ON RADIATION PROCESSES IN PLASMAS

T. BIRMINGHAM
J. DAWSON
C. OBERMAN

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T. Birmingham*
J. Dawson*
C. Oberman

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*Princeton University, Plasma Physics Laboratory, Princeton, N.J., U.S.A.

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TRIESTE

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Abstract

The problem of arbitrary test current sources embedded in an infinite spatially homogeneous Vlasov plasma is considered. By computing the rate at which the test sources do work on the plasma, the energy emission into transverse and longitudinal waves is obtained. A variety of radiation problems can be handled by appropriate choice of the current sources. Use of the time varying dipole moment due to encounters between shielded electrons and ions leads to expressions for the bremsstrahlung spectra. If the currents are produced by the interaction of waves with density fluctuations, the scattering and coupling of waves is obtained. By equating the rate of emission of longitudinal or transverse waves to their rate of absorption, we obtain expressions for the steady state energy densities of such waves.
On Radiation Processes in Plasmas

by

T. Birmingham, J. Dawson, and C. Oberman
Plasma Physics Laboratory, Princeton University,
Princeton, New Jersey

I. INTRODUCTION

In recent years considerable attention has been directed toward the problems of the interaction of radiation with and the emission of radiation by plasmas. The earliest work\textsuperscript{1,2} on bremsstrahlung considered only bare binary interactions and neglected the collective aspects (proper shielding of the colliding particles and the modification of the emitted waves due to the dielectric properties of the plasma). Likewise work on the scattering\textsuperscript{3,4,5} of radiation by density fluctuations in plasmas has generally neglected the effects of the plasma dielectric properties on the incident and scattered waves (the longitudinal dielectric properties were used, however, to compute the density fluctuations).

In previous work two of the authors\textsuperscript{6,7} (hereafter denoted by D.O.) obtained the bremsstrahlung emission coefficients from a thermal plasma. This calculation utilized a simple (but valid) model of the plasma to compute the absorption coefficient and then invoked
Kirchhoff's law to obtain the emission. This work took proper account of the longitudinal and transverse dielectric properties and included the coupling of longitudinal and transverse waves and the scattering of longitudinal waves by density fluctuations.

Numerous attempts have been made to treat the plasma radiation problem from a kinetic description of both the particles and the radiation field. Only recently Dupree, utilizing a modification of the Klimontovich formalism, has been successful in this attempt by proceeding to second order in a systematic expansion in the plasma parameter. His results on bremsstrahlung are in agreement with those of D.O.

It is the purpose of this paper to show how a number of radiation problems can be treated utilizing a simple model. The term radiation as used here includes both transverse and longitudinal waves. The radiative processes considered here are: (1) the emission of radiation due to particle encounters (this is a generalization of the usual bremsstrahlung calculations and includes both transverse and longitudinal wave emission), (2) scattering of transverse and longitudinal waves by density fluctuations, (3) the coupling of longitudinal and transverse waves by density fluctuations, and (4) the generation of transverse radiation by the interaction of longitudinal waves with each other.
The calculations are given in the classical limit, i.e. when \( \omega \ll \frac{k}{c} \). This approximation is valid for a wide range of frequencies for hot thermonuclear plasmas. The correct quantum mechanical treatment (even in the presence of a uniform steady magnetic field) including effects due to the uncertainty and exclusion principles can be given a parallel treatment along the lines of Oberman and Ron\(^{13}\) utilizing the reduced Wigner distribution. We do not pursue these quantum mechanical modifications further here.

Our procedure is the following. First we compute the radiation, both transverse and longitudinal, from test sources embedded in a Vlasov plasma. We then assume that the sources are the currents due to accelerated particles, where the acceleration is due either to encounters between particles or has its source in coherent waves propagating in the plasma. The phase relations between the fields produced by different particles must be taken into account here. The acceleration due to encounters is obtained from the fluctuating electric field which a particle sees. The principle of the superposition of dressed particle fields due to Hubbard\(^{14}\) and Rostoker\(^{15,16}\) is used to obtain the fluctuating field.

Our procedure amounts to carrying Hubbard's\(^{14}\) method for obtaining the fluctuating fields in a plasma one step further in the plasma expansion. Hubbard computes these fields by adding up the shielded fields due to all the individual particles moving along their straight line orbits, now taking the shielded particles to be uncorrelated. We add to these fields the shielded fields due to the accelerations of the particles. The radiative processes we consider have their origin in the accelerative corrections to a particle's orbit. For bremsstrahlung the accelerations result from shielded binary interactions and are thus of higher order in the particle interaction expansion. The secondary fields produced by these events are themselves in turn modified (shielded) by the dielectric properties of the plasma.
This work was to a large extent stimulated by the approach of Mercier to the problem of bremsstrahlung. Mercier computed radiation from the fluctuating dipole moment per unit volume as obtained by Hubbard. For plasmas free of static magnetic fields this is the dominant radiative process. We shall see that this is equivalent to summing up the radiation due to individual shielded electron ion encounters.
We consider an infinite homogeneous Vlasov plasma consisting of mobile electrons with an isotropic velocity distribution and an infinitely massive ion background. We consider the perturbation induced by given charge and current densities \( \rho_s(\mathbf{r}, t) \) and \( \mathbf{j}_s(\mathbf{r}, t) \) (i.e., \( \rho_s, \mathbf{j}_s \) are not regarded at this time as belonging to the plasma). We take the fields in the plasma to be given by the linearized Maxwell-Vlasov equations:

\[
\frac{\partial f}{\partial t} + \mathbf{v} \cdot \nabla f - \frac{e \mathbf{E}}{m} \cdot \frac{\partial f}{\partial \mathbf{v}} = -\epsilon f, \tag{2.1}
\]

\[
\nabla \cdot \mathbf{E} = -4\pi e n_0 \int f \, d^3 \mathbf{v} + 4\pi \rho_s, \tag{2.2}
\]

\[
\nabla \cdot \mathbf{B} = 0, \tag{2.3}
\]

\[
\nabla \times \mathbf{E} = -\frac{1}{c} \frac{\partial \mathbf{B}}{\partial t}, \tag{2.4}
\]

\[
\nabla \times \mathbf{B} = \frac{1}{c} \frac{\partial \mathbf{E}}{\partial t} + \frac{4\pi e n_0}{c} \int f \, d^3 \mathbf{v} + \frac{4\pi}{c} \mathbf{j}_s. \tag{2.5}
\]

Here \( n_0 \) and \( f^0 \) are the unperturbed electron density and distribution function with \( f^0 \) normalized to unity. The small damping term \( \epsilon f \) has been introduced in the usual way only for mathematical convenience in deciding the proper paths for contour integration and is ultimately put to zero. If we now Fourier analyze in both space
and time, we obtain

\[ f(k, \omega, v) = \frac{i e}{m} \frac{E(k, \omega)}{\omega - k \cdot v + i \varepsilon} \cdot \frac{\partial}{\partial v} \frac{\varepsilon}{\partial v} \quad , \quad (2.6) \]

\[ k \cdot E(k, \omega) = 4\pi \left[ \int_{0}^{\infty} f(k, \omega, v) \, d^{3}v - \rho_{s}(k, \omega) \right] \quad , \quad (2.7) \]

\[ k \cdot B(k, \omega) = 0 \quad , \quad (2.8) \]

\[ k \times E(k, \omega) = \frac{\omega}{c} B(k, \omega) \quad , \quad (2.9) \]

\[ k \times B(k, \omega) = -\frac{\omega}{c} E(k, \omega) + \frac{4\pi i}{c} \int_{0}^{\infty} f(k, \omega, v) \, d^{3}v - j_{s}(k, \omega) \quad , \quad (2.10) \]

We now decompose \( E \) and \( j_{s} \) into longitudinal and transverse parts and write, for example,

\[ E(k, \omega) = \hat{E}(k, \omega) \cdot \hat{k} \hat{k} - \hat{k} \times \left[ \hat{k} \times E(k, \omega) \right] \quad , \quad (2.11) \]

where \( \hat{k} \) is a unit in the direction of \( k \). We thus find

\[ k \cdot E(k, \omega) = -\frac{4\pi i}{D_{L}(k, \omega)} \rho_{s}(k, \omega) \quad , \quad (2.12) \]

\[ k \times E(k, \omega) = \frac{4\pi i}{k^{2} c^{2}} \frac{k \times j_{s}(k, \omega)}{D_{T}(k, \omega)} \quad , \quad (2.13) \]

where the longitudinal dielectric function \( D_{L}(k, \omega) \) is given by

\[ D_{L}(k, \omega) = 1 + \frac{\omega^{2}}{k^{2}} \int \frac{k \cdot \frac{\partial}{\partial v}}{\omega - k \cdot v + i \varepsilon} \, d^{3}v \quad , \quad (2.14) \]
and the transverse dielectric function \( D_T(k, \omega) \) by

\[
D_T(k, \omega) = 1 - \frac{\omega^2}{c^2 k^2} + \frac{\omega^2 \omega}{k^2 c^2} \int \frac{f_0}{\omega - k \cdot v + i \epsilon} \, d^3 v. \tag{2.15}
\]

We have assumed the sources satisfy charge continuity to write

\[
\omega \rho_s(k, \omega) = k \cdot j_s(k, \omega) \quad \text{in (2.12).}
\]

We now compute the energy emitted by the source. That is, we compute

\[
W = \lim_{T \to \infty} \int_{-T}^{+T} \left( \int \mathbf{E}(\mathbf{r}, t) \cdot \mathbf{j}(\mathbf{r}, t) \, d^3 r \, dt \right). \tag{2.16}
\]

By Parseval's theorem Eq. (2.16) can be written as

\[
W = \frac{\text{Re}}{2(2\pi)^4} \int \mathbf{E}(k, \omega) \cdot \mathbf{j}^*(k, \omega) \, d^3 k \, d\omega, \tag{2.17}
\]

while the energy emitted per unit frequency is

\[
W_\omega = \frac{\text{Re}}{(2\pi)^4} \int \mathbf{E}(k, \omega) \cdot \mathbf{j}^*_s(k, \omega) \, d^3 k. \tag{2.18}
\]

We can and shall restrict ourselves to the case of \( \omega > 0 \). However the Fourier transform is defined for \(-\infty \leq \omega \leq +\infty\). We make this change by multiplying Eq. (2.17) by a factor of 2 and hereafter interpret \( \omega \) as \( |\omega| \).

