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The Relativistic Structure of SU(6)

by

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It is shown that a relativistic basis for invariance under SU(6) exists only if the group structure is extended to $U^+(6) \otimes U^-(6)$ for any interaction terms. The notion of inhomogeneous extension $U^+_{\omega}(6) \otimes U^-_{\omega}(6)$ is introduced. This extension leaves the kinetic energy terms invariant, though it still does not provide a fully satisfactory theory.

THE RELATIVISTIC STRUCTURE OF SU(6)

1. Introduction

We wish to examine in this note the relativistic basis[1] of recent generalisations of Wigner's supermultiplet theory[2] to elementary particle physics. We start with the assumption that so far as the relativistic and internal-symmetry structures are concerned it is sufficient to start with an elementary multiplet of Dirac spinors -- elementary in the sense that it corresponds to the fundamental representation of the internal symmetry concerned. More specifically for the internal symmetry group U(3), this fundamental representation corresponds to a Dirac set of three (Sakata-like) quarks. Our assumption then amounts to saying that so far as group theory is concerned all particles can be considered as composed from Dirac quarks. There are three questions to be studied:

(A) The structure of the "algebras" formed from the Dirac matrices and the internal symmetry generators T^i.

(B) For which ones of these "algebras" are the kinetic energy and the mass terms in a free Dirac Hamiltonian invariant?

(C) The types of interaction Hamiltonians (if any) invariant for each "algebra".

Gk gilmrw

2. The Structure of the Combined Algebras

Given a set of hermitian Dirac matrices γ^A and internal symmetry generators [3] T^i, note that

$$[\gamma^A T^i, \gamma^B T^j] = \frac{1}{2}\{\gamma^A, \gamma^B\}[T^i, T^j] + \frac{1}{2}[\gamma^A, \gamma^B]\{T^i, T^j\} .$$

For the fundamental (n - fold) representation of any unitary group U(n), the $n \times n$ matrices T^i span the entire hermitian basis and therefore both $[T^i, T^j]$ and $\{T^i, T^j\}$ are expressible as the linear sums of the T^i's themselves. So is trivially the case also with the full set of the 16 Dirac matrices. Specialising to U(3) (*i.e.* for the nine matrices T^i, $i = 0, 1, \ldots, 8$), it is clear from the above that the 144 matrices $\gamma^A T^i$ ($A = 1, \ldots, 16$; $i = 0, 1, \ldots, 8$) in general provide the set of generators for a U(12) structure.

It is easy to see from the results for the anticommutators of the Dirac γ's given in the Appendix that the general U(12) group contains two U(6) sub-groups each generated by the 36 matrices,

$$\begin{array}{lll} \mathcal{U}^+(6) : & \frac{1}{2}(1+i\gamma_5)T^i , & \frac{1}{2}(1+i\gamma_5)\,\sigma_{\mu\nu}\,T^i \\ \mathcal{U}^-(6) : & \frac{1}{2}(1-i\gamma_5)T^i , & \frac{1}{2}(1-i\gamma_5)\,\sigma_{\mu\nu}\,T^i \end{array} \qquad (1)$$

The crucial remark is that since ($1 + i\gamma_5$)$\sigma_{\mu\nu}$ is a set of antisymmetric self-dual matrices, there are only three independent ones among these; and likewise for ($1 - i\gamma_5$)$\sigma_{\mu\nu}$ Clearly a U(6)-invariant parity-conserving theory must necessarily possess $\mathcal{U}^+ \leftrightarrow \mathcal{U}^-$ symmetry. $U^+(6)$ and $U^-(6)$ clearly are straight generalisations of $SU^+(2)$ and $SU^-(2)$ -- the two sub-groups into which the (Euclidean) group of rotations in 4-dimensions splits.

The relevant matrices are unimodular

for a Lorentz metric and one must therefore first resort to the "unitary trick" of Weyl, i.e., go to a Euclidean metric, generalise $U^{+}(2)$, $U^{-}(2)$ to $U^{+}(6)$ and $U^{-}(6)$ and then pass back to the Lorentz metric [4]. Adopting an obvious nomenclature we shall call $U^{\pm}(6)$ the homogeneous symmetry group in contrast to the inhomogeneous groups we consider in the next section.

