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ICTP 64/11

INTERNATIONAL ATOMIC ENERGY AGENCY

INTERNATIONAL CENTRE FOR THEORETICAL PHYSICS

U(12) AND BROKEN SU(6) SYMMETRY

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1964

PIAZZA OBERDAN

TRIESTE

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(Submitted to Physical Review Letters)

Trieste

17 December 1964

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§ 1. Starting with a spin $\frac{1}{2}$ quark model the most general algebraic structure is the U(12) ring of matrices $\gamma^{\mu}\tau^{i}$[1]. We wish to point out that this U(12) structure can be used to give a direct <u>covariant</u> formulation of the SU(6) symmetry of GÜRSEY, RADICATI and SAKITA[2], provided that for the physically realised multiplets one writes not only the composite field operators but also their conjugate momentum operators as "independent" components within the same multiplet. ~~The~~ motivation of our remark is as follows: a number of authors[3] have recently suggested that the SU(6) symmetry of ref. 2 may be looked upon as a non-covariant approximation to a symmetry $W(6) = U_L(6) \times U_R(6)$ which itself is a straightforward generalisation of the $U_L(2) \times U_R(2)$ symmetry associated with the <u>homogeneous</u> Lorentz group[4]. Starting with this, a number of examples of interaction Lagrangians invariant for W(6) have been written down.

Now there are serious difficulties in elaboration of these ideas. First, the right and left split of the basic quark implies that $m_q = 0$ and therefore W(6) must be badly broken. Second, physical particles correspond to representations of the inhomogeneous Lorentz group, and since kinetic energy terms are not invariant for $U_L(6) \times U_R(6)$ (in contrast to the Lorentz $U_L(2) \times U_R(2)$ case) it has so far been possible to develop theories of physical states at zero momenta[5] only. A third difficulty is related to the second; so long as there is no analogue of the inhomogeneous Lorentz group structure, it is impossible to assign physical particles unambiguously to the multiplets of W(6); thus baryon octet and decimet can belong equally to (56,1) + (1,56) or to (6,21) + (21,6).

For the 4-component Dirac equation, which includes the mass term, the passage to the inhomogeneous group is made in the well-known fashion by extending the sub-algebras $U_L(2) \times U_R(2)$ (with six generators $\sigma^{\mu\nu}$) to the full Dirac algebra U(4). This takes place essentially because U(4) contains in addition to the $\sigma^{\mu\nu}$, the four (translation-like) matrices γ^{μ}'s with commutation rules

$$[\gamma^{\mu}, \sigma^{\nu\lambda}] = 2i(g^{\mu\nu}\gamma^{\lambda} - g^{\mu\lambda}\gamma^{\nu}) \qquad (1)$$

allowing one to write equations invariant for the inhomogeneous group;

$$(\gamma^\mu p^\mu - m)\psi = 0 \quad , \quad w^\mu p^\mu \psi = 0$$

where $$w^\mu = \frac{i}{4}\gamma^5[\gamma^\mu, \gamma^\nu p^\nu] \quad . \qquad (2)$$

Note in passing that $i\gamma^\mu\gamma^5 w^\mu = \frac{3}{2}\gamma^\mu p^\mu$, so that the first equation may be written in the form

$$[\gamma^\mu(p^\mu + i\gamma^5 w^\mu) + \tfrac{1}{2}m]\psi = 0 \quad . \qquad (3)$$

What we wish to emphasise is that there is a close analogy between the group completion $U_L(2) \times U_R(2) \leftrightarrow U(4)$ and $U_L(6) \times U_R(6) \leftrightarrow U(12)$. The generators for $U_L(6) \times U_R(6)$ are the 72 matrices $\sigma^{\mu\nu}T^i$, T^i , $\gamma^5 T^i$. In addition to these U(12) contains another set of 72 matrices,

$$\Gamma^A = \gamma^\mu T^i \ , \quad \Gamma^{A5} = i\gamma^\mu\gamma^5 T^i \ ; \ A = (\mu\, i) \cancel{,\ i = 0,\ldots 8,\ \mu = 0,\ldots}$$

with the typical commutation rules (similar to (1)):-

$$[\gamma^\lambda T^i, \sigma^{\mu\nu}T^j] = i d^{ijk}(g^{\lambda\mu}\gamma^\nu - g^{\lambda\nu}\gamma^\mu)T^k + f^{ijk}\epsilon^{\lambda\mu\nu\rho}\gamma^\rho\gamma^5$$

$$[\gamma^\lambda\gamma^5 T^i, T^j] = i f^{ijk}\gamma^\lambda\gamma^5 T^k \quad .$$

Defining a 72-component vector (P^A, W^A) [6] once again one may write an "(inhomogeneous)W(6)" invariant equation

$$[\Gamma^A(P^A + i\gamma^5 W^A) + M]\psi = 0 \qquad (4)$$

Note that $\delta[P^A P^A - W^A W^A] = 0$ and also $\delta[p^\mu p^\mu - w^\mu w^\mu] = 0$ where $p^\mu = P^{\mu o}$, $w^\mu = W^{\mu o}$. It is worth stressing too that the Lagrangian mass term remains invariant as well[7].