When the values of the electric field are substituted from Eqs. (2.12) and (2.13)

\[
W_\omega = \frac{1}{4\pi^3} \text{Im} \int \left\{ \left( \frac{|k \cdot j_s(k, \omega)|^2}{\omega k^2 D_L(k, \omega)} - \frac{\omega |k \times j_s(k, \omega)|^2}{k^4 c^2 D_T(k, \omega)} \right) \right\} \, d^3 k. \tag{2.19}
\]
Of particular interest is the case where \( j_s \) is the current density of a test dipole of moment \( p \) embedded in the plasma,

\[
\dot{j}_s (r, t) = \dot{p} (t) \delta [r - r_0] \quad .
\] (2.20)

When Eq. (2.20) is Fourier decomposed and the result inserted in Eq. (2.18), the energy spectrum reduces to

\[
W_\omega = \frac{\omega^2}{4\pi^3} \text{Im} \int \left\{ \frac{|k \cdot p(\omega)|^2}{\omega k^2 D_L(k, \omega)} + \frac{\omega [ |k \cdot p(\omega)|^2 - k^2 |p(\omega)|^2 ]}{k^4 c^2 D_T(k, \omega)} \right\} d^3 k .
\] (2.21)

The energy emission at frequency \( \omega \) and wave number \( k \) is given by the integrand of Eq. (2.21), viz,

\[
W_{\omega, k} = \frac{\omega^2}{4\pi^3} \text{Im} \left\{ \frac{|k \cdot p(\omega)|^2}{\omega k^2 D_L(k, \omega)} + \frac{\omega [ |k \cdot p(\omega)|^2 - k^2 |p(\omega)|^2 ]}{k^4 c^2 D_T(k, \omega)} \right\} .
\] (2.22)

We note from Eq. (2.13) that the transverse wave is always polarized with its electric field in the plane determined by \( k \) and \( j_s \).

Equation (2.21) gives the total energy expended by the dipole. This includes the energy transfer to individual plasma electrons by encounters with the dipole (witness the logarithmic divergence of the first integral of (2.21) for large \( k \), see D. O.) as well as the emission of transverse and longitudinal waves. The wave emission is manifested by the contributions to the integral (2.21) from the spikes occurring because of the near vanishing of \( D_T \) and \( D_L \) for certain values of \( k \) and \( \omega \) (see Fig. 1). This wave emission is obtained only for
phase velocities large compared to the electron thermal velocity. Hence to obtain these contributions we may utilize the asymptotic forms of \( D_T \) and \( D_L \) for \( kv/\omega \ll 1 \) (where \( \bar{v} \) is the rms value of \( v \))

\[
D_T(k, \omega) \approx 1 - \frac{\omega^2}{c^2k^2} + \frac{\omega^2}{c^2k^2} + i \text{Im} D_T , \quad (2.23)
\]

\[
D_L(k, \omega) \approx 1 - \frac{\omega^2}{\omega^2} - \frac{\omega^2}{\omega^2} \frac{3k^2u_0^2\omega^2}{\omega^4} + i \text{Im} D_L . \quad (2.24)
\]

Here \( \text{Im} D_T \) and \( \text{Im} D_L \) are small slowly varying functions of \( k \) and \( \omega \) in the regime considered. Equation (2.23) admits of solution for all \( \omega > \omega_p \), while D.O. have found that the simultaneous near-vanishing of the real and imaginary parts of Eq. (2.24) occurs only in the approximate frequency range \( \omega_p \leq \omega \leq 1.4 \omega_p \) (for an equilibrium plasma).

Integration of Eq. (2.21) over the resonances yields for the wave emission at frequency \( \omega \)

\[
W_\omega = \frac{1}{2\pi} \left( \frac{\omega_p^3}{9(3)^{1/2}u_0^3} \right) \left( \omega^2 - \omega_p^2 \right)^{1/2} \left| p(\omega) \right|^2 . \quad (2.25)
\]

The upper coefficient represents the energy emission in longitudinal oscillations, while the lower describes the transverse spectrum.

While the longitudinal emission is restricted to a narrow band near the plasma frequency, in this regime its effect dominates that of transverse waves by a factor of order \( \left( \frac{c}{u_0} \right)^3 \).
III. FLUCTUATIONS IN THE CURRENT DENSITY OF A SYSTEM OF CHARGES

The radiation emitted by a plasma may be written in terms of effective current sources within the plasma. (This can in fact be taken as a definition of the effective current sources.) We hypothesize that these current sources are just the currents arising from the accelerated motion of the charges due to their interactions with the other particles of the plasma. (We also include acceleration due to waves propagating through the plasma when we consider scattering and coupling of waves.) For very low plasma densities, where the plasma does not influence the radiative processes, this is clearly correct. It has also been shown to be correct for the Čerenkov emission of longitudinal waves by fast particles moving along their straight line trajectories. 15,16

The current produced by a system of \( n \) charges is

\[
j(r, t) = \sum_{\ell=1}^{n} q_{\ell} v_{\ell}(t) \delta (r - r_{\ell}(t)),
\]

(3.1)

where \( q_{\ell} \) is the charge on the \( \ell^{th} \) particle and \( v_{\ell}(t) \) and \( r_{\ell}(t) \) are its velocity and position. If (3.1) is Fourier analyzed in space we obtain

\[
j(k, t) = \sum_{\ell=1}^{n} q_{\ell} v_{\ell}(t) \exp\{-i k \cdot r_{\ell}(t)\},
\]

(3.2)
Now as mentioned above if straight line orbits are used in (3.1), one obtains Cerenkov radiation of longitudinal waves. (The transverse waves have phase velocities greater than light for the case we consider, so there is no Cerenkov emission of these waves.) Since we are here primarily interested in radiation due to particle acceleration, we shall not consider this but look directly at the \( \mathbf{j} \) resulting from the acceleration. Thus we look at \( \frac{d\mathbf{j}(k, t)}{dt} \).

\[
\frac{d\mathbf{j}(k, t)}{dt} = \sum_{\ell=1}^{n} q_\ell \left[ \mathbf{v}_\ell + \mathbf{v}_\ell \frac{d}{dt} \right] \exp\left\{ -i \cdot \mathbf{k} \cdot \mathbf{x}_\ell(t) \right\}. \tag{3.3}
\]

The dominant part of the \( \frac{\mathbf{v}_\ell}{dt} \) term in (3.3) arises from the straight line motion of the particles.
IV. THE EMISSION OF LONG WAVELENGTH RADIATION

Equation (3.3) is the general equation for the rate of change of \( \mathbf{j} \) with respect to time and is too complicated to employ directly. We shall begin by looking at radiation at long wavelengths or small \( k \)'s. This radiation is generated by the small \( k \) part of \( \mathbf{j} \). To the extent that we are looking only at the radiation produced by the acceleration, the \( \nu \) \( d/dt \) term in (3.3) can be neglected (its ratio to the first term is roughly \( k \cdot \nu / \omega \), see Appendix I). Substituting \( q \mathbf{E}(r)/m \) for \( \nu \) we thus write (3.3) as follows

\[
\frac{d\mathbf{j}(k, t)}{dt} = \sum_{l=1}^{\infty} \frac{q_l^2}{m_l} \mathbf{E}(r_l) \exp \left\{ -i \cdot k \cdot r_l(t) \right\}. \tag{4.1}
\]

Let us first consider \( \mathbf{E} \) as being produced by particle encounters. We write for \( \mathbf{E}(r_l) \)

\[
\mathbf{E}(r_l) = \sum_{\ell} \mathbf{E}_{\ell l}, \quad \tag{4.2}
\]

where \( \mathbf{E}_{\ell l} \) is the electrostatic field at \( \ell \) produced by \( \ell \). Substituting (4.2) into (4.1) gives

\[
\frac{d\mathbf{j}(k, t)}{dt} = \sum_{\ell l} \frac{q_l^2}{m_l} \mathbf{E}_{\ell l} \exp \left\{ -i \cdot k \cdot r_l(t) \right\}
\]

\[
= \frac{1}{2} \sum_{\ell l} q_l \mathbf{E}_{\ell l} \left[ \frac{q_l}{m_l} \exp \left\{ -i \cdot k \cdot r_l(t) \right\} - \frac{q_l}{m_l} \exp \left\{ -i \cdot k \cdot r_l(t) \right\} \right]. \tag{4.3}
\]
where use has been made of the fact that

\[ q_l \mathbf{E}_{\mathbf{r}_l} = -q_{l'} \mathbf{E}_{\mathbf{r}_{l'}} \]  \hspace{1cm} (4.4)

Now the dominant contribution to \( \mathbf{E}(\mathbf{r}_l) \) will come from particles within a few Debye lengths. This will be particularly true when we analyze in \( \omega \), since distant encounters do not give high frequency contributions to \( \mathbf{E} \). For those \( l \) and \( l' \) which contribute to (4.3) we may treat \( \mathbf{k} \cdot \mathbf{r}_l \) as equal to \( \mathbf{k} \cdot \mathbf{r}_{l'} \), to lowest order in \( \mathbf{k} \). Thus we write

\[ \frac{d_j(k, t)}{dt} \approx \frac{1}{2} \sum_{l, l'} q_l \mathbf{E}_{l,l'} \left[ \frac{q_l}{m_l} - \frac{q_{l'}}{m_{l'}} \right] \exp \left\{ -i \mathbf{k} \cdot \mathbf{r}_l(t) \right\} \]  \hspace{1cm} (4.5)

We see immediately that the interactions between like particles make no contributions to (4.5). Equation (4.5) amounts to the dipole approximation used by Mercier. The like particle interactions go out because they produce no net acceleration of the dipole moment.

If we have electrons and one species of ions of charge \( Z_e \) and mass \( M \) then (4.5) becomes

\[ \frac{d_j(k, t)}{dt} \approx \frac{Z_e^2}{m_e} \left( 1 + \frac{Z m_e}{M_i} \right) \sum_{e, i} E_{i,e} \exp \left\{ -i \mathbf{k} \cdot \mathbf{r}_i(t) \right\} \]  \hspace{1cm} (4.6)

where sums are over electrons \( e \) and ions \( i \) and \( E_{i,e} \) is the electric field at the \( i \)th ion due to the \( e \)th electron.
We now take the ions to be infinitely massive so that the ion positions are no longer time dependent. The source term which we will insert in Eq. (2.19) thus becomes

$$j_s(k,\omega) = \frac{i\mu(r)}{\mu^2} \Sigma E_{ie} \exp\{-i\cdot k \cdot r_i\}.$$  

(4.7)

In accord with the superposition principle of dressed particles of Hubbard and Rostoker, the total fluctuating electric field of all the electrons is the sum of the shielded fields (shielded by electrons only) from all electrons when these electrons are treated as statistically independent. We consequently approximate Eq. (4.7) by

$$j_s(k,\omega) = \frac{iZe^2}{m_e \omega} \Sigma \tilde{E}_{ie} \exp\{-i\cdot k \cdot r_i\}.$$  

(4.8)

where \(\tilde{E}_{ie}\) is the shielded electric field of the \(e^{th}\) electron at the site of the \(i^{th}\) ion.