"Algebras" of the second kind based on the inhomogeneous rather than the homogenous Lorentz group are generated if we combine general relativistic "spin operators" with the T^{i}'s. The "spin operators" are products of the Dirac Matrices with momentum; one example is the set of the Pauli-Lubanski operator (which in the rest-frame of a particle give its intrinsic spin),

$$w_{\mu} = \frac{1}{4}\epsilon_{\mu\nu\rho\kappa}\sigma_{\nu\rho}p_{\kappa} \quad , \quad (p_{\mu}w_{\mu}=0). \tag{2}$$

Since,

$$\begin{aligned} [w_{\mu}, w_{\nu}] &= i\,\epsilon_{\mu\nu\rho\kappa}\,p_{\rho}w_{\kappa}\,, \\ \{w_{\mu}, w_{\nu}\} &= 2\,(p_{\mu}p_{\nu} - p^{2}g_{\mu\nu}). \end{aligned} \tag{3}$$

A new U(6) algebra [4] is generated by the 36 quantities T^{i}, $w_{\mu}T^{i}$. Since w_{κ} (like $\sigma_{\mu\nu}$) commute with γ_{5}, it is also possible to set up the groups $\mathcal{U}^{\pm}_{w}(6)$ with the generators

$$\mathcal{U}^{\pm}_{w}(6) : \qquad \frac{1}{2}(1 \pm i\gamma_{5})T^{i} \;,\; \frac{1}{2}(1 \pm i\gamma_{5})w_{\mu}T^{i}$$

Now the w's are only one example of the general class of "spin-operators". Another operator has been described by Calogero [5] this is the tensor

$$\begin{aligned} w_{\alpha\beta} &= -i\,\epsilon_{\mu\nu\alpha\beta}\,\gamma_{5}\gamma_{\mu}p_{\nu} \\ w'_{\alpha\beta} &= \frac{1}{2}\epsilon_{\alpha\beta\gamma\delta}\,w_{\gamma\delta} = -i\gamma_{5}(\gamma_{\alpha}p_{\beta} - \gamma_{\beta}p_{\alpha}) \end{aligned}$$

For a free Dirac particle, Calogero shows that the "even" components of the w_μ and $w_{\mu\nu}$ (in the Foldy-Wouthuysen sense) represent respectively

$$w_b = E\sigma_b^l + m\sigma_b^t \quad , \qquad w_0 = p\cdot\sigma^l$$
$$\tfrac{1}{2}\epsilon_{abc}w_{bc} = m\sigma_a^l + E\sigma_a^t \quad , \qquad w_{a0} = p\times\sigma^t$$

Here σ^l and σ^t are the longitudinal (along p) and transverse components of spin σ . In the rest-frame therefore w_μ and $w_{\mu\nu}$ possess the same physical significance. In a future paper we hope to come back to the complete algebra of these spin operators.

3. Invariance of the Lagrangians

(A) The Free Lagrangian

The Pauli-Lubanski operator w_μ and Calogero operator $w_{\alpha\beta}$ possess the remarkable property that the Dirac operator D = $\gamma\cdot p - m$ commutes with these. Thus

$$\mathcal{L}_0 = \bar\psi(\gamma p - m)\psi$$

is invariant for $U_W(6)$ (with generators T^i , $w_\mu T^i$). Likewise defining $\psi_{L,R} = \frac{1}{2}(1\pm i\gamma_5)\psi$, the terms $\bar\psi_L\gamma p\psi_L$ and $\bar\psi_R\gamma p\psi_R$ are invariant for $U^\pm_W(6)$ (generators T^i, $w_\mu^\pm T^i$) respectively. This is of course not true of the mass term $m(\bar\psi_R\psi_L + \bar\psi_L\psi_R)$.

Consider now the algebras $U^\pm(6)$ generated by the T^i's and the Dirac matrices. The transformations [6]

$$\psi_L' = \left(1 + \tfrac{i}{2}\alpha^j_{\mu\nu}T^j\sigma_{\mu\nu} + i\alpha^j T^j\right)\psi_L$$
$$\psi_R' = \left(1 + \tfrac{i}{2}\beta^j_{\mu\nu}T^j\sigma_{\mu\nu} + i\beta^j T^j\right)\psi_R \qquad (4)$$

even when taken in conjunction with the Lorentz transformation

$$p'_\mu = p_\mu + \epsilon^0_{\mu\nu} p_\nu$$

do not leave the free Dirac Lagrangian invariant [7]. In fact

$$\delta(\bar{\Psi}_L \gamma_p \psi_L) = \alpha^i_{\mu\nu} \bar{\Psi}_L T^i (\gamma_\mu p_\nu - \gamma_\nu p_\mu) \psi_L$$

where the summation is to be carried over $i = 1$ to 8. The conclusion therefore is that so far as the free Lagrangian is concerned, the group $U_W(6)$ is the only one which leaves the Lagrangian invariant; $U^{\pm}_W(6)$ leaves the kinetic energy term unchanged but not the mass term, while for the covariant groups $U^{\pm}(6)$ (defined with Dirac matrices $\sigma_{\mu\nu}$ etc., rather than the spin matrix W_μ) the free Lagrangian possesses no specially desirable transformation character.