It is perfectly possible now to write a covariant $U_L(6) \times U_R(6)$ S-matrix formalism, using the U(12) algebra in complete analogy with a Lorentz covariant formalism for spin $\frac{1}{2}$ particles which utilises the U(4) algebra. The chief problem is the passage to the physical limit of such S-matrix elements, the physical limit being defined[8] as $P^A \to p^\mu$, all other components of P and W vanishing. This last step will naturally break the $U_L(6) \times U_R(6)$ symmetry in a well-defined and determinate manner leaving a formalism which is fully Lorentz covariant. The symmetry breaking is well defined in the sense that we know precisely the transformation properties of the broken vector $(p^\mu, 0)$.

§ 2. Consider now the problem of higher representations. Starting from a single Dirac field γ^μ and a 4-component spinor one generates successively higher multiplets and their algebras by taking outer products[9]

$$\gamma^\mu_{(r)} = 1 \times 1 \times \ldots \times \gamma^\mu \times 1 \times \ldots \times 1 .$$

The first concrete example of this is the 4x4 representation of DUFFIN and KEMMER[10] with the associated algebra $\beta^\mu = \frac{1}{2}(\gamma^\mu \times 1 + 1 \times \gamma^\mu)$. This gives rise in the well-known manner to particles of spin one (10 components) and spin zero (5 components) within one multiplet. The crucial point is not that this is obviously the "natural" formalism for extension to U(6) ideas in that it combines zero spin and spin one; it is more, for by imposing the requirement that the field quantity satisfies a linear equation, Kemmer could show that the spin one field is composed of the potential A^μ as well as the field tensor[11,12] $F^{\mu\nu}$. Likewise the spin zero part consists of ϕ as well as its conjugate momentum $\partial_\mu \phi$. Altogether the spin decomposition is $16 = 10 \oplus 5 \oplus 1$. (The "1" does not correspond ~~to any~~ dynamical situation and is called the trivial representation of the algebra.)

The next algebra is generated by the matrices[13]

$$\alpha^\mu = \gamma^\mu \times 1 + 1 \times \beta^\mu$$

the reducible representations describing particles of spins $\frac{1}{2}$ and $\frac{3}{2}$. In a future paper this decomposition will be exhibited in detail; like for the case of the β-algebra, both the field operators as well as their conjugate momenta occur together in the description of a physical system.

The extension of the above to include unitary spin (passage from U(4) to U(12)) presents no essential complications though the formalism gets tedious as is well known from past experience of calculations involving for example β-formalism for mesons. But the compensations are two-fold; first the ambiguities of $U_L(6) \times U_R(6)$ assignments for the same physical multiplet are avoided[14]; the formalism incorporates them all in a specific manner. Second, using the methods of §1 a broken but covariant SU(6) formalism can readily be constructed. In practice since one is hardly ever going to work out the dynamics of particles of spins $> 3/2$; we hope one can set up the necessary formal machinery once and for all. This will be treated elsewhere.

Our thanks are due to Dr. M. A. Rashid for numerous helpful suggestions.

References and Footnotes

1. The nine T^i are the 3x3 Hermitian generators of U(3), while the sixteen Dirac γ's generate U(4).

2. F. GÜRSEY and L. A. RADICATI, Phys. Rev. Letters 13, 173 (1964).
B. SAKITA, Phys. Rev. (to be published).

3. A. SALAM, Phys. Letters (to be published).
R. DELBOURGO, A. SALAM and J. STRATHDEE, submitted to Phys. Rev.
R. P. FEYNMAN, M. GELL-MANN and G. ZWEIG, Phys. Rev. Letters 13, 678 (1964).
K. BARDAKCI, J. M. CORNWALL, P. G. O. FREUND and B. W. LEE, Phys. Rev. Letters 13, 698 (1964).
T. FULTON and J. WESS, preprint (Vienna).

4. Adopting Weyl's "unitary trick", one considers a pseudo Euclidian metric for space-time. For a Dirac particle the homogeneous Lorentz group is then generated by $U_L(2) \times U_R(2) \approx (1 \pm i\gamma^5)\sigma^{\mu\nu}$.

