu + \frac{m^6}{\rho_0^6} \rho^t L_{\mu \nu} + \frac{i}{2m} \epsilon^{-1} \rho^t \rho'_\mu$$  \hspace{1cm} (53)

$$P'_\mu = \epsilon^{-1} \rho^{-1} \rho'_\mu + \frac{i}{2m} \epsilon^{-1} \rho^{-2} \rho'_\mu$$  \hspace{1cm} (54)

In all of these formulae $i, j = 1, 2, 3$.

A. Let $\epsilon^2 \rho^t \to 1$. Then the contraction gives an irreducible, unitary representation of the 4+1 Poincaré group. Reducing this to the ordinary Poincaré group we have a sum of irreducible, unitary representations of the latter, involving every mass $m^2 = \rho^2 - m^2 + \rho^2 + \rho^2$, $0 < \rho^2 < \infty$, and a finite number of spin values (the same set of spins for every mass). There is no evidence of a spin-mass correlation.

B. Next let $\epsilon^2 \rho^t \to 0$ but $\epsilon^{-1} \rho^{-t} \to 1$. Then (54) shows that we have to let

$$\chi \equiv \rho'_0 \epsilon^{-1} \rho^{-\frac{1}{2}}, \quad -\infty < \chi < +\infty$$  \hspace{1cm} (55)
remain finite. This gives the usual expressions for $L_i$ and $L_{0}$, and in addition

$$p_{\mu} = \partial_{\mu}, \quad p_{\mu}^{\prime} = m$$

(56)

$$p_{\mu}^{\prime} = -i \gamma_{\mu} \partial_{x}$$

(57)

$$p_{\rho} = x$$

(58)

These operators satisfy (47), rather than (45). Hence from our representations of SU, we can obtain representations of contraction "C", but not contraction "B".

If we let $t \rightarrow 0$ we obtain the same as above. The mass is unique, and equal to $m$. The spin situation is exactly as under "A". The states are labelled by spin, momentum and by the additional continuous quantum number $X$, which appears to have no physical interpretation.

Although the symmetry under SU 4 tends to couple a finite number of particles with different spins $A_{\text{max}}, A_{\text{max}}-1, \ldots, i \rightarrow 0$, no discrete spectrum of masses is found to emerge. This is not really unexpected, since no charge space symmetry group has yet been included. The next task is to include isotopic spin by enlarging $\mathcal{P} \otimes \text{SU}_{2}$ to some subgroup $G_{2}$ of SU 8. The final step is to enlarge $\mathcal{P} \otimes \text{SU}_{3}$ to a subgroup $G_{12}$ of SU 12, as discussed in the introduction.

The group SU 4 also has another interpretation. It is the analogue of SU 12 in a space time of 2 dimensions, with an SU 2 charge space symmetry group included.

Another possible application of noncompact groups should perhaps be mentioned. With the number of resonances ever increasing, perhaps even the charge space symmetry group is non compact.

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1. Some of the papers that have recently appeared on this subject are:
   Rev. Letters 13, 698 (1964), T. Fulton and J. Wess, preprint
   (Vienna), and others quoted below.

   L. Michel, preprint
   and W. D. McGlinn, Phys. Rev. Letters 12, 467 (1947), have pointed
   out some important limitations on what can be achieved.


4. See for example R. Delbourgo, A. Salam and J. Strathdee, ICTP
   preprint (to be published),

5. The most complete work on the conformal group is probably the
   An incomplete list of other recent literature is L. Gross,

6. This interpretation, in connection with SU6, was suggested by
   Aghassi and Roman, ref. 4.

7. This "contraction" procedure has been discussed at length by

8. This feature of the 4+1 De Sitter group was pointed out by
   (Quoted by F. Gürsey, Proceedings of the 1962 Istanbul
   International Summer School of Physics).

9. A somewhat more detailed discussion of the same procedure, as
   applied to the 3+2 De Sitter group was given by the author in
10. V. Bargmann, Ann. of Math. 48, 568 (1947) discusses only one and two valued representations. If $\Phi = -\rho^{-1}m$ is integer or half integer then $\mathcal{D}(\Phi^*)$ is the same as Bargmann's $\mathcal{D}_{\frac{1}{2}}$. Another paper on the representations of the 2+1 Lorentz group will be published shortly by the author.


12. The details of the physical interpretation for nonvanishing $\rho$ are not important here. They will be discussed in a forthcoming paper on the De Sitter group. The states $|\rho, \nu>, |\rho', \nu'>$ are not orthogonal for $\rho \neq \rho'$. Instead we have

$$<\rho, \nu | \rho', \nu'> = \tilde{f}(\rho, \rho', \nu, \nu') (\rho \rho' + m)^{-\sigma}$$

where $\rho \rho' = \rho \rho' - \bar{\rho} \bar{\rho}' = \bar{\rho} \rho'$ and $\sigma = \rho^{-1}m - k, -k$. Instead we have

$$<\rho, \nu | \rho', \nu'> = \tilde{f}(\rho, \rho', \nu, \nu') (\rho \rho' + m)^{-\sigma}$$

where $\rho \rho' = \rho \rho' - \bar{\rho} \bar{\rho}' = \bar{\rho} \rho'$ and $\sigma = \rho^{-1}m - k, -k$. Instead we have

Properly normalized this inner product tends to $\tilde{f}(\rho, \rho') (\rho \rho' + m)^{-\sigma}$ as $\rho \to 0$, provided $\sigma > 0$. Note that the condition $\sigma > 0$ is just the (necessary) condition of unitarity (25).
Figure 1. Above is the three-dimensional weight diagram for $D(m, k_1, k_2)$, with $k_1 = 1$ and $k_2 = 2$. The coordinate that runs into the page is $L_{36}$. The weights form a rectangle for each value of $L_{36}$; and the projection of the lowest rectangle is shown below in an $L_{36}, L_{12}$ projection. The lowest value of $P_6 + \frac{1}{3} D_{05}$ is $m$. 