SOME ASPECTS OF NONCOMMUTATIVE INTEGRABLE SYSTEMS à LA MOYAL

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Abstract

Besides its various applications in string and D-brane physics, the non commutativity of space (-time) coordinates, based on the $*$-product, behaves as a more general framework providing more mathematical and physical information about the associated system. Similar to the Gelfand-Dickey framework of pseudo differential operators, the non commutativity à la Moyal applied to physical problems makes the study more systematic. Using these facts, as well as the backgrounds of Moyal momentum algebra introduced in previous works [25, 26], we look for the important task of studying integrability in the noncommutativity framework. The main focus is on the noncommutative version of the Lax representation of two principal examples: the noncommutative $sl_2$ KdV equation and the noncommutative version of Burgers systems. Important properties are presented.
1 Introduction

It has been known for several years that \((1 + 1)\)-dimensional integrable models \cite{1} in connection with conformal field theories \cite{2} and their underlying lower \((s \leq 2)\) \cite{3, 4, 5, 6, 7} and higher \((s \geq 2)\) \cite{8, 9} spin symmetries, play a central role in various areas of research. In this sense the Virasoro algebra \(W_2\) defines the second Hamiltonian structure for the KdV hierarchy \cite{10}, \(W_3\) rely to the Boussinesq \cite{11} and \(W_{1+\infty}\) is associated to the KP hierarchy \cite{12} and so on.

Note by the way that the methods developed for the analysis of integrable models can also be used to study the well known problem related to hyper-Kahler metrics building programme. This is an important question of hyper-Kahler geometry, which can be solved in a nice way in the harmonic superspace, see for instance \cite{13}.

In the Gelfand-Dickey framework of integrable systems, one usually introduces the following operators \cite{14}

\[
\mathcal{L}_n = \sum_{j \in \mathbb{Z}} u_{n-j} \partial^j,
\]

These pseudo-differential Lax operators, allowing both positive as well as nonlocal powers of the differential \(\partial^j\), are those used to establish the correspondence between KdV hierarchies and extended conformal symmetries. The fields \(u_s\) of conformal spin \(s\) can be related to general primary fields upon some covariantization procedure \cite{15}.

During the last several years, there has been a growth in the interest in non-commutative geometry (NCG), which appears in string theory in several ways \cite{16}. Much attention has been paid also to field theories on noncommutative (NC) spaces and more specifically Moyal deformed space-time, because of the appearance of such theories as certain limits of string, D-brane and M-theory \cite{17}. Non-commutative field theories emerging from string (membrane) theory stimulate actually a lot of important questions about the non-commutative integrable systems and how they can be described in terms of star product and Moyal bracket \cite{18, 19, 20, 21, 22, 23, 24, 25, 26, 27}.

Recall that in the Moyal momentum algebra\(^1\) the ordinary pseudo differential Lax operators (1) are naturally replaced by momentum Lax operators

\[
\mathcal{L}_n = \sum_{j \in \mathbb{Z}} u_{n-j} \star p^j,
\]

satisfying a non-commutative but associative algebra inherited from the star product. Note by the way that a consistent description of this Moyal momentum algebra and its application in

\(^1\)The Moyal momentum algebra, as a notion, was introduced first by the authors of \cite{21, 22}, and systematically studied later by the authors of \cite{25, 26} with some application to conformal field theory and noncommutative integrable models
integrable models and 2d-conformal field theories is presented in our previous works [25, 26].

On the same lines of our previous contributions, we try in the present work to provide new insights into integrable models and their associated Lax representation in the non-commutative geometry framework. The key step towards achieving this aim is via the Moyal momentum algebra that we note as \( \hat{\Sigma}(\theta) \). We will use our convention notation and the properties of the Moyal momentum algebra \( \hat{\Sigma}(\theta) \) to study some implications of this algebra in noncommutative integrable hierarchies and their Lax representation. We will concentrate on the noncommutative \( sl_2 \) KdV equation and the noncommutative version of the Burgers equation and their Lax representations. The derived properties may naturally be derived for the more general case namely the \( sl_n \) KdV hierarchy.

In fact, we actually know that the \( sl_n \)-momentum Lax operators play a central role in the study of integrable models and more particularly in deriving higher conformal spin algebras (\( w_\theta \)-algebras) from the \( \theta \)-extended Gelfand-Dickey second Hamiltonian structure [23, 25, 26]. Since they are also important in recovering 2d conformal field theories via the Miura transformation, we guess that it is possible to extend this property, in a natural way, to the non-commutative case and consider the \( \theta \)-deformed analogue of the well-known 2d conformal models namely: the \( sl_2 \)-Liouville field theory and its \( sl_n \)-Toda extensions and also the Wess-Zumino-Novikov-Witten conformal model.

This work is presented as follows: We give in section 2 our convention notations with a recall of the basic lines of the Moyal momentum algebra. Section 3 is devoted to a set up of the Lax pair representation of special noncommutative integrable systems namely the noncommutative KdV and Burgers systems and section 4 is for concluding remarks.

