GAUGE APPROXIMATIONS IN MESODYNAMICS

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1964
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Trieste
December 1964
Abstract

In a non-perturbative theory for vector and scalar mesodynamics of the kind proposed by Salam it is found that the 'gauge approximations' introduced in that approach are the most sensible of a set of approximation schemes that preserve the scalar gauge invariance of the approximated unitarity equations. In showing this, it is found to be most convenient to work with one-photon irreducible and one-photon one-meson irreducible Green's functions. The generalised identities of the Ward-Takahashi kind that these Green's functions satisfy are exhibited, as well as the decomposition of the general n-point function in terms of these Green's functions.
A gauge technique has been proposed in recent papers (1,2) that gives a non-perturbative renormalisable theory of vector and scalar mesodynamics, which can be summarised as follows:

The scalar gauge character of the theory implies generalised Ward-Takahashi identities between \( \mathbb{M}^{(n)}(l) \) and \( \mathbb{M}^{(n)}(l) \) where \( \mathbb{M}^{(n)}(l) \) is an \( l \)-meson (incident mesons and \( l \) emitted mesons), \( n \)-photon Green's function. Given \( \mathbb{M}^{(1)}(l) \), \( \mathbb{M}^{(n+1)}(l) \) can be decomposed into longitudinal and transverse parts in such a way that the longitudinal part is a functional of \( \mathbb{M}^{(n)}(l) \) alone. Beginning with \( \mathbb{M}^{(n)}(l) \) for any \( l \), the following sequence can be constructed.

\[
\begin{align*}
\mathbb{M}^{(n)}(l) &= \mathbb{M}^{(n)}(l) + \mathbb{M}^{(n)}(l) \\
\mathbb{M}^{(2)}(l) &= \mathbb{M}^{(2)}(l) + \mathbb{M}^{(2)}(l) + \mathbb{M}^{(2)}(l) + \mathbb{M}^{(2)}(l), \\
\end{align*}
\]

etc.

In conventional notation \( \mathbb{M}^{(n)}(l) \equiv \Delta, \mathbb{M}^{(1)}(l) \equiv \Gamma, \mathbb{M}^{(2)}(l) \equiv C \).

If \( \mathcal{D} \) is the photon propagator the set of Green's functions forms a basis set, in that any Green's function can be expressed as a functional of these in a Dyson-Schwinger sense. Instead of this set, let the set of Green's functions \( \mathcal{D}, \{ \mathbb{M}^{(n)} \} \) be taken as the set from which all further Green's functions are to be constructed.

Having expressed all higher Green's functions in this way, the unitarity equations for \( \mathbb{M}^{(n)}(l) \) can be written in the form

\[
\mathbb{M}^{(n)}(l) = \mathcal{F}[\mathcal{D}, \Delta, \Gamma, C, \ldots, \mathbb{M}^{(m)}(l), \ldots].
\]  (1.2)

The zeroth gauge approximation is to take

\[
\Gamma = \Gamma^R[\Delta], \quad C = C^R[\Delta], \ldots.
\]  (1.3)
where $\Delta$ is obtained by explicitly solving the unitarity equations

$$\Delta = G[D, \Delta, \Gamma, C, \ldots, \gamma^{(m)}] \approx G[D, \Delta, \gamma^{(a)}[\Delta], C^{\alpha\alpha}[\Delta], \ldots]$$

$$= G[D, \Delta], \quad (1.4a)$$

$$D = H[D, \Delta, \Gamma, C, \ldots, \gamma^{(m)}] \approx H[D, \Delta, \gamma^{(a)}[\Delta], C^{\alpha\alpha}[\Delta], \ldots]$$

$$= H[D, \Delta], \quad (1.4b)$$

for $\Delta$ and $D$. (Solved by iteration beginning with one-meson many-photon unitarity in (1.4a), reducing (1.4a) to the form

$$\Delta = G[\Delta]$$

for example)

$$\Gamma^{BE}, \ C^{BE}, \ldots \text{ etc. are then determined by their unitarity equations as}$$

$$\Gamma^{BE} = -\Gamma^{A} + K[D, \Delta, \Gamma, C, \ldots] \approx -\Gamma^{A} + K[D, \Delta, \gamma^{(a)}, C^{\alpha\alpha}, \ldots]$$

$$(1.5a)$$

$$C^{\alpha\beta} = -C^{\alpha\alpha} + L[D, \Delta, \Gamma, C, \ldots] \approx -C^{\alpha\alpha} + L[D, \Delta, \gamma^{(a)}, C^{\alpha\alpha}, \ldots]$$

$$(1.5b)$$

etc.

In the next gauge approximation, the procedure is analogous, except that three equations have to be solved explicitly, those for $\Delta, D$, and $\Gamma$.
Symbolically, these equations are

$$\Delta = G \left[ D, \Delta, \nabla, C^{\alpha\alpha} + C^{\alpha\beta}, \ldots \right]$$ \hspace{1cm} (1.6a)

$$\mathcal{D} = \mathcal{H} \left[ D, \Delta, \nabla, C^{\alpha\alpha} + C^{\alpha\beta}, \ldots \right]$$ \hspace{1cm} (1.6b)

$$\mathcal{T} = \mathcal{K} \left[ D, \Delta, \nabla, C^{\alpha\alpha} + C^{\alpha\beta}, \ldots \right]$$ \hspace{1cm} (1.6c)

and corresponding terms for higher functions are then determined by their unitarity equations at

$$C^{\alpha\beta} = - (C^{\alpha\alpha} + C^{\alpha\beta}) + L \left[ D, \Delta, \nabla, C^{\alpha\alpha} + C^{\alpha\beta}, \ldots \right]$$ \hspace{1cm} (1.7)

etc.

Higher order gauge approximations are analogously defined. It is the purpose of this paper not to show under which conditions the solution of equations (1.4) to (1.7) is feasible, but to show how the n-point functions can be expressed in terms of the above Green's functions, to derive the identities that these Green's functions satisfy, to enumerate the approximation schemes whereby the approximated unitarity equations maintain their gauge invariance, and to show that this invariance is maintained by the gauge approximation schemes outlined above. These topics are dealt with in \( \frac{2}{5} \) 4, \( \frac{2}{5} \) 2, 3, and 5, and \( \frac{2}{6} \) 6 respectively. Since the motivation of this paper is a non-perturbative solution of \( \mathcal{N} \)-dynamics, the approach will be as far removed from perturbation theory as possible, although a perturbative approach like that of the \( \frac{2}{5} \)-limiting process \(^{(3)}\) would yield the identities in a much shorter space. For convenience the terminology of scalar mesodynamics will be used, since the generalisation of notation to the vector meson case is readily apparent.
§ 2. The n-point Functions

The identities satisfied by the connected parts of the n-point functions are first established. Introduce the operator-valued generating functional of time-ordered operator products

\[ \hat{T}\{J, \bar{\eta}, \eta\} = T \exp \left( i \int [ J_\mu(x) A_\mu(x) + \bar{\eta}(x) \phi(x) + \eta(x) \phi^*(x) ] dx \right), \tag{2.1} \]

where \( J_\mu(x), \bar{\eta}(x), \eta(x) \) are c-number functions of \( x \).