We now insert Eq. (4.8) as the source in Eq. (2.19). The wave emission is then extracted by local integration over the resonances. Taking an ensemble average of this result over the ions and electrons, we obtain
As in Sec. II, the upper coefficient represents the emission of longitudinal waves and the lower the emission of transverse. The remaining integration in each case is to be carried out over all directions of emission. The resonant values of $|k_L|$ and $|k_T|$ are determined by the near vanishing of Eqs. (2.23) and (2.24) respectively and are given by

$$|k_L| = \frac{1}{\sqrt{3(\frac{3}{2})^2 u_0}} \frac{1}{\omega} \sum \exp \left\{ i k_L \cdot (r_1 - r_4) \right\}$$

(4.10a)

$$|k_T| = \frac{1}{c} \frac{1}{3} \frac{1}{\omega} \sum \exp \left\{ i k_T \cdot (r_1 - r_4) \right\}$$

(4.10b)
To obtain the shielded field of an electron, we solve the test particle problem of an electron moving through an infinite uniform plasma with fixed ions. The plasma is described by the Vlasov equations (2.1 - 2.5), where the source is an electron moving uniformly along a straight line. We retain only the longitudinal electric field, the transverse interaction being relativistically small and therefore negligible if the thermal energy is small compared to the electron rest mass. The longitudinal electric field is obtained from Eq. (2.12)

\[ E_L(k, \omega) = \frac{4\pi i k \rho_s(k, \omega)}{k^2 D_L(k, \omega)} \]  

and is given by

The source charge density is given by

\[ \rho_s = -e \delta \left( r - r_{eo} - v_{eo} t \right) \]  

with \( r_{eo} \) and \( v_{eo} \) the initial position and velocity of the test particle.

Fourier analysis of Eq. (4.12) gives

\[ \rho_s(k, \omega) = -2\pi e \exp \left( \frac{-i k \cdot \mathbf{r}_{eo}}{\omega - k \cdot v_{eo}} \right) \delta(\omega - k \cdot v_{eo}) \]  

If we now substitute Eq. (4.13) into Eq. (4.11) and invert the \( k \) transform, we obtain

\[ \tilde{E}(r_1, r_{eo}, \omega) = \frac{ie}{\pi} \int d^3k \frac{k}{k^2 D_L(k, \omega)} \exp \left[ \frac{i k \cdot (r_1 - r_{eo})}{\omega - k \cdot v_{eo}} \right] \delta(\omega - k \cdot v_{eo}) \]  

(4.14)
To evaluate 4.9 we must compute \( \langle \exp \{i \mathbf{k} \cdot (\mathbf{r}_i - \mathbf{r}_{i'}) \} \tilde{E}_e(\mathbf{r}_{i'}, \omega) \rangle \). The superposition principle of dressed particles insures the statistical independence of the \( \tilde{E}_e \)'s. In order to perform the average over \( i \) and \( i' \) we must know the ion correlations. D. O. have pointed out that the presence of non-thermal ion density correlations, as may be induced, for example, by large amplitude ion waves or instabilities, can significantly enhance the absorption coefficient and consequently, in the steady state, the wave emission. This can also be seen from Eq. (4.9),

\[
W_\omega = \frac{Z^2 e^4}{8\pi^2 m_e^2} (\omega^2 - \omega_p^2) \frac{1}{3} \int d\Omega_{k_L, T} \left\{ \frac{1}{\omega_p e} \sum_{k_L, T} \frac{1}{\omega} \left\langle \left| \tilde{E}_e \right|^2 \right\rangle \right. \\
\left. \frac{1}{c^3 \omega} \sum_{k_T} \left\langle \left| \tilde{E}_e \right|^2 \right\rangle \right\}
\]

where \( n_+ \) is the ion density and the ensemble average is to be extended over ion positions.

As an example of the procedure by which ion correlations can be included in the formalism we have worked out in Appendix II the case in which the ions are thermally correlated. In the present context, however, we assume that the ions are uncorrelated.

With this assumption of uncorrelated ions, the only contribution to Eq. (4.9) occurs when \( i = i' \). After integrating over all directions,
the wave contribution is thus given by

\[ W_\omega = \frac{Z^2 e^4}{2\pi m_e^2} (\omega^2 - \omega_p^2)^{1/2} \left( \frac{1}{(3)^{3/2} \omega p_o^3} \right) \sum_{ie} \left| \frac{E_e (r_i, \omega)}{2/3 \omega c^3} \right| ^2 \]  

(4.16)

the summations to be carried out over all ions and electrons of the plasma.

The wave emission from a single encounter can be obtained by extracting one term from the double summation appearing in Eq. (4.16) and inserting the expression for the shielded field derived as Eq. (4.14). We do so, choosing the origin of time to be that time corresponding to closest approach in a collision; hence \((r_i - r_{eo})\) is the impact parameter, \(b_{ie}\), in the straight line approximation. The result is

\[ W_\omega = \frac{Z^2 e^6}{2\pi^3 m_e^2} (\omega^2 - \omega_p^2)^{1/2} \left( \frac{1/9 (3)^{3/2} \omega p_o^3}{2/3 \omega c^3} \right) \int d^3 k d^3 k' \frac{k \cdot k' \exp \left[ i (k - k') \cdot b_{ie} \right]}{k^2 k'^2 D_L (k, \omega) D_L^* (k', \omega)} \delta (\omega - k \cdot v_{eo}) \delta (\omega - k' \cdot v_{eo}) \]  

(4.17)

The total number of electron-ion collisions per unit time per unit volume, characterized by impact parameter \(b_{ie}\) and electron velocity \(v_e\), is

\[ dN = n_+ n_0 \left| v_e \right| d\phi b_{ie} dB_{ie} f(v_e) d^3 v_e \]  

(4.18)
In Eq. (4.18) \( \phi \) represents the orientation of \( \mathbf{b}_{i\ell} \) in the plane transverse to \( \mathbf{v}_e \), and \( n_+ \) and \( n_- \) are the average number densities of ions and electrons respectively.

The total power radiated per unit volume is obtained by multiplying Eq. (4.17) by Eq. (4.19) and performing the indicated integrations.

\[
P(\omega) = \left( \frac{1}{9} \frac{\omega^3}{2 \omega_c^3} \right) \left( \omega^2 - \frac{\omega_p^2}{2} \right) \frac{\rho^6}{2\pi^3 m^2} \int d^3 \mathbf{v}_e \frac{f(\mathbf{v}_e)}{|\mathbf{v}_e|} \int d\phi \\
\int d\mathbf{b}_i \int d^3 \mathbf{k} \int d^3 \mathbf{k}' \frac{\mathbf{k} \cdot \mathbf{k}' \exp\left[i(\mathbf{k} - \mathbf{k}') \cdot \mathbf{b}_{i\ell}\right]}{k^2 k'^2 D(k, \omega) D_{L*}(k', \omega)} \delta(\omega - \mathbf{k} \cdot \mathbf{v}_e) \delta(\omega - \mathbf{k}' \mathbf{v}_e).
\]

The geometry of the problem is depicted in Fig. 2. The \( \delta \)-functions occurring in Eq. (4.19) insure the identity of the components of \( \mathbf{k} \) and \( \mathbf{k}' \) parallel to \( \mathbf{v}_e \), since

\[
\delta(\omega - \mathbf{k} \cdot \mathbf{v}_e) \delta(\omega - \mathbf{k}' \cdot \mathbf{v}_e) = \frac{\delta(k_{||} - k'_{||})}{|\mathbf{v}_e|} \delta(\omega - \mathbf{k} \cdot \mathbf{v}_e).
\]

Interchange of the \( \mathbf{k}, \mathbf{k}' \) with the \( \phi, \mathbf{b}_{i\ell} \) integrations generates the two dimensional \( \delta \)-function,

\[
\int_0^{2\pi} d\phi \int d\mathbf{b}_i \int d\mathbf{b}_{i\ell} \exp\left[i(\mathbf{k} - \mathbf{k}') \cdot \mathbf{b}_{i\ell}\right] = (2\pi)^2 \delta(k_{\perp} - k'_{\perp}).
\]
Eq. (4.19) therefore simplifies to

\[ P(\omega) = \left( \frac{1}{9(3)^{\frac{1}{3}}} \frac{\omega}{\omega_c} \right)^3 \left( \omega^2 - \frac{\omega}{p} \right)^{\frac{1}{3}} \frac{4Z^2 e^6 n_+ n_0}{\pi m^2} \]

\[ \int \int d^3 k d^3 v_e f(v_e) \frac{\delta(\omega - k \cdot v_e)}{k^2 |D_L(k, \omega)|^2} \] (4.22)

The transverse part of Eq. (4.22) is in agreement with the results of Mercier. For the thermal equilibrium case, it reduces to the expression obtained by D. O.
V. SCATTERING AND COUPLING OF WAVES
BY DENSITY FLUCTUATIONS

The electrons of a plasma are accelerated by waves propagating through it. The accelerated electrons in turn emit radiation (both transverse and longitudinal) in accordance with Eq. (2.19). If the plasma were completely uniform the radiation emitted would simply reproduce the wave (Huygen's principle). However, the presence of density fluctuations results in scattering and coupling of the two types of waves. (Appendix V).

Since only the electrons are appreciably accelerated, only electron density fluctuations are involved. The electron density fluctuations can have their origin in a number of ways. First the electrons tend to follow the ions so as to maintain charge neutrality and so there will be electron density fluctuations associated with ion density fluctuations. Second there are electron density fluctuations due to the random motion of the electrons. These are important for scattering of waves whose wavelength is short compared to a Debye length. Third there are density fluctuations associated with longitudinal electron oscillations.

Here we shall confine ourselves to the scattering and coupling of long wavelength waves. We shall assume that we have a spectrum of waves. If we take the plasma to be contained in a box of volume $L^3$, then we may write the electric field in the form
\[ \mathbf{E}(r,t) = \sum_k \mathbf{E}(k) \exp\left\{ i [k \cdot r - \omega(k)t] \right\} . \quad (5.1) \]

The waves can be either longitudinal or transverse depending on whether \( \mathbf{E}(k) \) is parallel or perpendicular to \( k_\parallel \omega(k) \), of course, depends on the type of wave we have.