(B) <u>The Interaction Lagrangian</u>

It is at this stage that our difficulties start. It has so far appeared impossible to construct an interaction Lagrangian involving a product of a finite number of field operators which is invariant for $U_W(6)$ or $U^{\pm}_W(6)$. It would seem therefore that if U (6) is a relatively exact symmetry of nature, only an S-matrix type of theory can be constructed for it.

On the other hand for the group structures $U^{\pm}(6)$, even though the free Lagrangian is not invariant, one can write invariant interaction terms. For example,[8] in the Euclidian sense (with $\bar{\Psi} \equiv \psi^{+}$), the interaction part of the parity conserving Lagrangian

$$\bar{\Psi}(\gamma_p - m)\psi + (\bar{\Psi}\gamma_\mu T^i \psi)(\bar{\Psi}\gamma_\mu T^i \psi) - (\bar{\Psi}\gamma_\mu \gamma_5 T^i \psi)(\bar{\Psi}\gamma_\mu \gamma_5 T^i \psi) \quad (5)$$

is invariant for $U^{+}(6) \otimes U^{-}(6)$ transformations. There is a total of 72 currents of which only the SU(3) ninefold $\bar{\Psi}\gamma_\mu T^i \psi$ is conserved. The divergences of these currents are listed in the Appendix.

4. Invariance of $SU^{+}(3) \otimes SU^{-}(3)$ under SU(6)

The considerations of section 3 leave us with a dilemma. What is the U(6) group of Gürsey, Radicati and Sakita, if it is not $U_W(6)$? If we are willing to give up covariance of the group-structure (though of course not of the basic Lagrangian) a non-covariant sub-group of the structure $U^{+}(6) \otimes U^{-}(6)$ is provided by the 36 generators.

$$T^i , \quad \sigma_{ab} T^i \qquad (i, j = 0, 1, \ldots, 8 \quad ; \quad a, b = 1, 2, 3)$$

This structure coincides with the little-group $U_W(6)$ for the rest frame $\mathbf{p} = 0$. Thus the set of transformations,

$$\psi' = \left(1 + i\, 6^j T^j + \tfrac{i}{2}\, \epsilon^{j}_{ab}\, \sigma_{ab} T^j\right)\psi \qquad (6)$$

$$p_0' = p_0 \;, \quad p_a' = p_a + \epsilon^{0}_{ab}\, p_b \quad \text{(space-rotation)}$$

leaves the interaction $(\bar{\Psi}\gamma_\mu T^i \psi)(\bar{\Psi}\gamma_\mu T^i \psi) + (\bar{\Psi}\gamma_\mu \gamma_5 T^i \psi)(\bar{\Psi}\gamma_\mu \gamma_5 T^i \psi)$ as well so the mass term as well as $\psi^{+} p_0 \psi$ invariant, changing the kinetic energy term by

$$\delta \mathcal{L} = 2\epsilon^{i}_{ab}\, \bar{\Psi}\gamma_a p_b T^i \psi , \qquad (i = 1, \ldots, 8)$$

The differential conservation law does not hold for the 24 currents

$$\bar{\Psi}\gamma_\mu \sigma_{ab} T^i \psi \qquad i = 1, \ldots, 8 \quad ; \quad ab = 1, 2, 3 ,$$

Note quite generally that $\delta \mathcal{L} = i \sum \epsilon\, \partial_\mu j_\mu$.

Now the Lagrangian (5) postulated above is precisely the type of Lagrangian previously written down in a different connection - in connection with what has been called the $[SU(3)]_L \otimes [SU(3)]_R$ theory [9]. It is perhaps instructive to write the equations of the gauge version of this theory in detail as well as the transformations involved. We shall use the Lorentz metric and not adopt the trick of passing to the Euclidean space.