Define \( A_{\mu_1, \ldots, \mu_n}^\mu(x_1, \ldots, x_n) \) by

\[ A_{\mu_1, \ldots, \mu_n}^\mu(x_1, \ldots, x_n) = \frac{\delta^{(n-m-n)} \hat{T}}{\hat{T} \delta J_{\mu_1}(x_1) \cdots \delta \bar{\eta}(y_i) \cdots \delta \eta(z_i)} \tag{2.2} \]

and define \( \xi_{\mu_1, \ldots, \mu_n}^\mu(x_1, \ldots, x_n; y_1, \ldots, y_m; z_1, \ldots, z_m) \) by

\[ \xi_{\mu_1, \ldots, \mu_n}^\mu(x_1, \ldots, x_n; y_1, \ldots, y_m; z_1, \ldots, z_m) = (-i)^{n+2m} \left| \left. \left\langle A_{\mu_1, \ldots, \mu_n}^\mu(x_1, \ldots, x_n; y_1, \ldots, y_m; z_1, \ldots, z_m) \right| \right|_{\bar{\eta} = \eta = 0} \right. \tag{2.3} \]

The generator \( \hat{H} \) of the \( \rho \)-functions, the connected parts of the \( \xi \)-functions, is defined by

\[ \exp \hat{H} \{ J, \bar{\eta}, \eta\} = \left\langle T\{J, \bar{\eta}, \eta\} \right\rangle \tag{2.4} \]
Starting from a renormalised gauge-invariant Lagrangian a differential recursion relation can be obtained between the $\tau$-functions. This is that

\[ \left( \frac{\partial}{\partial x_\mu} \right) \rho_{\mu_1 \ldots \mu_n} \left( x \right) = \sum_{\nu_1 \ldots \nu_m} \rho_{\nu_1 \ldots \nu_m} \left( x \right) \rho_{\mu_1 \ldots \mu_n} \left( x \right) - \sum_{\nu_1 \ldots \nu_m} \rho_{\nu_1 \ldots \nu_m} \left( x \right) \rho_{\mu_1 \ldots \mu_n} \left( x \right) \]

where $\rho$ is the renormalised charge.

On the l.h.s. of equation (2.5) one uses the recursion formula (arising from $\tau_{x} = \hat{A}_{x} \tau_{x}$)

\[ \left( \frac{\partial}{\partial x_\mu} \right) \rho_{\mu_1 \ldots \mu_n} \left( x \right) = \rho_{\mu_1 \ldots \mu_n} \left( x \right) + \sum_{\text{comb}} \rho_{\mu_1 \ldots \mu_n} \left( x \right) \rho_{\mu_1 \ldots \mu_n} \left( x \right) \times \]

\[ \times \rho_{\mu_1 \ldots \mu_n} \left( x \right) \rho_{\mu_1 \ldots \mu_n} \left( x \right) \]

A corollary of (2.7) is that

\[ \left( \frac{\partial}{\partial x_\mu} \right) \rho_{\mu_1 \ldots \mu_n} \left( x \right) = 0, \quad \text{all } n \]
Taking the full meson and photon propagators as \( i\Delta \) and \(-iD_{\mu\nu}\) respectively, \( \rho \) can be expressed as

\[
\rho_{\mu\nu}(x_1, \ldots, x_n; \gamma_1, \ldots, \gamma_n; z_1, \ldots, z_n) = i^{n-1}(-1)^n \mathcal{A} \prod_i dx_i^1 \prod_i dx_i^2 \left( i\Delta(x_i - y_i^{(1)}) \prod_i (a_{\mu\nu}^{(2)}) \times \right.
\]

\[
\times D_{\mu\nu}(x_i - x_i') \mathcal{M}^{(a)}_{(\alpha)}(x_1, \ldots, x_n; \gamma_1, \ldots, \gamma_n; z_1, \ldots, z_n))
\]

(2.8)

(i.e. removing the propagators from \( \rho \) gives \( i(-e)^n \mathcal{M}^{(a)}_{(\alpha)} \)).

\( \mathcal{M}^{(a)}_{(\alpha)}(x_1, \ldots, x_n; \gamma_1, \ldots, \gamma_n; z_1, \ldots, z_n) \) has the Fourier transform

\[
\mathcal{M}^{(a)}_{(\alpha)}(p_1, \ldots, p_n; p_1', \ldots, p_n') \text{ with } \sum \mathbf{p}' = \sum \mathbf{p} + \sum \mathbf{k}
\]

where the incoming and outgoing mesons have charge + e and momenta \( \mathbf{p} \) and \( \mathbf{p}' \) respectively, and the photons have incoming momenta \( \mathbf{k} \) and polarisations \( \nu \).

C and CPT invariance give

\[
\mathcal{M}^{(a)}_{(\alpha)}(p_1, \ldots, p_n; \gamma_1, \ldots, \gamma_n; k_1, \ldots, k_n) = (-1)^n \mathcal{M}^{(a)}_{(\alpha)}(-p_1, \ldots, -p_n; \gamma_1, \ldots, \gamma_n; k_1, \ldots, k_n)
\]

\[
= \mathcal{M}^{(a)}_{(\alpha)}(-p_1, \ldots, -p_n; \gamma_1, \ldots, \gamma_n; k_1, \ldots, k_n)
\]

(2.9a)

In addition,

\[
\mathcal{M}^{(a)}_{(\alpha)}(p_1, \ldots, p_n, p_1', \ldots, p_n'; k_1, \ldots, k_n) = \mathcal{M}^{(a)}_{(\alpha)}(p_1, \ldots, p_n, p_1', \ldots, p_n'; k_1, \ldots, k_n)
\]

(2.9b)

and

\[
\mathcal{M}^{(a)}_{(\alpha)}(p_1, \ldots, p_n, p_1', \ldots, p_n'; k_1, \ldots, k_n) = \mathcal{M}^{(a)}_{(\alpha)}(p_1, \ldots, p_n, p_1', \ldots, p_n'; k_1, \ldots, k_n)
\]