5. M. BEG and A. PAIS, submitted to Phys. Rev. We are grateful to the authors for sending us a preprint.

6. A 36-component vector of this type was first considered by FULTON and WESS (ref. 3). For the case we are considering, it is necessary however that the vector have 72 components.

7. In fact the mass term is invariant also for a full U(12) transformation

$$\delta\psi = i(\epsilon^j + \epsilon^{j5} + \epsilon^{j\mu}\gamma^\mu + i\epsilon^{j\mu 5}\gamma^\mu\gamma^5 + \tfrac{1}{2}\epsilon^{j\mu\nu}\sigma^{\mu\nu})\psi .$$

if all the ϵ are real.

8. We realise that the precise meaning of the Hilbert space operators $\mathcal{P}^A$ and $\mathcal{W}^A$ (with eigenvalues P^A and W^A) is, to say the least, obscure. It is interesting nonetheless to observe that the W^A occurring in eq. (4) could be identified with the generalised Pauli-Lubanski-Bargmann spin operators if one relates (in complete analogy

with the case of the normal Dirac equation (eq. (3)) W^A and P^A by the implicit equation

$$W^A(P) = \text{constant } X \; i\gamma_5 \left[\Gamma^A , \Gamma^B \left(P^B + i\gamma^5 W^B \right) \right] \qquad (5)$$

For the physical limit $P^A \to p^\mu$, eq. (5) solves to give

$$W^{\mu i}(p) \;\propto\; \gamma_5 \left[\gamma^\mu T^i , \gamma^\nu p^\nu \right]$$

Clearly, $p^\mu W^{\mu i}(p) = 0$ for all i , and when i = o, $W^{\mu o}(p)$ is just the normal spin pseudovector. More generally the 36 operators $W^{\mu i}(p)$ and T^i generate the little group $\mathcal{G}_6$ (or $U_W(6)$ of ref. 3). We have thus demonstrated that the solutions of the free equation $\left[\Gamma^A (P^A + i\gamma^5 W^A) + M \right] \psi = 0$ in what we have called the physical limit $P^A \to p^\mu$ can be labelled by the eigenvalues of p^μ, $W^{\mu i}(p)$ and T^i. Note that

$P^{\mu o} P^{\mu o} - W^{\mu o} W^{\mu o} =$ constant now implies with this interpretation of $W^{\mu o}$, the relation:-

$$p^\mu p^\mu = \text{constant} + j(j+1)$$

9. F. J. BELINFANTE, H. A. KRAMERS, J. K. LUBANSKI, Physica 8, 597 (1941).

10. R. J. DUFFIN, Phys. Rev. 54, 1114 (1938).
N. KEMMER, Proc. Roy. Soc. A, 173, 91 (1939).

11. For a Lie-group gauge theory $F^{\mu\nu}$ would correspond to the "generalised" conjugate momentum. For a Yang-Mills type of theory for example $\underline{F}^{\mu\nu} = \partial^\mu \underline{A}^\nu - \partial^\nu \underline{A}^\mu + 2\, \underline{A}^\mu \times \underline{A}^\nu$.

12. N. KEMMER, Helv. Phys. Acta, 23, 829 (1960), has stressed that both types of quantities A^μ as well as $F^{\mu\nu}$ are completely on par so far as dynamics is concerned and neither of the quantities is in any sense more "fundamental" than the other.

13. A detailed worked-out example of a slightly modified version of the above is the algebra generated by the matrices $\alpha^\mu = \gamma^\mu \times 1 + \gamma^5 \times \beta^1$ See HARISH-CHANDRA, Proc. Roy. Soc. 192, 195 (1947). This theory describes also spin $\frac{1}{2}$ and $\frac{3}{2}$ particles. The description of the spin $\frac{1}{2}$ particle is given essentially by the derivative of a spinor $\partial_\mu \psi$

rather than by a fundamental spinor ψ.

14. Note $12 \times \overline{12} = 143 \oplus 1 = 9 \times 10 + 8 \times 5 + 5$ (barring the 9 nine trivial components). Also,

$12 \times 12 \times 12 = 220 \oplus 2\,(572) \oplus 364$, where the $U_L(6) \times U_R(6)$ content of the multiplet is as follows,

$$220 = (20,1) + (15,6) + (6,15) + (1,20)$$
$$572 = (70,1) + (21,6) + (15,6) + (6,15) + (6,21) + (1,70)$$
$$364 = (56,1) + (21,6) + (6,21) + (1,56)$$