Start with the Lagrangian

$$\mathcal{L} = \frac{1}{2} \sum_{1,2} \left[\tfrac{1}{2} Z^i_{\mu\nu} (\partial_\mu Z^i_\nu - \partial_\nu Z^i_\mu + q f^{ijk} Z^j_\mu Z^k_\nu) - \tfrac{1}{4} Z^i_{\mu\nu} Z^i_{\mu\nu} - \tfrac{1}{2}\mu^2 Z^i_\mu Z^i_\mu \right] +$$

$$+ \bar{\psi}_L \gamma\cdot(p + q Z^i_1 T^i)\psi_L + \bar{\psi}_R \gamma\cdot(p + q Z^i_2 T^i)\psi_R - m \bar{\psi}\psi \quad , \tag{7}$$

where $Z^i_{1\mu}$ and $Z^i_{2\mu}$ are 18 gauge vector fields which can be expressed as sums and differences of vector and axial vector fields,

$$Z_1 = V + A \quad , \qquad Z_2 = V - A .$$

Now if we specialise to the Weyl representation of the γ-matrices the $U^+(6) \otimes U^-(6)$ transformation written in eq.(4) we have

$$\left.\begin{aligned} \psi'_L &= \left(1 + i\,\alpha^i T^i + i\,\underline{\alpha}^i\cdot\underline{\sigma}\, T^i\right)\psi_L \\ \psi'_R &= \left(1 + i\,\beta^i T^i + i\,\underline{\beta}^i\cdot\underline{\sigma}\, T^i\right)\psi_R \end{aligned}\right\} \tag{8}$$

or

$$\psi' = \left[1 + i\,(\epsilon^i + i\gamma_5 \eta^i) T^i + i\,\underline{\sigma}\cdot(\underline{\epsilon}^i + i\gamma_5 \underline{\eta}^i) T^i\right]\psi$$

with $\alpha = \epsilon + i\eta$, $\beta = \epsilon - i\eta$. Then it is easy to check that the Yukawa-like interaction terms in (7) are invariant providing

$$\left.\begin{aligned} Z^{i\,\prime}_{10} &= Z^i_{10} + f^{ijk}\,(\alpha^j Z^k_{10} + \underline{\alpha}^j\cdot \underline{Z}^k_1) \\ \underline{Z}^{i\,\prime}_1 &= \underline{Z}^i_1 + f^{ijk}\,(\alpha^j \underline{Z}^k_1 + \underline{\alpha}^j Z^k_{10}) - d^{ijk}\underline{\alpha}^j \times \underline{Z}^k_1 \end{aligned}\right\} \tag{9}$$

$$\left.\begin{aligned} Z^{i\,\prime}_{20} &= Z^i_{20} + f^{ijk}\,(\beta^j Z^k_{20} - \underline{\beta}^j\cdot \underline{Z}^k_2) \\ \underline{Z}^{i\,\prime}_2 &= \underline{Z}^i_2 + f^{ijk}\,(\beta^j \underline{Z}^k_2 - \underline{\beta}^j Z^k_{20}) - d^{ijk}\underline{\beta}^j \times \underline{Z}^k \end{aligned}\right\} \tag{10}$$

However the free Lagrangian changes by

$$\delta \mathcal{L}_0 = 2\,\overline{\Psi}_L \underline{\alpha}^i \times \underline{\gamma} \cdot \underline{p}\, T^i \psi_L + 2\,\overline{\Psi}_R \underline{\beta}^i \times \underline{\gamma} \cdot \underline{p}\, T^i \psi_R + + i m \left(\underline{\alpha}^i - \underline{\beta}^i\right) \cdot \left(\overline{\Psi}_R \underline{\sigma} T^i \psi_L - \overline{\Psi}_L \underline{\sigma} T^i \psi_R\right) + \tag{11}$$

+ meson terms

Clearly the mass term $\overline{\Psi}\psi$ is invariant only if the γ_5 containing part of the transformation vanishes ($\eta = 0$ or $\alpha = \beta$). Also as stated before, the kinetic energy term is at least invariant for the part of the transformation (8) corresponding to pure rotations, viz.

$$\psi \rightarrow \left(1 + i\,\epsilon^i T^i + i\,\underline{\epsilon}^0 \cdot \underline{\sigma}\right)\psi \quad , \quad \underline{p} \rightarrow \underline{p} + \underline{\epsilon}^0 \times \underline{p}$$

Note also that $Z_\mu Z_\mu$ is invariant in the Euclidean sense, i.e. $\delta(Z_0^2 + \underline{Z}^2) = 0$. We have omitted writing the meson equivalents of the fermion kinematic-energy in eq.(11) for the sake of brevity.