(2.9c)

A similar expression exists for interchange of \( p_1', p_1 \), etc.
Taking Fourier transforms of (2.7), it is seen that the following relation holds for all \( n \) and \( l \) except \( l = 0 \), and \( n = 0, l = 1 \), for which the h.s. of (2.10) is not defined.

\[
\begin{align*}
\sum_i \Delta^{-1}(p \pm) \Delta (p \pm + k) M^{(n)}_{(\mu)}(p \pm; p \pm; \ldots; k, k, \ldots, k) &= \\
\sum_i \Delta^{-1}(p \pm) \Delta (p \pm - k) M^{(n)}_{(\mu)}(p \pm; p \pm; \ldots; k, k, \ldots, k).
\end{align*}
\tag{2.10}
\]

The case \( n = 0, l = 1 \), is the Ward – Takahashi identity

\[
k_{\mu} \gamma^{\mu} (p, p') = \Delta'(p') - \Delta'(p)
\tag{2.10a}
\]

A corollary of (2.10) is that for all mesons on the mass shell

\[
k_{\mu} M^{(n^+)}_{(\mu)}(\mu, \ldots; k, k, \ldots, k) \equiv 0
\tag{2.11}
\]

as required by gauge invariance.

It is necessary to consider the case \( l = 1 \) in detail.
§ 3. The Green's Functions of $\mathcal{M}^{(n)}_{(a)}$

Let the one-photon irreducible part of $\mathcal{M}^{(m)}_{(a)}$ be $\mathcal{M}^{(1)}_{(a)}$ (that is, all diagrams of $\mathcal{M}^{(m)}_{(a)}$ containing photon poles $\mathcal{B}(k)$, where $k$ is a sum of external photon momenta, are omitted). It follows from graphical considerations that $\mathcal{M}^{(1)}_{(a)}$ satisfies (2.10) for $l = 1$. Denoting the one-photon, one-meson irreducible vertex with $m$-photons and one incident and emitted meson by $i(-\epsilon)^m M^{(m)}_{(1)}$, $\mathcal{M}^{(1)}_{(a)}$ can be expressed in terms of $\mathcal{M}^{(n)}_{(a)}$, $n = 1, 2, \ldots, n$

$$\mathcal{M}^{(1)}_{(a)}(\mathbf{p}, \mathbf{p}', k_1, \ldots, k_n) = \sum_{\text{comb}} M^{(n)}_{(1)}(\mathbf{p}, \mathbf{p}_0; k_1, \ldots, k_n) \left[ -\Delta(\mathbf{p}_0) \right] M^{(a)}_{(a)}(\mathbf{p}_0, \mathbf{p}_0; k_1', \ldots, k_n') \times$$

$$\ldots \times \left[ -\Delta(\mathbf{p}_0) \right] M^{(a)}_{(a)}(\mathbf{p}_0, \mathbf{p}_0; k_1', \ldots, k_n')$$

(3.1)

where

$$\sum_i n_i = n$$

and

$$\mathbf{p}_r = \mathbf{p}_{r-i} + \sum_{k_i} k_i,$$

and where $k_1', \ldots, k_n'$ is the permutation of $k_1, \ldots, k_n$, the summation being taken over all possible partitions. (The photon indices have again been omitted for convenience).

The symmetry properties of $\mathcal{M}^{(n)}_{(1)}$ are the same as for $\mathcal{M}^{(n)}_{(a)}$ and

$$M^{(n)}_{(1)}(\mathbf{p}, \mathbf{p'}) = \mathcal{T}^I_{\mu}(\mathbf{p}, \mathbf{p'})$$

(3.2a)

$$M^{(2)}_{(1) \mu

\nu}(\mathbf{p}, \mathbf{p}', k, k') = \mathcal{C}_{\mu\nu}(\mathbf{p}, \mathbf{p}', k, k')$$

(3.2b)

in the notation of references 1 and 2.

In a perturbation sense, $\mathcal{M}^{(n)}_{(1)}$ is that function obtained by the insertion of $n$ photon vertices in all possible ways in a meson proper self-energy blob.
The identities satisfied by the $\mathcal{K}_i^{(n)}$ can now be obtained.

Firstly, the following definitions are needed.

Let

$$D^{(n)}_{\mu_1...\mu_n}(\mathcal{P}, \mathcal{P}', k_1...k_n)$$

when there can be no confusion, abbreviated to $D^{(n)}_{\mathcal{P}, \mathcal{P}'}$

 denote any diagrams which contain a single meson line of incoming momentum $\mathcal{P}$ and outgoing momentum $\mathcal{P}'$ with photon vertices of incoming momenta $k_1...k_n$ and polarizations $\mu_1...\mu_n$ in which all general vertices are one-meson one-photon irreducible (i.e. any term on the r.h.s. of (3.1)). Then $D^{(n)}_{\mu_1...\mu_n}(\mathcal{P}, \mathcal{P}', k_1...k_n)$ can be said to be symbolically of the form

$$D^{(n)}_{\mathcal{P}, \mathcal{P}'} = \left[ \langle n_1 \rangle \langle n_2 \rangle ... \langle n_s \rangle \right] ,$$

(3.3)

where

$$\sum_{i=1}^{s} n_i = n$$

For every such $D^{(n)}_{\mu_1...\mu_n}(\mathcal{P}, \mathcal{P}', k_1...k_n)$, $D^{(n+1)}_{\mu_1...\mu_n}(\mathcal{P}, \mathcal{P}', k, k_1...k_n)$

(abbreviated as $D^{(n+1)}_{\mu}(\mathcal{P}, \mathcal{P}', k, k$) ) is defined as the sum of the set of diagrams obtained from $D^{(n)}_{\mu_1...\mu_n}(\mathcal{P}, \mathcal{P}', k_1...k_n)$ by the insertion of a photon vertex of incoming momentum $k$ and polarization $\mu$ in all possible ways (i.e. the removal of a photon vertex $\mu$ from $D^{(n+1)}_{\mu}(\mathcal{P}, \mathcal{P}', k, k_1...k_n$) gives a set of diagrams topologically equivalent to $D^{(n)}_{\mathcal{P}, \mathcal{P}'}$). Fig. 1. is an example of this procedure (straight lines denote mesons and wavy lines denote photons).