In the long wavelength limit the current produced by the acceleration of the \( \ell \)th electron is

\[ j_{\ell} = \delta(r - r_{\ell}(t)) \sum_k \frac{i e^2 \mathbf{E}(k)}{m_e \omega(k)} \exp\left\{ i [k \cdot r_{\ell} - \omega(k)t] \right\} . \quad (5.2) \]

The \( k^{th} \) Fourier component of \( j \) due to all electrons is given by

\[ j(k,t) = \sum_{\ell, k'} \frac{i e^2 \mathbf{E}(k')}{m_e \omega(k')} \exp\left\{ i [k' \cdot r_{\ell} - \omega(k')t] \right\} . \quad (5.3) \]

Now

\[ \sum_{\ell} \exp\{ -i k \cdot r_{\ell}(t) \} = n_{e}(k,t) , \]

where \( n_{e}(k,t) \) is the \( k^{th} \) Fourier component of the electron number density. We may thus write Eq. (5.3) in the form

\[ j(k,t) = \sum_{k'} \frac{i e^2 \mathbf{E}(k')}{m_e \omega(k')} n_{e}(k - k' , t) e^{-i \omega(k')t} . \quad (5.4) \]

Fourier analyzing \( j(k,t) \) in time gives

\[ j(k,\omega) = \sum_{k'} \frac{i e^2 \mathbf{E}(k')}{m_e \omega(k')} n_{e}(k - k' , \omega - \omega(k')) . \quad (5.5) \]

Substituting (5.5) into (2.19) and extracting the wave contribution by integration over the resonant values of \( |k| \), we obtain
\[
W \omega = \frac{e^{4(\omega^2 - \omega_p^2)^{1/2}}}{8\pi^2 m_e^2} \int d\Omega_{k_L, T} \sum_{k', k''} \frac{\omega_p}{3^{3/2} u_o^3 \omega(k')} |\hat{k}_L \cdot E(k')|^2 n_e(k_L - k', \omega(k_L) - \omega(k')) e^{i(k_L - k'' + \omega(k'') - \omega(k_L))}
\]

\[
\left\{ \frac{\omega_p}{3^{3/2} u_o^3 \omega^2(k')} |\hat{k}_L \cdot E(k')|^2 |n_e(k_L - k', \omega(k_L) - \omega(k'))|^2 \right\}
\]

\[
\left\{ \frac{\omega}{c^3 \omega^2(k')} |\hat{k}_T \cdot E(k')|^2 |n_e(k_T - k', \omega(k_T) - \omega(k'))|^2 \right\}
\]

The integration is to be carried out over angles of longitudinal emission \(d\Omega_{k_L}\) for the upper multiplying factor and angles of transverse propagation \(d\Omega_{k_T}\) for the lower. The values of \(|k_L|\) and \(|k_T|\) are those given by Eqs. (4.10a) and (4.10b).

If we ensemble average (5.6) and assume that waves of different \(k\) are randomly phased then the only terms which contribute to (5.6) are those with \(k'\) equal to \(k''\).

\[
\overline{W} \omega = \frac{e^{4(\omega^2 - \omega_p^2)^{1/2}}}{8\pi^2 m_e^2} \int d\Omega_{k_L, T} \sum_{k'} \frac{\omega_p}{3^{3/2} u_o^3 \omega^2(k')} |\hat{k}_L \cdot E(k')|^2 |n_e(k_L - k', \omega(k_L) - \omega(k'))|^2
\]

\[
\left\{ \frac{\omega}{c^3 \omega^2(k')} |\hat{k}_T \cdot E(k')|^2 |n_e(k_T - k', \omega(k_T) - \omega(k'))|^2 \right\}
\]

If the box size is taken very large then the sum over \(k'\) may be converted into an integral in the usual way.
Equation (5.7) gives the total energy emitted at frequency \( \omega \). If we further assume the process goes on from \( t = -T/2 \) to \( t = T/2 \), then we obtain the average power emitted per unit volume by dividing (5.7) by \( L^3T \),

\[
\overline{P}_\omega = \frac{e^4(\omega^2 - \omega_p^2)}{64\pi^5 m_e^2 \omega L^3 T^2} \int \frac{d^n\kappa_L}{\pi} \int d^3k' \frac{1}{(2\pi)^{3/2}} \frac{e^{-\frac{k'^2}{2\sigma^2}}}{(\kappa_L - k', \omega(\kappa_L) - \omega(k'))^2}
\]

\[
\left\{ \frac{\omega}{c^3 \omega^2(k')} \left| \frac{k_L \cdot \vec{E}(k')}{\omega_L^2} \right|^2 \left| n_e(k_L - k', \omega(k_L) - \omega(k')) \right|^2 \right\} \left\{ \frac{\omega}{c^3 \omega^2(k')} \left| \frac{k_T \times \vec{E}(k')}{\omega_L^2} \right|^2 \left| n_e(k_T - k', \omega(k_T) - \omega(k')) \right|^2 \right\}.
\]

In general \( |n_e(k, \omega)| \) will be proportional to \( T \), as we shall see shortly for some special examples, and hence \( \overline{P}_\omega \) is independent of \( T \).

For frequencies \( \omega \) much higher than the plasma frequency where dielectric shielding can be neglected, that part of Eq. (5.8) which represents the scattering of transverse waves reduces to the results obtained by Rosenbluth and Rostoker. However, the dielectric corrections are important near the plasma frequency.

We now proceed to consider the particular cases where the electron density fluctuations appearing in Eq. (5.8) result from (1) electron neutralization of ion density fluctuations and (2) longitudinal electron oscillations on the plasma.
Case I: Electron Neutralization of Ion Density Fluctuations

The electron density fluctuations can here be related to the ion density fluctuations. We have done so in Appendix III, once again disregarding ion motions so that the perturbed electron distribution (perturbed by interactions with discrete ions) is independent of time. The result is

\[ n_e(k, \omega) = n_o \int f(k, \omega) \, d^3v = - \frac{2\pi Z \omega}{k^2 D_L(k, o)} \sum p \frac{\delta(\omega)}{\partial \nu} \, \nu \, dv, \quad (5.9) \]

where \( n_o(k) \) is the Fourier transformed ion number density, \( F \) is the one-dimensional electron distribution normalized to unity, \( D_L(k, o) \) is the static longitudinal dielectric function, and the principal value of the integral over \( \nu \) is to be taken.

If we insert Eq. (5.9) for the electron density fluctuations, Eq. (5.8) then gives the average power per unit volume scattered and coupled by such density fluctuations.

\[ \bar{P} = \frac{Z^2 e^4 \langle \omega - \omega_p \rangle^{1/2}}{16\pi^3 m_e^2 T} \left\{ \int \frac{1}{\nu} \frac{\partial F}{\partial \nu} \, dv \right\} \int d\Omega_{k_L, T} \int d^3 k' \]

\[ \begin{bmatrix} \omega_p^2 \left( \frac{3}{2} \frac{u_o^3 \omega^2(k)}{k_L \cdot E(k')} \right)^2 & n_o(k_L - k') \frac{\delta(\omega(k_L) - \omega(k'))}{|k_L - k'|^2 D_L(|k_L - k'|, 0)} \\ \frac{\omega}{c^2 \omega^2(k')} \left( k_T \times E(k') \right)^2 & n_o(k_T - k') \frac{\delta(\omega(k_T) - \omega(k'))}{|k_T - k'|^2 D_L(|k_T - k'|, 0)} \end{bmatrix} \]  

(5.10)
Here the limiting procedure which is formally represented by \( \delta^2(\omega - \omega_0) \) is to be interpreted as \( \frac{T}{2\pi} \delta(\omega - \omega_0) \) where \( T \) is the duration of the emission process (cf. Appendix IV). The power emission is thus seen to be independent of \( T \). For a single wave the integral \( \int d^3k' \) is replaced by \( \frac{L^3}{(2\pi)^3} \).

If the ions are uncorrelated, the ensemble averaged value of \(|n_+(k_L, T - k_0)|^2\) is just equal to \( N \), the number of ions in the volume \( L^3 \) being considered. However, as mentioned in conjunction with the bremsstrahlung calculation, the presence of strong ion correlations (ion waves) can enhance this factor by many orders of magnitude.

Equation (5.10) is valid for any isotropic distribution of electrons. For the particular case where \( F \) is a Maxwellian,

\[
P \int \frac{1}{v} \frac{\partial F}{\partial v} \, dv = -\frac{1}{u_0^2} \, , \tag{5.11}
\]

and

\[
D_L(k, o) = 1 + \frac{k^2}{L^2} \, , \tag{5.12}
\]

where \( \kappa = \frac{P}{u_0} \) is the reciprocal Debye length.

Thus for a single wave with wave vector \( k_0 \) and frequency \( \omega(k_0) \) interacting with density fluctuations associated with the interaction of a Maxwellian electron distribution with discrete ions, we can write Eq. (5.10) as
$$\bar{P}_\omega = \frac{Z^2 e^4 (\omega^2(k_o) - \omega_p^2)^{1/2}}{4 \pi m_e^2 \omega(k_o)} n_+ \kappa^4 \int d\Omega_{L, T}$$

\[
\left\{ \begin{array}{c} \frac{|k_L \cdot E(k_o)|^2}{(3)^{3/2} u_o^3} \frac{1}{[|k_L - k_o|^2 + \kappa^2]^2} \delta(\omega(k_L) - \omega(k_o)) \\ \frac{|k_T \times E(k_o)|^2}{c^3} \frac{1}{[|k_T - k_o|^2 + \kappa^2]^2} \delta(\omega(k_T) - \omega(k_o)) \end{array} \right\}.
\] (5.13)

For long wavelength radiation $|k_{L, T} - k_o| \ll \kappa$ and hence can be neglected in evaluating the integrals in Eq. (5.13). Doing this and integrating over frequencies, we pick up only the resonances at

$$\omega(k_{L, T}) = \omega(k_o),$$

$$\bar{P}_\omega(k_o) = \frac{Z^2 e^4 (\omega^2(k_o) - \omega_p^2)^{1/2}}{m_e^2 \omega(k_o)} n_+ |E(k_o)|^2 \left\{ \frac{1}{9(3)^{1/2} u_o^3} \right\}.$$

(5.14)

If $E(k_o)$ represents a longitudinal wave, the upper coefficient gives the wave scattering, while the lower gives the coupling to transverse waves of frequency $\omega(k_o) \neq \omega_p$. The energy density per frequency interval of a longitudinal wave of frequency $\omega(k_{L, T})$ and wave vector $k_L$ can be expressed in terms of its electric field amplitude by

\[
\]
The first term in Eq. (5.15) represents the energy density associated with
the electric field, while the second is the energy density of particle motion.
The factor \( \frac{k_L^2}{2\pi} \frac{d k_L}{d\omega} \) represents the density of \( k_L \) states per frequency
interval. If we sum Eq. (5.14) over all waves in a frequency interval
\( d\omega \), we determine the scattered power to be

\[
\begin{align*}
\bar{P}_{\omega_L} \ d\omega &= \frac{Z e^2}{9(3)^{1/2}} \frac{\omega_p}{m_e u_0} \sqrt{(\omega^2 - \omega_p^2)} \ \mathcal{E}_L(\omega) \ d\omega \, , (5.16)
\end{align*}
\]

while the power coupled to transverse waves is

\[
\begin{align*}
\bar{P}_{\omega_T} \ d\omega &= \frac{Ze^2}{3} \frac{\omega_p}{m_e c^3} \sqrt{(\omega^2 - \omega_p^2)} \ \mathcal{E}_L(\omega) \ d\omega \, . (5.17)
\end{align*}
\]

In deriving Eq. (5.17) we have noted the fact that the longitudinal
wave couples to only one of the two transverse polarizations, viz. to
that wave polarized in the plane determined by the wave vectors of the
incident longitudinal and outgoing transverse waves. D. O. have derived
these results from absorption calculations and detailed balance argu-
ments. Our results are in agreement with theirs.