5. Conclusions

To summarise the situation in respect of combining the Lorentz with the internal symmetry groups, we succeed in writing down a complete field-theoretic formalism provided we extend the algebra of the homogeneous, and not the inhomogeneous Lorentz group. The operators ($1 \pm i\gamma_5$) $\sigma_{\mu\nu}$ correspond to the two independent (angular momentum) operators which generate the homogeneous Lorentz group.

The group structures $U^{+}(6) \otimes U^{-}(6)$ are a direct generalisation from $SU^{\pm}(2)$ to $U^{\pm}(6)$ of spinors of each kind. However we note that these generalisations do not leave the kinetic energy terms invariant. The physically significant group with 36 generators is then the generalisation of the homogeneous Lorentz group, this generalisation consisting of 36 generators of space rotations σ and the unitary transformations τ^{i}, $\sigma\tau^{i}$. For the rest frame of a single particle, this group coincides with $U_{W}(6)$, the "little group" of Gürsey, Radicati and Sakita.

The authors are deeply indebted to Prof. P. T. Matthews and Dr. J. Charap for stimulating discussions. They have developed the $U^{+}(6) \otimes U^{-}(6)$ formalism independently. After completion of this work the authors have also seen a preprint by M. A. B. Beg and A. Pais on the subject and a reference therein to a preprint by Lee, Cornwall, Freund and Bardakci.

APPENDIX

We list here the commutators and anticommutators that arise in the algebra of U(12). For the U(3) spin part we have

$$[T^i, T^j] = i f^{ijk} T^k \quad , \quad \{T^i, T^j\} = d^{ijk} T^k.$$

The f and d are the same as those of Gell-Mann (Phys. Rev. 125, 1067 (1962)) and $\lambda = \frac{1}{2} T$ establishes the correspondence with his notation.

For the Dirac algebra we use the 16 matrices

$$\gamma^A = \quad 1 \ , \ \gamma_\mu \ , \ \sigma_{\mu\nu} = \frac{i}{2}[\gamma_\mu, \gamma_\nu] \ , \ \sigma_{\mu 5} = i\gamma_\mu\gamma_5 \ , \ \gamma_5$$

for which $\overline{\psi}\gamma^A\psi$ is real. Then listing the results,

$$[1, \sigma_{\mu\nu}] = 0$$
$$\{1, \sigma_{\mu\nu}\} = 2\sigma_{\mu\nu}$$
$$[\gamma_\lambda, \sigma_{\mu\nu}] = 2i(g_{\lambda\mu}\gamma_\nu - g_{\lambda\nu}\gamma_\mu)$$
$$\{\gamma_\lambda, \sigma_{\mu\nu}\} = -2\,\epsilon_{\lambda\mu\nu\rho}\sigma_{\rho 5}$$
$$[\sigma_{\kappa\lambda}, \sigma_{\mu\nu}] = 2i(g_{\kappa\nu}\sigma_{\lambda\mu} + g_{\lambda\mu}\sigma_{\kappa\nu} - g_{\kappa\mu}\sigma_{\lambda\nu} - g_{\lambda\nu}\sigma_{\kappa\mu})$$
$$\{\sigma_{\kappa\lambda}, \sigma_{\mu\nu}\} = 2(g_{\kappa\mu}g_{\lambda\nu} - g_{\lambda\mu}g_{\kappa\nu}) - 2\,\epsilon_{\kappa\lambda\mu\nu}\gamma_5$$
$$\{\sigma_{\lambda 5}, \sigma_{\mu\nu}\} = -2\,\epsilon_{\lambda\mu\nu\rho}\gamma_\rho$$
$$[\sigma_{\lambda 5}, \sigma_{\mu\nu}] = 2i(g_{\lambda\mu}\sigma_{\nu 5} - g_{\lambda\nu}\sigma_{\mu 5})$$
$$[\gamma_5, \sigma_{\mu\nu}] = 0$$
$$\{\gamma_5, \sigma_{\mu\nu}\} = \epsilon_{\mu\nu\rho\lambda}\sigma_{\rho\lambda}$$