The following Lemma then holds:

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Lemma

If, for \( n = 1, 2, \ldots, m \)

\[
k\mu M^{(n+1)}_{\mu, \mu_n} (\tau, q^* k; k, k, \ldots, k_n) = M^{(n)}_{\mu, \mu_n} (\tau, q^* k; k, k, \ldots, k_n) - M^{(n)}_{\mu, \mu_n} (\tau, q^* k; k, k, \ldots, k_n)
\]

then

\[
k\mu D^{(n)}_{\mu, \mu_n} (\tau, q^* k; k, k, \ldots, k_n) = e \Delta^{-1}(\tau k) \Delta(\tau) D^{(n)}_{\mu, \mu_n} (\tau, q^* k; k, k, \ldots, k_n) -
\]

\[
e D^{(n)}_{\mu, \mu_n} (\tau k, q^* k; k, k, \ldots, k_n) \Delta(\tau k) \Delta^{-1}(\tau)
\]

for all \( D^{(n)} \) for which \( n = 1, 2, \ldots, m \).

Proof. a) If \( D^{(n)}(\tau, \tau') \) is of the form \([n]_n\), then

\[
k\mu D^{(n)}_{\mu}(\tau, q^* k; k) = -i (-e)^{n+1} [k\mu M^{(n+1)}_{\mu, \mu_n} (\tau, q^* k) - k\mu M^{(n)}_{\mu, \mu_n} (\tau, q^* k) M^{(n)}_{\mu, \mu_n} (\tau, q^* k) -
\]

\[
- k\mu M^{(n)}_{\mu, \mu_n} (\tau, q^* k) \Delta(\tau) M^{(n)}_{\mu, \mu_n} (\tau, \tau') -
\]

\[
e \Delta^{-1}(\tau k) \Delta(\tau) D^{(n)}_{\mu}(\tau, q^* k) -
\]

\[
e D^{(n)}_{\mu}(\tau k, q^* k) \Delta(\tau k) \Delta^{-1}(\tau)
\]

for \( n = 1, 2, \ldots, m \).
b) If $D^{(n)}(p, p')$ is of the form $[c_8]...[c_n]$ where $1 < s < n$ assume that (3.5) holds for $n=1,2,...$ where $n < m$. Then $D^{(n+1)}(p, p')$ is of the form:

\[ D^{(n+1)}(p, p') = -\left(-e\right)^s M_{\mu_1}(c_1) \Delta(p''k) D^{(n+2)}_{\mu}(p''k,k) + e\left(-e\right)^s M_{\mu_1}(c_1) \Delta(p''k) D^{(n-1)}_{\mu}(p''k) \]

\[ \quad - e\left(-e\right)^s \sum_{\mu_1} \Delta(p''k) M_{\mu_2}(c_1) \Delta(p''k) D^{(n-1)}_{\mu_2}(p''k) \]

(3.8)

This gives:

\[ D^{(n+2)}_{\mu}(p''k,k) = -\left(-e\right)^s M_{\mu_1}(c_1) \Delta(p''k) D^{(n+1)}_{\mu}(p''k,k) \]

\[ + e\left(-e\right)^s M_{\mu_1}(c_1) \Delta(p''k) D^{(n)}_{\mu}(p''k,k) \]

\[ - e\left(-e\right)^s \sum_{\mu_1} \Delta(p''k) M_{\mu_2}(c_1) \Delta(p''k) D^{(n)}_{\mu_2}(p''k,k) \]

(3.9)

which, under the hypothesis made, gives

\[ kD^{(n+2)}_{\mu}(p''k,k) = eD^{(n+1)}(p''k) \Delta(p''k) \Delta(p) - eD^{(n)}(p''k) \Delta(p''k) \Delta(p) \]

(3.9)

Equation (3.9) is easily seen to be true for $D = [c_1, c_2]$ and hence the Lemma is true for D's of type b). Thus the Lemma is true. The following theorem can now be shown to be true.
Theorem 1

If $M_{(o)}^{(n+1)}$ is the one-photon one-meson irreducible part of $M_{(o)}^{(n)}$ where $M_{(o)}^{(n)}$ is defined as in (2.8), then

$$k_{\mu} M_{(o)\mu np_1...p_n}(p,k'_1,k_2...k_n) = M_{(o)\mu np_1...p_n}(p+k,p'k_1,k_2...k_n) -$$

$$- M_{(o)\mu np_1...p_n}(p,p'k_1,k_2...k_n)$$

for all $n > 0$.

Proof

The previous results are that

$$\Delta(p) k_{\mu} M_{(o)\mu np_1...p_n}(p+k,p'k_1,k_2...k_n) \Delta(p') =$$

$$= \Delta(p+k) M_{(o)\mu np_1...p_n}(p+k,p'k_1,k_2...k_n) \Delta(p') - \Delta(p) M_{(o)\mu np_1...p_n}(p,p'k_1,k_2...k_n) \Delta(p')$$

(3.11)

for all $n$,

and that if (3.10) holds for $n=1,2,...$ on then (3.5) holds for $n=1,2,...$.

Assume that (3.10) is true for $n=1,2,...$. Take $n=m+1$ in (3.11).

Using the decomposition (3.1) for $M_{(o)}^{(n+2)}$ and $M_{(o)}^{(n+1)}$ and the Lemma, (3.11) reduces, for this value of $n$, to the form

$$k_{\mu} M_{(o)\mu np_1...p_n}(p+k,p'k_1,k_2...k_n) - k_{\mu} \Gamma_{\mu}(p,p+k) M_{(o)\mu np_1...p_n}(p+k,p'k_1,k_2...k_n) -$$

$$- k_{\mu} \Gamma_{\mu}(p,p'k_1,k_2...k_n) M_{(o)\mu np_1...p_n}(p,p'k_1,k_2...k_n) =$$

$$= \Delta'(p+k) \Delta(p') M_{(o)\mu np_1...p_n}(p,p'k_1,k_2...k_n) -$$

$$- \Delta(p+k) \Delta'(p) M_{(o)\mu np_1...p_n}(p+k,p'k_1,k_2...k_n)$$

(3.12)
equivalent diagrammatically to Fig. 2. Ward's identity immediately gives

\[ k_\mu M^{(n-2)}_{\mu\nu..\mu_{n-2}}(p, p', k, k_1, k_2, ..., k_{n-1}) = M^{(n-1)}_{\mu\nu..\mu_{n-1}}(p, k, p', k_1, k_2, ..., k_{n-1}) - M^{(n-1)}_{\mu\nu..\mu_{n-1}}(p_1, p', k, k_1, k_2, ..., k_{n-1}). \]

Thus, if (3.10) holds for \( n=1,2,...,m \), it holds for \( n=m+1 \).

Equation (3.10) can easily be shown to be true for \( n=2 \). Hence the theorem is proved.