In a very similar manner we can compute the scattering and
coupling when \( E(k) \) is a transverse wave. For a given polarization
the energy density of such waves is

\[
\begin{align*}
\mathcal{E}_T(\omega) \ d\omega &= \frac{1}{4\pi} \frac{\omega_p^2}{|E(k_T)|} \ \mathcal{E}_T(\omega) \ \frac{k_T^2}{2\pi} \ \frac{d k_T}{d\omega} \ d\omega \, . (5.18)
\end{align*}
\]
Equation (5.14) includes the energy associated with the electric field, magnetic field, and particle motion. The latter two combine to give a contribution just equal to that due to the electric field alone.

From Eq. (5.18) we find the power coupled to longitudinal waves to be

$$P_{\omega_L} \, d\omega = \frac{Z e^2}{9(3)^{1/2}} \frac{\omega}{m_e c} \left( \omega^2 - \frac{\omega_p^2}{\omega} \right)^{1/2} \mathcal{E}_T(\omega) \, d\omega , \tag{5.19}$$

and that scattered as transverse waves (again remarking that the prevailing transverse wave can couple to only one of the two possible polarizations of the scattered transverse wave) to be

$$P_{\omega_L} \, d\omega = \frac{Z e^2}{6} \frac{\omega}{m_e c^3} \left( \omega^2 - \frac{\omega_p^2}{\omega} \right)^{1/2} \mathcal{E}_T(\omega) \, d\omega . \tag{5.20}$$

The latter two results are in agreement with those obtained by Berk, who extended the D. O. model to include finite wavelength as well as electromagnetic corrections to the absorption coefficient.

We remark in passing that by applying the principle of detailed balance to a thermal equilibrium plasma, we may equate the coupled powers, Eqs. (5.17) and (5.19), to find that the steady state wave energy densities, $\mathcal{E}_L(\omega)$ and $\mathcal{E}_T(\omega)$, are in the ratio

$$\frac{\mathcal{E}_L(\omega)}{\mathcal{E}_T(\omega)} = \left( \frac{c}{n_o} \right)^3 \frac{1}{6(3)^{1/2}} \mathcal{E}_T(\omega) , \tag{5.21}$$

and that we have equipartition, viz.
In Eq. (5.21) $\bar{E}_T(\omega)$ includes both polarization of the transverse wave.

We shall verify Eq. (5.21) directly in Sec. VI by balancing emission against absorption.
Case II: Longitudinal Electron Oscillations on the Plasma

As a second example illustrating the application of Eq. (5.8) we look at the generation of transverse waves by the interaction of two or more longitudinal waves. This problem has been investigated by Sturrock\textsuperscript{20} and quite recently by Aamot and Drummond\textsuperscript{21} as well as Boyd.\textsuperscript{22}

These latter authors solved the problem of the coupling of longitudinal to transverse waves in an infinite homogeneous collisionless plasma. They considered only the case where the longitudinal wave energy is much greater than the transverse wave energy so that they could neglect the interaction of the transverse waves back on the longitudinal waves or with each other. Aamot and Drummond computed the rate of increase of electric and magnetic field energy in the transverse waves and equated this to the rate of emission of radiation. This is actually in error, for there is also particle kinetic energy associated with these waves. This correction is included in our calculations.

The density fluctuations appearing in Eq. (5.8) are now those electron density fluctuations associated with the passage of longitudinal waves through the plasma. They can therefore be determined from a solution of Poisson's equation

\[
\rho_e(k, \omega) = -\frac{i k \cdot E(k, \omega)}{4\pi e}. \tag{5.23}
\]

If we assume that we have the spectrum of longitudinal waves given by Eq. (5.1), then $E(k, \omega)$ is given by
\[ \Sigma(k, \omega) = (2\pi)^4 \sum_{k''} \frac{\Sigma(k''') \delta(k - k'') \delta(\omega - \omega(k'''))}{k''} \]  (5.24)

By combining Eqs. (5.23) and (5.24), we find that the quantity
\[ n_{e}(k_{L,T} - k', \omega(k_{L,T}) - \omega(k')) \] which appears in Eq. (5.8) can be written as
\[ n_{e}(k_{L,T} - k', \omega(k_{L,T}) - \omega(k')) = - \frac{i}{e} \frac{4\pi}{\varepsilon} \frac{k_{L,T} - k'}{\sum_{k''} \frac{\Sigma(k'') \delta(k_{L,T} - k' - k'') \delta(\omega(k_{L,T}) - \omega(k') - \omega(k'')}{k''}} \]

\[ = - \frac{i}{e} \frac{4\pi}{\varepsilon} \frac{k_{L,T} - k'}{\sum_{k''} \frac{\Sigma(k'') \delta(k_{L,T} - k' - k'') \delta(\omega(k_{L,T}) - \omega(k') - \omega(k''))}{k''}} \]

\[ = \frac{i}{e} \frac{4\pi}{\varepsilon} \frac{k_{L,T} - k'}{\sum_{k''} \frac{\Sigma(k'') \delta(k_{L,T} - k' - k'') \delta(\omega(k_{L,T}) - \omega(k') - \omega(k''))}{k''}} \]  (5.25)

We have already assumed that \( k'' \) is the wave vector of a longitudinal wave (and therefore \( k'' \) is parallel to \( \Sigma(k'') \) in Eq. (5.25)). We now further confine ourselves to the situation where \( k' \) also represents a longitudinal wave. The \( \delta \)-functions appearing in Eq. (5.25) physically represent the conservation of momentum and energy for the "collision" of these two longitudinal waves.

\[ k_{L,T} = k' + k'' \]  (5.26)

\[ \omega(k_{L,T}) = \omega(k') + \omega(k'') \]  (5.27)

Since \( k' \) and \( k'' \) are both longitudinal waves, \( \omega(k') \) and \( \omega(k'') \) each lie in the range \( \omega_{p} \leq \omega \leq 1.5 \omega_{p} \). The quantity \( \omega(k_{L,T}) \) is thus at least as large as \( 2 \omega_{p} \). Since longitudinal waves can not
propagate at this frequency, we see that it is energetically impossible for two longitudinal waves to couple to a third. We thus drop the subscript $L$ in Eqs. (5.26) and (5.27).

We further note that since $k'$ and $k''$ are much greater in magnitude than $k_T$ (except for relativistically energetic plasmas), transverse waves can only be produced by the nearly "head-on collision" of the two longitudinal waves.

If we introduce Eq. (5.25) for the density fluctuations appearing in Eq. (5.8), we obtain for the transverse emission

$$\bar{\rho} = \frac{e^2}{4m_e^2} \frac{\varphi^2(\omega^2 |k_T|^2 - \omega^2)^{1/2}}{c^3} \omega(|k_T|) \int d\Omega \int d^3k' \frac{\left| k_T \times E(k') \right|^2}{\omega^2(k')},$$

$$\sum_{k'',k''''} k'''' \left| E(k'') \right| \left| E^{*}(k'') \right| \delta(k_T - k' - k''') \delta(k_T - k' - k''''')$$

$$\delta(\omega(k_T) - \omega(k') - \omega(k'')) \delta(\omega(k_T) - \omega(k') - \omega(k''')) \delta(\omega(k_T) - \omega(k') - \omega(k''')).$$  

(5.28)

Assuming that the $k''''s$ and the $k'''''s$ are dense, we can convert the summations appearing in Eq. (5.28) into integrals and use the $\delta$-functions on $k''$ and $k'''''$ to carry out these integrals. The result is

$$\bar{\rho} = \frac{e^2}{4m_e^2} \frac{\varphi^2(\omega^2 |k_T|^2 - \omega^2)^{1/2}}{c^3} \omega(|k_T|) \frac{L}{(2\pi)^6} \int d\Omega k_T \int d^3k'$$

$$\frac{\left| k_T \times E(k') \right|^2}{\left| k_T \right|^2 \omega^2(k')} \left[ \frac{k_T - k' \left| E(k_T - k') \right|^2}{\omega(k_T - k')} \right]^{2} \delta(\omega(k_T) - \omega(k') - \omega(k')) \delta(\omega(k_T) - \omega(k') - \omega(k'')).$$

(5.29)
Since \( k' \) is a variable of integration, we can rewrite the integrand of Eq. (5.29) as

\[
\left\{ \left[ \frac{k_T \times E(k')}{|k_T|} \right] \left[ \frac{E(k_T - k')}{|k_T - k'|} \right] \delta(\omega(k_T) - \omega(k') - \omega(k_T - k')) \right\}^2 \]

In so splitting the \( k' \) integrand we are considering in pairs the effects produced by wave \( k' \) scattering from density fluctuations produced by wave \( k_T - k' \) and wave \( k_T - k' \) scattering from density fluctuations produced by wave \( k' \). Since \( E(k') \) and \( E(k_T - k') \) represent longitudinal waves, Eq. (5.30) can be reexpressed in the form

\[
\left\{ \frac{|k_T \times E(k')|}{|k_T| \omega(k')} \left[ \frac{|k_T - k'| |E(k_T - k')|}{|k_T - k'| \omega(k')} \delta(\omega(k_T) - \omega(k') - \omega(k_T - k')) \right] \right\}^2
\]

\[
\left\{ \frac{1}{2} \left[ \frac{|k_T \times E(k')|}{|k_T| \omega(k')} \left[ \frac{|k_T - k'| |E(k_T - k')|^2}{|k_T - k'| \omega(k')} \right] \right\}^2 \delta(\omega(k_T) - \omega(k') - \omega(k_T - k')) .
\]  \quad (5.31)

From Eq. (5.31) we note that since \( k_T \ll k' \), the two collisional processes interfere destructively. To lowest order in the ratio

\[
\frac{|k_T|}{|k'|} ,
\]

Eq. (5.31) can be written as
If we now insert Eq. (5.32) into Eq. (5.29) and carry out the integration over directions of emission of transverse waves, we obtain

\[
\overline{\mathcal{F}} \omega = \frac{e}{60\pi} \frac{2}{m_e c^5} \frac{\omega^2 (|k_T| - \omega_p)}{L^3} \int d^3 k' \frac{1}{\omega^2(k')} \frac{|\mathcal{E}(k')|^2 |\mathcal{E}(-k')|^2}{\delta^2(\omega(k_T) - 2\omega(k'))},
\]

(5.33)

where again \( \delta^2(\omega(k_T) - 2\omega(k')) \) is to be interpreted as \( \frac{T}{2\pi} \delta(\omega(k_T) - 2\omega(k')) \) (cf Appendix IV). We have substituted for \( |k_T| \) the value given by Eq. (4.10b).