We also include here the currents which generate $U^{+}(6) \otimes U^{-}(6)$. Thus under (4) we have

$$\delta\mathcal{L} = \partial_\lambda \left[\frac{\partial \mathcal{L}}{\partial(\partial_\lambda \psi_L)} \delta\psi_L + \frac{\partial \mathcal{L}}{\partial(\partial_\lambda \psi_R)} \delta\psi_R \right] + \quad \text{meson contributions}$$

$$= -\partial_\lambda \Big[\bar{\psi}_L \gamma_\lambda \left(\tfrac{1}{2} i \alpha^j_{\mu\nu} \sigma_{\mu\nu} T^j + i \alpha^j T^j \right) \psi_L +$$

$$+ \bar{\psi}_R \gamma_\lambda \left(\tfrac{1}{2} i \beta^j_{\mu\nu} \sigma_{\mu\nu} T^j + i \beta^j T^j \right) \psi_R + \cdots \Big]$$

$$= -\tfrac{i}{2} \alpha^j_{\mu\nu} \partial_\lambda J^j_{L\mu\nu,\lambda} - \tfrac{i}{2} \beta^j_{\mu\nu} \partial_\lambda J^j_{R\mu\nu,\lambda} - i \alpha^j \partial_\lambda J^j_{L\lambda} - i \beta^j \partial_\lambda J^j_{R\lambda}$$

where

$$J^j_{L\mu\nu,\lambda} = \bar{\psi}_L \gamma_\lambda \sigma_{\mu\nu} T^j \psi_L + \cdots , \qquad J^j_{L\lambda} = \bar{\psi}_L \gamma_\lambda T^j \psi_L + \cdots$$

and similarly for J_R. Since the only change in the Lagrangian is that of the free part ,

$$\partial_\lambda J^j_{L\lambda} = \partial_\lambda J^j_{R\lambda} = 0 , \qquad j = 0, 1, \ldots, 8 .$$

$$\left. \begin{aligned} \partial_\lambda J^j_{L\mu\nu,\lambda} &= 2 \bar{\psi}_L (\gamma_\nu \partial_\mu - \gamma_\mu \partial_\nu) T^j \psi_L + \cdots \\ \partial_\lambda J^j_{R\mu\nu,\lambda} &= 2 \bar{\psi}_R (\gamma_\nu \partial_\mu - \gamma_\mu \partial_\nu) T^j \psi_R + \cdots \end{aligned} \right\} \quad j = 1, \ldots, 8$$

and $\partial_\lambda J^0_{R\mu\nu,\lambda} = \partial_\lambda J^0_{L\mu\nu,\lambda} = 0 \qquad \text{if} \quad m = 0.$

1. A. Salam, Phys. Letters, to be published.
   F. Gürsey and L. A. Radicati, Phys. Rev. Letters 13, 173 (1964).
   R. P. Feynman, M. Gell-Mann and G. Zweig, Phys. Rev. Letters 13, 678 (1964). Unfortunately, the authors did not have the benefit of seeing this paper while the work was in progress.

2. E. Wigner, Phys. Rev. 51, 106 (1937).

3. For details see A. Salam, Lectures in Theoretical Physics, Boulder, Colorado, 1959. (Interscience, New York, 1960).

4. Strictly speaking, this is only an algebra in the rest frame.

5. F. Calogero, Il Nuovo Cimento 10, 280 (1961).

6. Note that the factors $\frac{1}{2}(1 \pm i\gamma_5)$ which appeared in the generators $(1 \pm i\gamma_5)T^i$, $(1 \pm i\gamma_5)\sigma_{\mu\nu}T^i$ in (1) is now incorporated in the $\psi_{L,R}$ in writing (4).

7. If we extended the notion of p_μ to a 72-dimensional vector $p_\mu^{j\pm}$ transforming according to the $(1,35) \oplus (35,1)$ representation then the invariance of the free Lagrangian would be restored.

8. $$\delta(\bar{\psi}_L \gamma_\mu T^i \psi_L) = \frac{i}{2} \alpha^j_{\nu\lambda} \bar{\psi}_L \left[i(g_{\mu\nu}\gamma_\lambda - g_{\mu\lambda}\gamma_\nu) d^{ijk} T^k - i\epsilon_{\mu\nu\lambda\rho}\gamma_\rho f^{ijk} T^k \right] \psi_L$$

9. A. Salam and J. C. Ward, Il Nuovo Cim. 19, 167 (1961) and Phys. Rev. 136 B, 763 (1964).
   M. Gell-Mann, Phys. Rev. 125, 1067 (1962).
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