A corollary to the theorem is that (3.5) is true for all \( n > 0 \). If the meson is on the mass shell initially and finally, then

\[ k_\mu D^{(n)}_{\mu} (p, p', k, k_1) \equiv 0 \quad \text{for all} \quad n > 0. \]
8.4. The Decomposition of \( \Pi_{(2)}^{(n)} \)

The case \( l > 1 \) is now considered in detail, using further properties of functional derivatives.

For the sake of convenience the notation will be adopted that repetition of the same space-time variables implies integration over them, and that, where not expressed explicitly, the space-time coordinates \( x, \gamma, \zeta \) correspond to functional differentiation with respect to \( \zeta, \gamma, \zeta \), and \( \zeta \) respectively.

In addition the functions \( \hat{\beta} \) will be defined as the functional derivatives of \( \hat{\Pi} \{ T, \zeta, \gamma \} \) with respect to external sources so that

\[
\hat{\beta} \Bigg|_{\zeta, \gamma} = \beta.
\]

With this notation, it is seen that

\[
\hat{\beta}_{\mu \nu} (x, \zeta) = \left. \frac{\delta \hat{\Pi}}{\delta \hat{\beta}_{\mu \nu} (x, \zeta)} \right|_{\zeta} = \frac{\delta \hat{\beta}_{\mu \nu}}{\delta \hat{\beta}_{\mu \nu} (x, \zeta)}.
\]  
\tag{4.2}

Differentiating the identity

\[
\delta \hat{\rho} (y, \zeta) = -\hat{\rho} (y, \zeta') \delta \beta'' (x, \gamma') \hat{\beta} (y, \zeta)
\]  
\tag{4.3}

\( n \) times with respect to \( \hat{\beta}_{\mu} \) and using the relation (4.2) a decomposition of \( \hat{\Pi}_{\mu \nu \ldots \mu n} (x, \ldots, x; y, \zeta) \) is obtained in terms of

\[
\hat{\Pi}_{\mu \nu \ldots \mu n} (x, \ldots, x; y, \zeta)
\]

where

\[
\hat{\beta}_{\mu \nu \ldots \mu n} (x, \ldots, x) = \frac{\delta}{\delta \hat{\rho}_{\mu \nu} (x, \zeta)} \ldots \frac{\delta}{\delta \hat{\rho}_{\mu \nu} (x, \zeta)} \hat{\beta}'' (x, \zeta). \]  
\tag{4.4}
and
\[ \hat{M}_{o}^{(\alpha)}(x_1, \ldots, x_r; \gamma, z) = i(-e)^r \frac{\delta}{\delta \gamma} \left( \frac{\delta}{\delta z} \right)^r \hat{\rho}^{-1}(\gamma, z). \quad (4.5) \]

If
\[ \sigma_{\nu, \ldots, \nu}^{(s)}(x_1, \ldots, x_2) = \sigma_{\nu, \ldots, \nu}^{(s)}(x_1, \ldots, x_2) \bigg|_{\gamma = 0} \quad (4.6) \]

and
\[ M_{o}^{(\alpha)}(x_1, \ldots, x_r; \gamma, z) = M_{o}^{(\alpha)}(x_1, \ldots, x_r; \gamma, z) \bigg|_{\gamma = 0}, \quad (4.7) \]

then \( \sigma^{(s)} \) is the one-photon irreducible \( s \)-photon purely-photon vertex and \( M_{o}^{(\alpha)} \) is the one-meson one-photon irreducible vertex whose Fourier transform is defined in \( \xi^3 \).

Because the form of the identities is one of photon-vertex insertion there is no need to begin from the basic identity (4.3) in the construction of \( M_{o}^{(\alpha)} \) and it is sufficient to start with \( \hat{\rho}(\gamma', \ldots, \gamma_k; z_1, \ldots, z_k) \) which can be expressed as

\[ \hat{\rho}(\gamma_1, \ldots, \gamma_k; z_1, \ldots, z_k) = x_1 \rho(\gamma_1, z_1) \cdots \rho(\gamma_k, z_k) \cdots \rho(\gamma_k, z_k) \times \hat{M}_{o}^{(\alpha)}(\gamma_1, \ldots, \gamma_k; z_1, \ldots, z_k), \quad (4.8) \]

where
\[ M_{o}^{(\alpha)}(\gamma_1, \ldots, \gamma_k; z_1, \ldots, z_k) = M_{o}^{(\alpha)}(\gamma_1, \ldots, \gamma_k; z_1, \ldots, z_k) \bigg|_{\gamma = 0} \quad (4.9) \]

Differentiating each side of (4.8) \( n \) times with respect to \( \gamma \) and using (4.2), (4.4) and (4.5) an expansion of \( \hat{\rho}_{\mu-\nu}(x_1, \ldots, x_r; \gamma, \gamma' \ldots, z_1, \ldots, z_k) \)
(and hence an expansion of \( \hat{M}^{(a)}_{(\alpha)} \) by stripping off the external
propagators) is obtained as a sum of simply connected diagrams,
each one expressible in terms of \( \hat{M}^{(a)}_{(0)}(x_1, \ldots, x_r; \gamma, \bar{z}) \)
and \( \hat{M}^{(a)}_{(\alpha)}(x_1, \ldots, x_r; \gamma, \bar{z}) \),
where

\[
\hat{M}^{(a)}_{(\alpha)}(x_1, \ldots, x_r; \gamma, \bar{z}) = (-e)^{m} \sum_{\delta} \frac{1}{\delta \tilde{\rho}_{\alpha}(x)} \int \sum_{\delta} \hat{M}^{(a)}_{(\alpha)}(\gamma, \bar{z}, x) \]

(4.10)

Define \( \hat{M}^{(a)}_{(\alpha)}(x_1, \ldots, x_r; \gamma, \bar{z}) \) by

\[
\hat{M}^{(a)}_{(\alpha)}(x_1, \ldots, x_r; \gamma, \bar{z}) = \hat{M}^{(a)}_{(\alpha)}(x_1, \ldots, x_r; \gamma, \bar{z}) \bigg|_{\tau = \frac{q}{q} = 0}
\]

Diagramatically it is easily seen that differentiation with
respect to \( \tau_{\mu} \) corresponds to the insertion of an external photon
propagator in all possible ways in a given diagram and that
differentiation with respect to \( \rho_{\alpha}(x) \) corresponds to the insertion
of a single photon vertex. Thus \( \hat{M}^{(a)}_{(\alpha)} \) corresponds to the insertion
of \( n \) photon vertices in \( M^{(a)}_{(\alpha)} \) (Note that \( M^{(a)}_{(\alpha)} \equiv M^{(a)}_{(\alpha)} \)).