## 2 Basic Notions

This section will be devoted to a set up of the background that will be used in our analysis. For more details about the origin of the notation used here we refer to [25, 26].

1. We first start by recalling that the functions often involved in the two dimensional phase-space are arbitrary functions which we generally indicate by \( f(x, p) \) where the variable \( x \) stands for the space coordinate while \( p \) describes the momentum coordinate.

2. With respect to this phase space, we have to precise that the constants \( f_0 \) are defined such that

\[
\partial_x f_0 = 0 = \partial_p f_0. \tag{3}
\]
3. The functions $u_i(x,t)$ depending on an infinite set of variables $t_1 = x, t_2, t_3, ...$, are independent from the momentum coordinates which means

$$\partial_p u_i(x,t) = 0.$$ (4)

The index $i$, describes the conformal weight of the field $u_i(x,t)$. These functions can be considered in the complex language framework as being the analytic (conformal) fields of conformal spin $i = 1, 2, ...$.

4. Other objects usually used are the ones given by

$$u_i(x,t) \star p^j,$$ (5)

which are objects of conformal weight $(i + j)$ living on the non-commutative space parametrized by $\theta$. Through this work, we will use the following convention notations $[u_i] = i$, $[\theta] = 0$ and $[p] = [\partial_x] = -[x] = 1$, where the symbol $[ \quad ]$ stands for the conformal dimension of the used objects.

5. The star product law, defining the multiplication of objects in the non-commutative phase space, is given by the following expression

$$f(x, p) \star g(x, p) = \sum_{s=0}^{\infty} \sum_{i=0}^{s} \frac{\theta^s}{s!} (-1)^i c_s^i (\partial_x^i \partial_p^{s-i} f)(\partial_x^{s-i} \partial_p^i g),$$ (6)

with $c_s^i = \frac{s!}{i!(s-i)!}$.

6. The conventional Moyal bracket is defined as

$$\{f(x,p), g(x,p)\}_\theta = \frac{f \star g - g \star f}{2\theta},$$ (7)

where $\theta$ is the noncommutative parameter, which, considered as a constant in this approach.$^2$

7. To distinguish the classical objects from the $\theta$-deformed ones, we consider the following convention notations $[25, 26]$:

a) $\hat{\Sigma}_m^{(r,s)}$: This is the space of momentum differential operators of conformal weight $m$ and degrees $(r, s)$ with $r \leq s$. Typical operators of this space are given by

$$\sum_{i=r}^{s} u_{m-i} \star p^i.$$ (8)

$^2$The non constant $\theta$ parameter is shown to be important in the noncommutative geometry framework related to current subject of string theory and D-brane physics, see for instance [24] and references therein.
b) $\hat{\Sigma}_{m}^{(0,0)}$: This is the space of functions of conformal weight $m$; $m \in \mathbb{Z}$, which may depend on the parameter $\theta$. It coincides in the classical limit $\theta \rightarrow \theta_{l}$, with the ring of analytic fields involved into the construction of conformal symmetry and $W$-extensions.

c) $\hat{\Sigma}_{m}^{(k,k)}$: Is the space of momentum operators type,

$$u_{m-k} \ast p^{k}.$$  

(9)

d) $\theta$-Residue operation: $\widetilde{\text{Res}}$

$$\widetilde{\text{Res}}(f \ast p^{-1}) = f.$$  

(10)

8. The Moyal Momentum algebra

We denote this algebra in our convention notation by $\hat{\Sigma}(\theta)$. This is the algebra based on arbitrary momentum differential operators of arbitrary conformal weight $m$ and arbitrary degrees $(r,s)$. It is obtained by summing over all the allowed values of spin (conformal weight) and degrees in the following way:

$$\hat{\Sigma}(\theta) = \bigoplus_{r \leq s} \bigoplus_{m \in \mathbb{Z}} \hat{\Sigma}_{m}^{(r,s)}.$$  

(11)

$\hat{\Sigma}(\theta)$ is an infinite dimensional momentum algebra which is closed under the Moyal bracket without any condition. A remarkable property of this space is the possibility to introduce six infinite dimensional classes of momentum sub-algebras related to each other by special duality relations. These classes of algebras are given by $\hat{\Sigma}_{s}^{\pm}$, with $s = 0, +, -$ describing respectively the different values of the conformal spin which can be zero, positive or negative. The $\pm$ upper indices stand for the values of the degrees quantum numbers, for more details see [7, 25, 26].

9. Algebraic structure of The space $\hat{\Sigma}_{m}^{(r,s)}$:

To start, let us precise that this space contains momentum operators of fixed conformal spin $m$ and degrees $(r,s)$ of type

$$\mathcal{L}_{m}^{(r,s)}(u) = \sum_{i=r}^{s} u_{m-i}(x) \ast p^{i}.$$  

(12)

These are $\theta$-differentials whose operator character is inherited from the star product law defined as in eq.(6). Using this relation, it is now important to precise how the momentum operators act on arbitrary functions $f(x,p)$ via the star product.