When we now integrate over frequencies of emission \( \omega \), we are left with

\[
\overline{\mathcal{F}} \omega = \frac{L^3}{60\pi^2} \frac{e}{m_e c^5} \left( \frac{L}{2\pi} \right)^3 \int d^3 k' \frac{(4\omega^2(k') - \omega_p^2)^{3/2}}{\omega(k')} \frac{|\mathcal{E}(k')|^2 |\mathcal{E}(-k')|^2}{\delta^2(\omega(k_T) - 2\omega(k'))}.
\]

(5.34)
If we estimate $\bar{P}_\omega$ by assuming that the average value of $\omega(k')$ for all waves is given by $\bar{\omega}(k') = \alpha \omega_p$, where $1 \leq \alpha \lesssim 1.5$, then

$$\bar{P}_\omega = \frac{L^3}{60\pi^2} \frac{(4\alpha^2 - 1)^{3/2}}{\alpha} \frac{e^2 \omega_p^2}{m_e c^5} \frac{L}{2\pi} \int d^3k' |\mathbf{E}(k')|^2 |\mathbf{E}(-k')|^2. \quad (5.35)$$

We see that for $\alpha = 1.5$, the power emission is a factor $\frac{32}{9}\left(\frac{2}{3}\right)^{1/2} \approx 2.9$ greater than for the choice of $\alpha = 1$. To achieve exact agreement with Aamodt and Drummond, when their result has been corrected to include particle energy density, we must choose $\alpha \cong 1.08$. 
VI. WAVE SPECTRA IN A STEADY STATE PLASMA

If we have a plasma maintained in the steady state, we can combine our emission calculations with the D.O. absorption calculations to find the steady state wave spectra in the plasma. Some steady state plasmas which might be considered are thermal equilibrium plasmas and non-equilibrium but steady discharges. Here we shall confine ourselves to isotropic electron distributions.

The absorption coefficient can be obtained from the plasma conductivity. With slight modification, the D.O. high frequency conductivity calculation is valid for any isotropic electron distribution. They find for the conductivity

\[ \sigma = \sigma_0 (1 + \tilde{\sigma}_1) \]  

(6.1)

where

\[ \sigma_0 = \frac{i \omega}{\mathcal{P}} \]  

(6.2)

If the ions are uncorrelated, \( \tilde{\sigma}_1 \) is given by

\[ \tilde{\sigma}_1 = \frac{2 Ze^2}{3 \pi m_e \omega^2} \int_0^{k_{\text{max}}} dk \frac{2}{k} \left( \frac{1}{D_L(k, \omega)} - \frac{1}{D_L(k, 0)} \right) \]  

(6.3)

The cut-off in Eq. (6.3) at \( k_{\text{max}} \sim \frac{m u_0}{Ze^2} (u_0 = \text{rms electron velocity}) \) is the usual cut-off introduced in order to prevent the divergence for small impact parameters \( \left( \frac{1}{k} \right) \).
In the paper by D. O. it is shown that the dispersion relation for transverse waves propagating through the plasma is

\[ k^2 c^2 - \omega^2 + 4\pi i \omega \sigma = 0. \quad (6.4) \]

Substituting \( \sigma \) from (6.1) and (6.2) into (6.4) gives

\[ k^2 c^2 - \omega^2 + \omega_p^2 + \omega_p^2 \sigma_1 = 0. \quad (6.5) \]

If Eq. (6.5) is solved for \( \omega \) for real \( k \), then the imaginary part of \( \omega \) gives the time rate of decay of the amplitude of this pure \( k \) mode (decay going as \( \exp(-\text{Im} \omega t) \)). Twice the imaginary part of \( \omega \) gives the rate of energy decay in the wave because the energy is quadratic in the amplitude. The \( e^{-1} \) time for the energy is thus \( \frac{1}{2 \text{Im} \omega} \). We may solve Eq. (6.5) for the imaginary part of \( \omega \) by making use of the fact that \( |\sigma_1| \) is small. We thus find

\[ \text{Im} \omega = \frac{1}{2} \frac{\omega_p^2}{\omega_p^2 + k^2 c^2} \text{Im} \sigma_1. \quad (6.6) \]

\[ \omega \approx \pm (\omega_p^2 + k^2 c^2)^{1/2} \]

Hence the decay time for transverse waves is

\[ \tau_T = \frac{1}{\omega_p^2 \text{Im} \sigma_1} = \frac{3\pi m_e}{2 Ze^2} \frac{\omega^3}{\omega} \frac{1}{\int_0^{k_{\text{max}}} \frac{d k k^2}{k} \text{Im} \frac{1}{D_L(k, \omega)}} \quad (6.7) \]

where we have used the fact that \( \text{Im} D_L(k, \omega) = 0 \).
In the steady state, the rate of absorption of transverse waves is balanced by their rate of emission provided we can neglect the escape of radiation from the plasma. The average rate of absorption of transverse wave energy per unit volume per unit frequency interval is just the steady state energy density $\bar{E}_{\omega_T}$ divided by the decay time for such transverse waves as given by Eq. (6.7). If we balance the absorption by the emission of transverse waves as represented by Eq. (4.22) we may write down the expression for $\bar{E}_{\omega_T}$:

$$\bar{E}_{\omega_T} = -\frac{4Ze^2n_e n_o \omega^2}{mc^3\omega_p^2} (\omega^2 - \omega_p^2)^{1/2} \int \frac{d^3k d^3v_e f(v_e) \delta(\omega - k \cdot v_e)}{k^2 |D_L(k, \omega)|^2} \int_0^{k_{max}} dk \frac{\text{Im} D_L^*}{|D_L(k, \omega)|^2},$$

(6.8)

Equation (6.8) can be somewhat simplified by integrating the numerator over velocity space and introducing the value of $\text{Im} D_L^*$ obtained from Eq. (2.14), viz.

$$\text{Im} D_L^*(k, \omega) = \frac{\pi \omega^2}{k^2} F \left(\frac{\omega}{k}\right).$$

(6.9)

In Eq. (6.9) $F$ is once again the one-dimensional distribution function normalized to unity. Equation (6.8) consequently becomes
We proceed in a quite analogous fashion to determine the steady state energy density of longitudinal waves. From Maxwell's equations we determine the dispersion relation for such waves propagating through the plasma to be

\[ \omega^2 = \omega^2_p(1 + \tilde{\sigma}_1) \],

(6.11)

where \( \tilde{\sigma}_1 \) has been given by Eq. (6.3). Since \( |\tilde{\sigma}_1| \ll 1 \), we see that such waves propagate only at frequencies near the plasma frequency, a fact that has been used repeatedly in this work.

The decay time for such longitudinal waves as determined by solving Eq. (6.11) for the imaginary part of \( \omega \) turns out to be

\[ \tau_L = \frac{1}{2 \text{Im} \omega} = \frac{1}{\omega_p \text{Im} \tilde{\sigma}_1} = \frac{3\pi m e}{2Z e^2} \left( \frac{1}{k_{\text{max}}} \int_0^k \frac{dk}{k |D_L(k, \omega)|^2} \right) \]

(6.12)

Landau damping has not been included in this calculation. For \( k^{-1} > 5 \lambda_D, \lambda_D = \text{Debye length} \), Landau damping is unimportant.
One could include it in these calculations. However, if one does this, Cerenkov emission of longitudinal waves (the inverse process to Landau damping) must also be included. From Eqs. (4.22) and (6.12) we thus obtain the steady state energy level of longitudinal oscillations.

\[ \overline{E}_{\omega_L} = \frac{-8Ze^4n_0}{3(3)^{1/2}m u_0^3} \frac{1}{\omega_p^2} \left( \omega^2 - \omega_p^2 \right)^{1/2} \int_0^{k_{\text{max}}} \frac{\omega}{k} \frac{F\left(\frac{\omega}{k}\right)}{k} \frac{1}{\left| D_L(k, \omega) \right|^2} dk, \quad (6.13) \]

In thermal equilibrium

\[ \int_0^{k_{\text{max}}} \frac{k_{\text{max}}}{\omega} \frac{F\left(\frac{\omega}{k}\right)}{k} \frac{1}{\left| D_L(k, \omega) \right|^2} \frac{dk}{d\omega} = -\frac{u_0^2}{\omega}, \quad (6.14) \]

and the transverse and longitudinal wave energy levels reduce to

\[ \overline{E}_{\omega_T} = \frac{1}{2} \frac{m u_0^2}{\pi^2 c^3} \omega(\omega^2 - \omega_p^2)^{1/2}, \quad (6.15) \]

\[ \overline{E}_{\omega_L} = \frac{1}{6(3)^{1/2}} \frac{m}{\pi^2 u_0^2} \frac{\omega(\omega^2 - \omega_p^2)^{1/2}}{\omega_p^2}. \quad (6.16) \]

From Eqs. (6.15) and (6.16), Eq. (5.21) can be immediately verified.
The density of modes $\rho(\omega)$ is given by

$$\rho(\omega) = \frac{k_0^2(\omega)}{2\pi^2} \frac{\partial k_0}{\partial \omega}.$$  

(6.17)

By dividing $\overline{E_{\omega_L}}$ and $\overline{E_{\omega_T}}$ by $\rho(\omega)$ and using the dispersion relations (4.10a) for longitudinal waves and (4.10b) for transverse and taking into account the two possible transverse polarizations, we find the average energy per mode is

$$\frac{\overline{E_{\omega_L, T}}}{\rho_{L, T}(\omega)} = m u_0^2 = \kappa T,$$

(6.18)

where $\kappa$ is Boltzmann's constant.
VII. DISCUSSION

We have shown how many problems involving radiation can be handled in terms of effective current sources embedded in a Vlasov plasma. We first compute the emission of radiation by arbitrary test sources. By appropriately choosing these sources, a variety of problems can be handled. If one uses as sources the time varying dipole moment due to encounters between shielded electrons and ions, one obtains the bremsstrahlung emission (including the emission of longitudinal waves). If one uses as sources the currents produced by the interaction of waves with density fluctuations or with other waves, one obtains the scattering of waves, the emission of transverse waves by longitudinal plasma oscillations, and the generation of longitudinal waves by transverse waves.

The method used in this paper can be extended to other radiation problems. At present we are calculating the quadrupole radiation due to electron-electron and to electron-ion encounters. This work will appear in a later paper. It is also possible to extend the method to anisotropic plasmas and to plasmas in magnetic fields. These problems are also under investigation.

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REFERENCES


FIGURE CAPTIONS

Figure 1. Graph of $\text{Im} \frac{k^2}{D_L(k, \omega)}$ against $\frac{1}{k}$ for various values of the ratio $\omega/\omega_p$.