Consider the above decomposition in more detail for a general
\( \rho \)-function. In this decomposition the \( \rho \)-function is expressed
as a sum of topologically distinct sets of simply connected
diagrams, the number of diagrams in each set being just sufficient
to yield the necessary symmetries in the external indices. For
a given value of \( n \) and \( \alpha \) let these distinct sets be labelled by
the index \( i = 0, 1, 2, \ldots, N(n, \alpha) \) and let the sum of the diagrams
in the \( i \)th set be \( G_{(\alpha)}^{(a)} \), have the requisite symmetries in the external indices and

\[
\rho_{\mu_1, \ldots, \mu_n}(x_1, \ldots, x_n; \gamma, \bar{z}, \ldots, \bar{z}) = \sum_{x=0}^{N} G_{(\alpha)}^{(a)}(x_1, \ldots, x_n; \gamma, \bar{z}, \ldots, \bar{z}) \]

(4.12)
The value $i = 0$ is given to the set consisting of the single diagram in which all photon insertions have been made into $M^{(n)}_{(e)}$.

That is,

$$C_{\mu_{\nu_1}...\mu_n}^{(n)}(x_1,...,x_n;\gamma_1,...,\gamma_n;z_1,...,z_n) = i(-e)^n \prod \rho_{\mu_1}^{(\nu_1)}(x_1'x_1) \prod \rho_{\gamma_1}(\gamma_1\gamma_1') \prod \rho_{\zeta_1}(\zeta_1\zeta_1') \times$$

$$\times M^{(n)}_{(e)}(x_1';...;x_n';\gamma_1';...;\gamma_n';z_1';...;z_n').$$

(4.13)

Thus for $l > 1$, all $C^{(n)}_{\mu_{\nu_1}...\mu_n}$ for which $i \neq 0$ contain only $M^{(n)}_{(e)}$ for which $m < n$. 

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6.5. The Generalized Identities

Reverting to momentum space, the Fourier transforms of the $M_{\omega}^{(n)}$ defined in §4 satisfy the following theorem.

Theorem II

If $\tilde{M}_{\omega}^{(n)}(\mu_1, \ldots, \mu_n; k)$ is the Fourier transform of $M_{\omega}^{(n)}(\mu_1, \ldots, \mu_n; x_1, \ldots, z_n)$ as defined in §4, where $x, y, k$ denote incoming momenta of mesons with charge $e$, outgoing momenta of mesons with charge $e$, and incoming momenta of photons with polarization $\mu$, respectively, then

$$k_\mu \tilde{M}_{\omega}^{(n)}(\mu_1, \ldots, \mu_n; k_1, \ldots, k_n) = \sum_i \tilde{M}_{\omega}^{(n)}(\mu_1, \ldots, \mu_n; k_1, \ldots, k_i', \ldots, k_n) - \sum_j \tilde{M}_{\omega}^{(n)}(\mu_1, \ldots, \mu_n; k_1, \ldots, k_j, \ldots, k_n)$$

(5.1)

for all $n, l$ except $n = 0$, $l = 1$ for which the r.h.s. of (5.1) is not defined (in which case we have the Ward-Takahashi identity (2.10)).
Proof.

Since Theorem I is the statement of Theorem II for the case $l=1$, we need only consider $l>1$. It is immediately seen from (2.10) and (2.10a) that (5.1) is true for $n=1$, all $l$. The proof for general $n$ is by induction in $n$.

Suppose (5.1) is true for all $l>1$ and $n=1, 2, \ldots, n-1$ for some $n>2$. Consider the decomposition of

$$
\rho_{\mu_1 \mu_2 \ldots \mu_n}(\tau_1, \tau_2, \ldots, \tau_n) \quad \text{for arbitrary } l>1
$$

From Eq. (4.10)

$$
\rho_{\mu_1 \mu_2 \ldots \mu_n}(\tau_1, \tau_2, \ldots, \tau_n) = \sum_{i} C_{\tau_1 \mu_1 \mu_2 \ldots \mu_n}^{i} (\psi_{\mu_1} \psi_{\mu_2} \ldots \psi_{\mu_n})
$$

(5.2)

where

$$
C_{\tau_1 \mu_1 \mu_2 \ldots \mu_n}^{i} (\psi_{\mu_1} \psi_{\mu_2} \ldots \psi_{\mu_n}) = i^{n-1} (-1)^{n} \left( \prod_{i} D_{\mu_i}(\kappa_i) \right) \left( \prod_{i} A(\tau_i) \right) \left( \prod_{i} A(\kappa_i) \right) \times
$$

$$
\times M^{(n)}_{\mu_1 \mu_2 \ldots \mu_n}(\tau_1, \tau_2, \ldots, \tau_n, \kappa_1, \kappa_2, \ldots, \kappa_n)
$$

(5.3)

If, for $i \neq 0$, $\rho_{\mu_1 \mu_2 \ldots \mu_n}(\tau_1, \tau_2, \ldots, \tau_n, \kappa_1, \kappa_2, \ldots, \kappa_n)$ is one of the diagrams in the $i$th set, then it has the form

$$
\rho_{\mu_1 \mu_2 \ldots \mu_n}(\tau_1, \tau_2, \ldots, \tau_n, \kappa_1, \kappa_2, \ldots, \kappa_n) =
$$

$$
i(-i)^{n-1} \left( \prod_{i} (\Delta(\tau_i) \Delta'(\kappa_i)) \right) \left( \prod_{i} \Delta(\tau_i) \right) \times
$$

$$
\times M^{(n)}_{\mu_1 \mu_2 \ldots \mu_n}(\tau_1, \tau_2, \ldots, \tau_n, \kappa_1, \kappa_2, \ldots, \kappa_n)
$$

(5.4)
where $\Sigma \sigma_i + \Sigma \sigma'_i + r = m$, and $\omega_1, \ldots, \omega_r$ are some subset of $k_1, \ldots, k_m$ with $\gamma_1, \ldots, \gamma_r$ the corresponding polarizations.

The functions $\mathcal{D}^{(n)}(p, q)$ appearing on the r.h.s of Eq. (5.4) correspond to simply connected diagrams obtained from $\mathcal{D}^{(n)}(p, q)$ as defined in 2.3 for some $n_0 \leq n$, by grafting on strings of purely-photon vertices in such a way that the total number of external photon vertices is $n$. The relationship between $M^{(r)}(\alpha)$ and $M^{(s)}(\alpha)$ for some $n_0 \leq r < m$ is similarly defined.

With a little labour, using the lemma to Theorem I, Eqs. (2.7a), (2.10a) and the induction assumption, it is seen that $q_i^{\alpha(n)}$ satisfies the identity

\[
\begin{align*}
&ik_{\mu} D_{\nu}^{(r)}(k) q^{\alpha(n)}_{\mu_{\alpha_1} \cdots \mu_{\alpha_r}}(\nu_{\alpha_1} \cdots \nu_{\alpha_r} \mu_{\alpha_1} \nu_{\alpha_2} \cdots \nu_{\alpha_r} \cdots k_1, k_2, \ldots, k_m) = \\
&= \sum_{j} q^{\alpha(n)}_{\mu_{\alpha_1} \cdots \mu_{\alpha_r}}(\nu_{\alpha_1} \cdots \nu_{\alpha_r} \mu_{\alpha_1} \nu_{\alpha_2} \cdots \nu_{\alpha_r} k, k_2, \ldots, k_m) - \\
&- \sum_{j} q^{\alpha(n)}_{\mu_{\alpha_1} \cdots \mu_{\alpha_r}}(\nu_{\alpha_1} \cdots \nu_{\alpha_r} \mu_{\alpha_1} \nu_{\alpha_2} \cdots \nu_{\alpha_r} k, k_2, \ldots, k_m),
\end{align*}
\]  

(5.5)

where $q^{\alpha(n)}_{\mu_{\alpha_1} \cdots \mu_{\alpha_r}}(\nu_{\alpha_1} \cdots \nu_{\alpha_r} \mu_{\alpha_1} \nu_{\alpha_2} \cdots \nu_{\alpha_r} k, k_2, \ldots, k_m)$ is the sum of the set of diagrams obtained from

$g_{\alpha(n)}(\nu_{\alpha_1} \cdots \nu_{\alpha_r} \mu_{\alpha_1} \nu_{\alpha_2} \cdots \nu_{\alpha_r} k, k_2, \ldots, k_m)$ by the insertion of a single photon propagator of momentum $k$ and external polarization $\mu$ in all possible ways.

Thus

\[
\mathcal{C}_{\alpha(n)}(\nu_{\alpha_1} \cdots \nu_{\alpha_r} \mu_{\alpha_1} \nu_{\alpha_2} \cdots \nu_{\alpha_r} k, k_2, \ldots, k_m)
\]  
satisfies an identical identity for all $i \neq 0$, where

\[
\mathcal{C}_{\alpha(n)} = \sum_{\gamma \in G} q^{\alpha(n)}_{\mu_{\alpha_1} \cdots \mu_{\alpha_r}}.
\]  

(5.6)

From Eqs (2.8) and (2.10) it is seen that $\rho_{\alpha_{\alpha_1} \cdots \alpha_{\alpha_r}}(\nu_{\alpha_1} \cdots \nu_{\alpha_r} \mu_{\alpha_1} \nu_{\alpha_2} \cdots \nu_{\alpha_r} k, k_2, \ldots, k_m)$ satisfies the identity
for all \( n \).

Hence from (5.2) \( G_{\omega_{\mu_{1}}...\mu_{n}}(p_{1},...,p_{k},...,p_{k_{1}},...,k_{n}) \) satisfies (5.5). From the form of \( G_{\omega_{n}} \) as given in (5.3) it is seen that this implies that (5.1) holds for all \( \lambda \) and \( n = m \). Thus the theorem is proved.

A corollary to Theorem II is that Eq. (5.5) holds for all values of \( \lambda \) and \( m \). Removing the external propagators from \( q_{\lambda,\mu_{1}}...\mu_{n} \) and \( q_{\lambda,\mu_{1}}...\mu_{n} \) to give functions \( \lambda_{\omega_{\mu_{1}}...\mu_{n}} \) and \( \lambda_{\omega_{\mu_{1}}...\mu_{n}} \) defined by

\[
\begin{align*}
q_{\lambda,\mu_{1}}...\mu_{n}(p_{1},...,p_{k},...,p_{k_{1}},...,k_{n}) & \left( \prod_{l} D_{\omega_{\mu_{l}}}(k_{l}) \right) = \\
& = (-1)^{s}(-i)^{n} \frac{1}{1!} \Delta(p_{1}) \frac{1}{1!} \Delta(p_{n}) \lambda_{\omega_{\mu_{1}}...\mu_{n}}(p_{1},...,p_{k},...,p_{k_{1}},...,k_{n})
\end{align*}
\]

(5.8)

and

\[
\begin{align*}
\lambda_{\omega_{\mu_{1}}...\mu_{n}}(p_{1},...,p_{k},...,p_{k_{1}},...,k_{n}) & \left( \prod_{l} D_{\omega_{\mu_{l}}}(k_{l}) \right) = \\
& = (-1)^{s}(-i)^{n} \frac{1}{1!} \Delta(p_{1}) \frac{1}{1!} \Delta(p_{n}) \lambda_{\omega_{\mu_{1}}...\mu_{n}}(p_{1},...,p_{k},...,p_{k_{1}},...,k_{n})
\end{align*}
\]

(5.9)

it is seen that for all the mesons on the mass shell

\[
\begin{align*}
k_{\lambda,\omega_{\mu_{1}}...\mu_{n}}(p_{1},...,p_{k},...,p_{k_{1}},...,k_{n}) & \equiv 0
\end{align*}
\]

(5.10)

and can be taken as a definition of the gauge invariance of the theory.
The function $M^{(n)}_{(a)}$ can be decomposed (non-uniquely) as

$$M^{(n)}_{(a)} = M^{(n)}_{(a)A} + M^{(n)}_{(a)B}$$

so that (symbolically)

$$k_i M^{(n)}_{(a)A} = \sum_{in} M^{(n-1)}_{(a)} - \sum_{out} M^{(n-1)}_{(a)}$$

$$k_i M^{(n)}_{(a)B} = 0$$

For a given $\mu, \ell$ define $M^{(m,n-m)}_{(a)}$ by the decomposition

$$M^{(n)}_{(a)A} = \sum_{\mu=1}^{n} M^{(m,n-m)}_{(a)}$$

$$M^{(n)}_{(a)B} = M^{(0,n)}_{(a)}$$

where

$$k_i M^{(m,n-m)}_{(a)} = \sum_{\mu=1}^{n} M^{(m-1,n-m)}_{(a)} - \sum_{\mu=1}^{n} M^{(m-1,n-m)}_{(a)}$$