Figure 2. Vector diagram of quantities appearing in Eq. (4.19).
Figure 1. Graph of $\text{Im} \frac{k^2}{D_L(k, \omega)}$ against $\frac{1}{k}$ for various values of the ratio $\frac{\omega}{\omega_p}$.
Figure 2. Vector diagram of quantities appearing in Eq. (4.19).
APPENDIX I

In this appendix we are interested in determining the relative magnitudes of the terms appearing in the equation

\[
\frac{d\mathbf{J}(k,t)}{dt} = \sum_{i=1}^{n} q_i \mathbf{v}_i \times \mathbf{v}_i \times \mathbf{r}_i(t) \exp \left\{ -i k \cdot \mathbf{r}_i(t) \right\} .
\]

In Sec. IV we considered only the term \[ \sum_{i=1}^{n} q_i \mathbf{v}_i \exp \left\{ -i k \cdot \mathbf{r}_i(t) \right\} \] (term a). We also considered only the acceleration of electrons and expressed \( \mathbf{v}_i \) as the sum \( \sum_{L} \mathbf{v}_{L} \), where \( \mathbf{v}_{L} \) represents the acceleration of the \( L \)th electron due to interaction with the \( L \)th ion. (Interactions between electrons produce no net change in \( \mathbf{J}(k,t) \) in the dipole approximation.) In the dipole approximation also, we assumed that for two interacting particles whose positions are \( \mathbf{r}_L(t) \) and \( \mathbf{r}_{L'}(t) \), \( k \cdot [\mathbf{r}_L(t) - \mathbf{r}_{L'}(t)] \ll 1 \). To this approximation we may therefore replace the time dependent position \( \mathbf{r}_L(t) \) in the exponent of Eq. (3.3) by the stationary position vector \( \mathbf{r}_L \) of the ion with which it interacts.

Doing this, we write the first term to lowest order (order 0) in the dipole approximation as

\[
\frac{d\mathbf{J}_{ao}(k,t)}{dt} = \sum_{L,L'} q_L \mathbf{v}_{L'} \exp \left\{ -i k \cdot \mathbf{r}_L \right\} .
\]

When Fourier transformed in time Eq. (I-1) may be written as

\[
\mathbf{J}_{ao}(k,\omega) = \sum_{L,L'} q_L \mathbf{v}_{L'}(\omega) \exp \left\{ -i k \cdot \mathbf{r}_L \right\} .
\]
We emphasize once more that in adopting the dipole approximation we have neglected terms of $\Theta[k \cdot (x_{k} - x_{\ell})]$. The next such term in the expansion of the first term of Eq. (3.3) would be

$$j_{a1}(k, \omega) = i \sum_{\ell \ell'} q_{\ell} \cdot \left[ (x_{\ell}' - x_{\ell}) \cdot \frac{\partial}{\partial \omega} \right] \exp\left\{ i k \cdot x_{\ell} \right\}.$$  \hspace{-1cm} (I-3)

Turning our attention now to the second term of Eq. (3.3) (term b), we reiterate the remark made in Sec. III that if only the unaccelerated electron velocities and positions are used, longitudinal Cerenkov emission is obtained. Since we are interested in computing bremsstrahlung, accelerative corrections to the electron orbits must be considered. If we express the velocity of the $\ell^{th}$ electron as

$$v_{\ell} = v_{\ell 0} + \sum_{\ell'} \delta v_{\ell \ell'},$$  \hspace{-1cm} (I-4)

where $v_{\ell 0}$ is the unperturbed velocity and $\delta v_{\ell \ell'}$, the perturbation produced by interaction with the $\ell'$ th particle (and here $\ell'$ can represent either an ion or another electron), the accelerative part of the second term in Eq. (3.3) can be written as

$$\frac{d j_{b}(k, t)}{dt} = -i \sum_{\ell \ell'} q_{\ell} \left\{ v_{\ell 0} \cdot \frac{\partial}{\partial \omega} v_{\ell \ell'} + \delta v_{\ell \ell'} \cdot \frac{\partial}{\partial \omega} v_{\ell 0} \right\} \exp\left\{ i k \cdot x_{\ell}(t) \right\}.$$  \hspace{1cm} (I-5)

We have, of course, assumed that $\sum_{\ell'} \delta v_{\ell \ell'} \ll v_{\ell 0}$ to linearize Eq. (I-5).
For the purpose of comparing Eq. (I-5) with (I-1), we take the \( F^{l} \) to represent only ions and once again replace \( -\ddot{r}_{k}(t) \) by the time invariant \( -\ddot{r}_{k} \). If we Fourier analyze the resultant expression and note that \( \delta \nu_{k}^{(l)}(\omega) = \frac{i \nu_{k}^{(l)}(\omega)}{\omega} \), we obtain

\[
\dot{j}_{00}(k, \omega) = \sum_{l} q_{k} \left\{ \frac{k \cdot \nu_{k}^{(l)}(\omega)}{\omega} + \frac{\nu_{k}^{(l)}(\omega)}{\omega} \right\} \exp\left\{ ik \cdot r_{k} \right\}.
\]

We thus see that the terms in Eq. (I-6) are in the ratio \( \frac{k \cdot \nu_{k}^{(l)}}{\omega} \) to those in Eq. (I-2). The phase velocities \( (\omega/k) \) of transverse waves and very long wavelength longitudinal waves are faster than the speed of light, so that the discarded terms are relativistically small. For very short wavelength longitudinal waves, however, these terms may be significant and even dominant.

It can be further shown that the currents represented by Eqs. (I-3) and (I-6) are of comparable magnitude and when introduced into Eq. (2.19) combine to give the longitudinal and transverse quadrupole emission spectra. We shall show this and consider such effects in a later paper. Electron-electron interaction effects can and should, of course, be included at this level.
APPENDIX II

We have derived as Eq. (4.15) an expression for the total energy emission, longitudinal and transverse, at frequency $\omega$ from collisions between shielded electrons and ions,

$$ W_{\omega} = \frac{2e^4}{8\pi^2 m_e^2} (\omega^2 - \omega_p^2)^{1/2} \int d\Omega_{k_L, T} \left\{ \begin{array}{c} \frac{1}{3(3)} \frac{1}{\omega} \sum_e \left( \frac{k_L \cdot (n_e + E_e)}{-k_L, \omega} \right)^2 \\ \frac{1}{c} \frac{1}{\omega} \sum_e \left( \frac{k_L \times (n_e + E_e)}{-k_L, \omega} \right)^2 \end{array} \right\} \right. $$

(4.15)

In Eq. (4.15) $n_e$ is the ion density and $E_e$ the shielded electric field due to the $e$th electron. The ensemble average is to be extended over ion positions. Use of the shielded electric fields due to the electrons has enabled us to neglect electron correlations.

In Sec. IV we have evaluated Eq. (4.15) for the case where the ions are also assumed to be uncorrelated. In this Appendix we work out the case where equilibrium correlations exist among the ions.

If we use Eq. (4.14) for $E_e$, then we can write down an expression for $n_e E_e$,

$$ n_e E_e = \frac{ie}{\pi} \sum_{i} \delta(r - r_i) \int d^3k \frac{k}{k^2 D_L(k, \omega)} \exp \left[ i k \cdot (r_i - r_{eo}) \right] \delta(\omega - k \cdot v_{eo}) $$

(II-1)

where the summation is to be extended over the positions of all ions.
If we Fourier analyze Eq. (II-1) in space, the quantity appearing in the longitudinal part of Eq. (4.15) (an exactly analogous procedure holds for the transverse emission) can be expressed as

\[ \langle \hat{\mathbf{k}} \cdot \left( n + \frac{\mathbf{E}}{\omega} \right) \rangle^2 \]

We are thus interested in evaluating

\[ \langle \sum_{i,j} \exp \left[ i \left( \mathbf{k} \cdot \mathbf{r}_i - \mathbf{k} \cdot \mathbf{r}_j \right) \right] \rangle = \sum_{i,j} \langle \exp \left[ i \left( \mathbf{k} \cdot \mathbf{r}_i \right) \right] \exp \left[ i \left( \mathbf{k} \cdot \mathbf{r}_j \right) \right] \rangle \]

In writing down Eq. (II-3) we have equated the ensemble average of the sum to the sum of the ensemble averages and have expressed the position of the \( j \)th ion in terms of the position of the \( i \)th ion and the distance between the two.

If we now arbitrarily select the \( i \)th ion and ask for the probability of finding another ion at distance \( r_{ij} \) away, then

\[ \langle \exp \left[ i \left( \mathbf{k} \cdot \mathbf{r}_i - \mathbf{k} \cdot \mathbf{r}_j \right) \right] \rangle = \exp \left[ i \left( \mathbf{k} \cdot \mathbf{r}_i \right) \right] \int P(r_{ij}) \exp \left[ i \left( \mathbf{k} \cdot \mathbf{r}_j \right) \right] d^3 \mathbf{r}_{ij} \]
In equilibrium the probability of finding a particular ion in a volume element \( d^3 \mathbf{r}_{ij} \) at a distance \( \mathbf{r}_{ij} \) from a given ion is

\[
P(\mathbf{r}_{ij}) d^3 \mathbf{r}_{ij} = \frac{d^3 \mathbf{r}_{ij}}{V} \exp \left( -\frac{Ze\phi}{m u_o} \right) \approx \frac{d^3 \mathbf{r}_{ij}}{V} \left\{ 1 - \frac{Ze\phi}{m u_o^2} \right\}, \tag{II-5}
\]

where \( V \) is the volume of the plasma and \( \phi \) the shielded potential of an ion given by

\[
\phi = \frac{Ze}{r_{ij}} \exp \left( -\kappa_T r_{ij} \right). \tag{II-6}
\]

In (II-6), \( \kappa_T \) is the inverse Debye length for both ions and electrons,

\[
\kappa_T^2 = \frac{(Z + 1) 4\pi ne^2}{m u_o^2}.
\]

When Eqs. (II-5) and (II-6) are inserted in Eq. (II-4), we obtain

\[
\langle \exp \left[ i\left( k - k' \right) \cdot \mathbf{r}_i - \left( k_L + k' \right) \cdot \mathbf{r}_{ij} \right] \rangle = \exp \left[ i\left( k - k' \right) \cdot \mathbf{r}_i \right]
\]

\[
\frac{1}{V} \int d^3 \mathbf{r}_{ij} \left\{ 1 - \frac{Z^2 e^2}{m u_o^2} \frac{1}{r_{ij}} \exp \left( -\kappa_T r_{ij} \right) \right\} \exp \left( -i(k_L + k') \cdot \mathbf{r}_{ij} \right). \tag{II-7}
\]

The first term in the integrand contributes only when \( j = i \).

(There are \( N \) of these terms corresponding to the number of ions in the volume \( V \).) The second term can be integrated directly. (There are \( N(N-1) \equiv N^2 \) of these corresponding to \( N-1 \) choices of \( j \) for each choice of \( i \).) If we carry out the integration in Eq. (II-7) and evaluate the double sum we obtain per unit volume
We can now insert Eq. (II-9) into Eq. (II-2) and in turn substitute this expression into Eq. (4.15). The summation over electrons is then carried out exactly as in Sec. IV. In carrying out the integration over directions of emission, we neglect $k_L$ by comparison with $k_1$ in Eq. (II-9), since the dominant contribution to the $k'$ integration arises from large $k'$ (small impact parameters).