Let us consider the process in which $m$ mesons of charge $+e$ interact with $s$ photons to produce $m$ mesons in the final state. Scalar gauge invariance implies that the amplitude for this process will vanish if all mesons are on the mass shell and if one photon is longitudinal. The connected and disconnected parts of this amplitude, each corresponding to a term on the r.h.s. of (2.6) will individually satisfy this requirement. Let these parts be enumerated by the index $j$. If each such part of the amplitude is then decomposed in the manner of 6.4 it has been shown that each subset of diagrams is individually gauge invariant.
There is thus almost unlimited scope in setting up approximation schemes that maintain the gauge invariance of the approximated amplitude (or the T-part of this amplitude in the context of unitarity equations). There seems little point, however, in working with sets of diagrams that do not have the full symmetry in the external momenta. The smallest subsets of diagrams that have this symmetry are the \( \mathcal{G}_{\ell} \). If \( \mathcal{M}^{(n)}_{\ell} \), the \( r \)-approximation for \( \mathcal{M}^{(n)}_{\ell} \) is defined as

\[
\mathcal{M}^{(n)}_{\ell} = \sum_{\ell = 0}^{\infty} \mathcal{M}^{(n-\ell)}_{\ell} \quad \text{for} \quad r < n,
\]

\[
\mathcal{M}^{(n)}_{\ell} = \mathcal{M}^{(n)}_{\ell} \quad \text{for} \quad r \geq n,
\]

then an approximation scheme that maintains gauge invariance of the amplitude (after its decomposition as in (4.1)) is given by

\[
\mathcal{M}^{(n)}_{\ell} = (\mathcal{T})_{i,j} \mathcal{M}^{(n)}_{\ell}.
\]

The indices \( i,j \) denote that \( \mathcal{M}^{(n)}_{\ell} \) belongs to the set \( \mathcal{G}_{\ell} \) in the \( \mathcal{T} \)-term of the expansion of the amplitude into disconnected parts. For a given \( (i,j) \), \( r \) is only dependent on \( \ell \).

Since there is no spectral representation for \( \mathcal{M}^{(\ell)} \) with \( \ell > 1 \), such \( \mathcal{M}^{(\ell)} \) must be expanded in terms of sets of diagrams that have the symmetry of \( \mathcal{M}^{(0)} \) in the external momenta and contain the \( \mathcal{M}^{(n)}_{\ell} \) and purely photon vertices only. The \( \mathcal{M}^{(n)}_{\ell} \) are obtained from these diagrams by grafting on the external propagators, making \( n \) photon-propagator insertions in the manner previously described, and then amputating all external propagators.

It is apparent that any approximation for \( \mathcal{M}^{(\ell)} \) in terms of \( \mathcal{M}^{(n)}_{\ell} \) that has the required symmetry in the external indices will lead to approximations for \( \mathcal{M}^{(n)}_{\ell} \) denoted by \( \mathcal{M}^{(n)}_{\ell} \) that have the symmetries of \( \mathcal{M}^{(n)}_{\ell} \) and which satisfy the identities for \( \mathcal{M}^{(n)}_{\ell} \).

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i.e.,

\[ k_i M^{(n)}_{(i)} = \sum_{n-1} M^{(n-1)}_{(i)} - \sum_{\omega \leq k} M^{(n-1)}_{(\omega)} \]  \hspace{1cm} (6.7)

for all \( k_i \).

If the expansion of \( M^{(n)}_{(i)} \) in terms of \( M^{(n)}_{(i)} \) has been carried out, the approximation schemes that maintain the gauge invariance of the amplitude are of the form

\[ M^{(n)}_{(i)} = (r)_{ij} M^{(n)}_{(i)} \]  \hspace{1cm} (6.8)

There are no criteria for constructing a sensible \( r \) dependence on \( i \) and \( j \). If \( r \) is independent of \( i, j \) the approximation schemes are

\[ M^{(n)}_{(i)} = (r) M^{(n)}_{(i)} \]  \hspace{1cm} (6.9)

with \( r = 0, 1, 2 \ldots \), and these are the gauge approximations discussed in \( \text{2.1} \). These gauge approximations thus maintain the gauge invariance of the approximated unitarity equations.
Conclusion

In this paper, the Green's functions chosen were not the simplest ones that were available to us. This choice of Green's functions has been made for two reasons. Primarily it has been made because scalar gauge invariance is easily understandable in terms of photon-propagator insertion and the operation of such insertion is more transparent when working with such Green's functions than when working with the connected parts of the \( n \)-point functions. Secondly, the identities satisfied by these Green's functions have been shown to be of the form

\[
\sum_{i=1}^{\infty} M^{(\infty}_{(i)} = \sum_{\text{conn}} M^{(\infty}_{(i)} - \sum_{\text{ext}} M^{(\infty}_{(i)}
\]

without the presence of external propagators, thus simplifying the construction of the part of \( M^{(\infty}_{(i)} \) that is a functional of \( M^{(\infty}_{(i)} \), were \( M^{(\infty}_{(i)} \) known. In the present context, it is only the \( M^{(\infty}_{(i)} \) that concern us (although it is beyond the scope of this paper to decide how the decomposition of \( M^{(\infty}_{(i)} \) into the \( M^{(\infty}_{(i)} \) is to be made, and to what extent an approximation for such a decomposition may be a "good" approximation) and so this advantage cannot be fully used.

Although it has been shown that there is wider scope for approximations that maintain the gauge invariance of amplitudes with these Green's functions than with the \( n \)-point functions, this greater choice is essentially spurious, since there are no sensible reasons for invoking it. The only sensible approximation schemes within this framework are the gauge approximations discussed in \( \frac{25}{1} \).

The application of the same approach in general Lie gauge theories will lead to approximation schemes of similar form, but this is also beyond the scope of this work and will be the subject of a subsequent paper.
Acknowledgements

The author would like to thank Professor A. Salam and the International Atomic Energy Agency for hospitality at the International Centre for Theoretical Physics. In addition, the author would like to thank Professor A. Salam for suggesting the work, and for his encouragement.
References

1) A. Salam. Phys. Rev. 132, 1287 (1963)

2) A. Salam and R. Delbourgo. Phys. Rev. 135, 1398 (1964)


Fig 1.

\[ D^{(3)}(p, p') = \]

\[ D^{(4)}(p, p' \pm k, k) = \]

Fig 2.

\[ p' \]

\[ p \]

\[ (m+1) \text{ photon} \]

\[ (m+1) \text{ photon} = \]

\[ p' \]

\[ p \]

\[ p \pm k \]

\[ (m+1) \text{ photon} \]

\[ (m+1) \text{ photon} \]