The results for both longitudinal and transverse radiation are

$$P(\omega) = \frac{1}{9(3)^{1/2}} \frac{\omega^3}{\omega_p u_o} \left( \frac{\omega^2 - \omega_p^2}{\omega_p} \right)^{1/2} \frac{4 Z^2 e^6 n_n + n_o}{\pi m^2},$$

$$\int \int d^3 k' d^3 v_e f(v_e) \frac{\delta(\omega - k' \cdot v_e)}{k'^2 |D_L(k', \omega)|^2} \left[ 1 - \frac{Z \kappa^2}{k'^2 + (Z+1)\kappa^2} \right].$$

(II-10)
APPENDIX III

We here evaluate the electron density fluctuations produced by interaction of the electrons with massive discrete ions. Since density fluctuations exist among such discrete ions (cf. Appendix II), the electron distribution in tending to follow the ions itself deviates from spatial homogeneity.

For immobile ions, the electron perturbation is assumed stationary in time and the Vlasov equation linearized in the electron-ion interaction becomes

\[ \nabla \cdot \frac{\partial f}{\partial r} + \frac{e \nabla \phi}{m_e} \cdot \frac{\partial f}{\partial v} = - \epsilon f \]  

(III-1)

The perturbed potential \( \phi \) is given by the Poisson equation with the ions inserted discretely as sources,

\[ \nabla^2 \phi = 4\pi e \left[ n_0 (1 + \int f \, d^3 v) - \sum \delta (r - r_i) \right] \]  

(III-2)

If we solve Eq. (III-1) for the Fourier transformed \( f_k \), we obtain

\[ f_k = \frac{-e}{m_e} \frac{\phi_k}{k \cdot v - i \epsilon} \frac{\partial f_0}{\partial v} \]  

(III-3)

When this value is inserted into Eq. (III-2), we may solve for \( \phi_k \) and obtain for \( k \neq 0 \)

\[ \phi_k = \frac{4\pi Ze n_0 (k)}{k^2 - \omega_p^2 - i\epsilon} \int \frac{k \cdot \frac{\partial f_0}{\partial v}}{k \cdot v - i \epsilon} d^3 v \]  

(III-4)
where \( n_+(k) \) has been defined as

\[
n_+(k) = \sum_i \exp\{-i k \cdot r_i\} \quad \text{(III-5)}
\]

In terms of the static longitudinal dielectric function (Eq. (2.14)), Eq. (III-4) can be written as

\[
\phi_k = \frac{4\pi Z e \ n_+(k)}{k^2 D_L(k,0)} \quad \text{(III-6)}
\]

and Eq. (III-3) as

\[
f_k = \frac{4\pi Z e^2}{m_e k^2 D_L(k,0)} n_+(k) \frac{\delta f_\circ}{k \cdot v - i\epsilon} \quad \text{(III-7)}
\]

The electron density fluctuations induced by Coulomb interactions with the ions thus turn out to be

\[
n_e(k) = n_0 \int f_k \, d^3v = \frac{4\pi Z e^2 n_0}{m_e k^2 D_L(k,0)} n_+(k) P \int \frac{1}{v} \frac{\delta f_\circ}{\delta v} \, dv \quad \text{(III-8)}
\]

In Eq. (III-8) \( F_\circ(v) \) is the one dimensional electron distribution normalized to unity. The pole contribution to the integration has disappeared because of the assumed isotropy of the electron distribution \( f_\circ \).

Since there is no time variation in the problem, the only \( \omega \)-component present is \( \omega = 0 \) and

\[
n_e(k,\omega) = \frac{-2\pi \omega^2 Z n_+(k)}{k^2 D_L(k,0)} \delta(\omega) P \int \frac{1}{v^2} \frac{\delta f}{\delta v} \, dv \quad \text{(III-9)}
\]
APPENDIX IV

In Sec. VI we have introduced the function $\delta^2(\omega)$. In this appendix we explain what we mean by this function.

We define the function $\delta(\omega)$ to be

$$
\delta(\omega) = \lim_{T \to \infty} \frac{1}{2\pi} \int_{-T/2}^{T/2} \exp i \omega t \, dt .
$$

(IV-1)

With $\delta(\omega)$ so defined, $\delta^2(\omega)$ can be written as

$$
\delta^2(\omega) = \lim_{T \to \infty} \frac{1}{(2\pi)^2} \int_{-T/2}^{T/2} \int_{-T/2}^{T/2} \exp i \omega (t + t') \, dt \, dt' .
$$

(IV-2)

Now both $\delta(\omega)$ and $\delta^2(\omega)$ by their definitions assume non-zero values only near $\omega = 0$, so that the integral

$$
I = \int f(\omega) \, \delta^2(\omega) \, d\omega
$$

is to good approximation given by

$$
I = f(0) \int \delta^2(\omega) \, d\omega ,
$$

(IV-3)

where $f(\omega)$ is a well-behaved function and the range of integration includes the point $\omega = 0$.

If we use the definition of $\delta^2(\omega)$ as given by (IV-2), $I$ can be written

$$
I = \lim_{T \to \infty} \frac{f(0)}{(2\pi)^2} \int_d \int_{-T/2}^{T/2} \int_{-T/2}^{T/2} \exp i \omega (t + t') \, dt \, dt' .
$$

(IV-5)
Since the \( \omega \)-integrand is spiked at \( \omega = 0 \) with negligible contribution elsewhere, we can extend the limits of integration to encompass the region \( -\infty \leq \omega \leq +\infty \) with negligible error. If we do this and invert the order of integrations so that the \( \omega \) integration is done first, we obtain

\[
I = \lim_{T \to \infty} \frac{f(\omega)}{(2\pi)^2} \int_{-T/2}^{+T/2} \int_{-T/2}^{+T/2} \int_{-\infty}^{+\infty} dt \, dt' \, dt'' \, d\omega \, \exp i \omega(t + t') \quad \text{(IV-6)}
\]

From the definition of the \( \delta \)-function, Eq. (IV-1), this can be rewritten as

\[
I = \lim_{T \to \infty} \frac{f(\omega)}{(2\pi)^2} \int_{-T/2}^{+T/2} \int_{-T/2}^{+T/2} \delta(t + t') dt \, \delta(t) \quad \text{(IV-7)}
\]

By comparing Eqs. (IV-4) and (IV-7) we see that integrating \( \delta^2(\omega) \) over its resonance value is equivalent to integrating \( \frac{T}{2\pi} \delta(\omega) \) over the resonance, \( T \) being large compared to \( 1/\omega \).
APPENDIX V

In this appendix we show that by using as source terms in the linearized Maxwell-Vlasov equations currents and charges produced by the interactions of first order waves we obtain solutions for $E$, $B$, and $f$ which are identical with those obtained by solving the non-linear Vlasov equation to second order.

We first consider the non-linear Maxwell-Vlasov equations. When Fourier transformed in space and time, these equations are

$$i(\omega - k \cdot v) f_k, \omega + \frac{e}{m} \left\{ \left( E + \frac{v \times B}{c} \right) \cdot \frac{\partial f}{\partial v} \right\}_k, \omega = 0,$$  \hspace{1cm} (V-1)

$$k \cdot E_k, \omega = -4\pi i \rho_k, \omega,$$  \hspace{1cm} (V-2)

$$k \cdot B_k, \omega = 0,$$  \hspace{1cm} (V-3)

$$k \times E_k, \omega = \frac{\omega}{c} B_k, \omega,$$  \hspace{1cm} (V-4)

$$k \times B_k, \omega = -\frac{\omega}{c} E_k, \omega - \frac{4\pi i}{c} j_k, \omega.$$  \hspace{1cm} (V-5)

If $E_1$, $B_1$, and $f_1$ are solutions of the linearized version of these equations, the second order quantities $E_2$, $B_2$, and $f_2$ are given by

$$i(\omega - k \cdot v) f_{2k}, \omega + \frac{e}{m} E_{2k}, \omega \cdot \frac{\partial f_0}{\partial v} + \frac{e}{m} \left\{ \left( E_1 + \frac{v \times B_1}{c} \right) \cdot \frac{\partial f_1}{\partial v} \right\}_k, \omega = 0,$$  \hspace{1cm} (V-6)
V-2

(where we have taken \( f_0 \) to be isotropic) and the second order Maxwell equations (i.e., Eqs. (V-2) through (V-5) with \( E_{2k,\omega}, B_{2k,\omega}, \rho_{2k,\omega} \) and \( j_{2k,\omega} \) as variables).

If we solve equation (V-6) for \( f_{2k,\omega} \)

\[
 f_{2k,\omega} = \frac{e}{m} \int \left[ \frac{\left( E_1 + \frac{v \times B_1}{c} \right) \cdot \frac{\delta f_1}{\delta v}}{\omega - k \cdot v} \right] \frac{\delta f_0}{\delta v} \, d^3v
\]

and evaluate

\[
 \rho_{2k,\omega} = -n_0 e \int f_{2k,\omega} \, d^3v \tag{V-8}
\]

and

\[
 j_{2k,\omega} = -n_0 e \int v f_{2k,\omega} \, d^3v \tag{V-9}
\]

then

\[
 k \cdot E_{2k,\omega} = -\frac{\omega^2}{D_L(k,\omega)} \int \left[ \frac{\left( E_1 + \frac{v \times B_1}{c} \right) \cdot \frac{\delta f_1}{\delta v}}{\omega - k \cdot v + i\varepsilon} \right] \, d^3v \tag{V-10}
\]

and

\[
 k \times E_{2k,\omega} = \frac{\omega^2}{k_p^2 c^2 D_T(k,\omega)} \int \left[ \frac{\left( E_1 + \frac{v \times B_1}{c} \right) \cdot \frac{\delta f_1}{\delta v}}{\omega - k \cdot v + i\varepsilon} \right] \, d^3v \tag{V-11}
\]

We now look at the linearized Maxwell-Vlasov equations, incorporating the effects of the first order waves as source terms. The sources are just those charges and currents produced by solving the non-linear equation (V-6) without shielding. If we thus take for the source charges
and currents

\[ \rho_{s, k, \omega} = -n_0 e \int \tilde{f}_{k, \omega} d^3 v \]  

(V-12)

and

\[ j_{s, k, \omega} = -n_0 e \int v \tilde{f}_{k, \omega} d^3 v \]  

(V-13)

where \( \tilde{f}_{k, \omega} \) is obtained from the equation

\[
i(\omega - k \cdot v) \tilde{f}_{k, \omega} + \frac{e}{m} \left\{ \left( E_1 + \frac{v \times B_1}{c} \right) \cdot \frac{\partial f_1}{\partial v} \right\} \nabla_{k, \omega} = 0,\]  

(V-14)

and insert these into the linearized Maxwell-Vlasov equations, we obtain Eqs. (V-10) and (V-11) as solutions. (In Eq. (5.2) we have used the test currents given by Eq. (V-13) in the long wavelength limit.) If we further take \( f_2 \) to be the sum of \( \tilde{f} \) (from Eq. (V-14)) and the \( f \) produced by \( j_s \) and \( \rho_s \) (the solution of the linearized Vlasov equation) we see that the \( f_2 \) obtained in this manner is identical with the \( f_2 \) obtained from Eq. (V-6).