Performing computations based on eq.(6), we find the following $\theta$-Leibnitz rules [25, 26]:

$$p^{n} \ast f(x,p) = \sum_{s=0}^{n} \theta^{s} c_{n}^{s} f^{(s)}(x,p) p^{n-s},$$  

(13)

$^{3}$ Usually the standard limit is taken such that $\theta_{limit} = 0$. In the present analysis, the standard limit is shifted by $\frac{1}{2}$ such that $\theta_{l} \rightarrow \theta_{limit} + \frac{1}{2}$. Thus taking the standard limit to be $\theta_{limit} = 0$ is equivalent to set $\theta_{l} = \frac{1}{2}$. The origin of this shift belongs to the consistent non commutative $W_{3}$-Zamolodchikov algebra construction [25, 26].
\[ p^{-n} \star f(x, p) = \sum_{s=0}^{\infty} (-)^s \theta^s c_n^{s-1} f^{(s)}(x, p) p^{-n-s} \]  

(14)

where \( f^{(s)} = \partial_x^s f \) is the prime derivative. We also find the following expressions for the Moyal bracket:

\[ \{p^n, f\}_\theta = \sum_{s=0}^{\infty} \theta^{s-1} c_n^s \left( \frac{1-(-)^s}{2} \right) f^s p^{n-s}, \]

\[ \{p^{-n}, f\}_\theta = \sum_{s=0}^{\infty} \theta^{s-1} c_n^{s+n-1} \left( \frac{(-)^s-1}{2} \right) f^s p^{-n-s}, \]

(15)

These equations do not contribute for even values of \( s \) as we can show in the following few examples

\[ \{p, f\}_\theta = f', \]

\[ \{p^2, f\}_\theta = 2f'p, \]

\[ \{p^3, f\}_\theta = 3f'p^2 + \theta^2 f''' \]

(16)

and

\[ \{p^{-1}, f\}_\theta = -f'p^{-2} - \theta^2 f'''p^{-4} - ... - \theta^{2k} f^{(2k+1)} p^{-2k-2} - ... \]

\[ \{p^{-2}, f\}_\theta = -2f'p^{-3} - 4\theta^2 f'''p^{-5} - ... - (2k + 2) \theta^{2k} f^{(2k+1)} p^{-2k-3} - ... \]

\[ \{p^{-3}, f\}_\theta = -3f'p^{-4} - 10\theta^2 f'''p^{-6} - ... - \frac{(2k+3)(2k+2)}{2} \theta^{2k} f^{(2k+1)} p^{-2k-4} - ... \]

(17)

Special Moyal brackets are given by

\[ \{p, x\}_\theta = 1 \]

\[ \{p^{-1}, x\}_\theta = -p^{-2} \]

(18)

Now, having derived and discussed some important properties of the Leibnitz rules, we can also remark that the momentum operators \( p^i \) satisfy the algebra

\[ p^n \star p^m = p^{n+m}, \]

(19)

which ensures the suspected rule

\[ p^n \star (p^{-n} \star f) = f \]

(20)

\[ (f \star p^{-n}) \star p^n = f. \]

3 Noncommutative Moyal Lax representation

The aim of this section is to present some results related to the Lax representation of noncommutative integrable hierarchies. Using the convention notations and the analysis presented previously and developed in [25, 26], we perform consistent algebraic computations, based on the Moyal-momentum analysis, to derive explicit Lax pair operators of some integrable systems in the noncommutative framework.
We underline that the present formulation is based on the (pseudo) momentum operators $p^n$ and $p^{-n}$ instead of the (pseudo) operators $\partial^n$ and $\partial^{-n}$ used in several works. We note also that the obtained results are shown to be compatible with the ones already established in literature [27].

Note by the way that the notion of integrability of the concerned nonlinear differential equations is defined in the sense that these equations may be linearizable.

To start, let us recall that the $sl_n$-Moyal KdV hierarchy is defined as

$$\frac{\partial L}{\partial t_k} = \{(L^k_\theta)_+, L\}_\theta. \tag{21}$$

Explicit computations related to these hierarchies are presented in [25, 26]. Working these hierarchies, we were able to derive, among others, for the $sl_2$ case up to the flow $t_9$, the following KdV-hierarchy equations

\begin{align*}
  u_{t_1} &= u', \\
  u_{t_3} &= \frac{3}{8} uu' + \theta^2 uu'', \\
  u_{t_5} &= \frac{15}{8} u^3 u' + 5\theta^2 (u'u'' + \frac{1}{2} uu'''') + \theta^4 u^{(5)}, \\
  u_{t_7} &= \frac{45}{16} u^3 u' + \frac{45}{8} \theta^2 (4 uu'u'' + u'^3 + u^2 uu'') + \frac{7}{2} (uu^{(5)} + 3 u'u^{(4)} + 5 uu'''') \theta^4 + \theta^6 u^{(7)},
\end{align*} \tag{22}

... Actually this construction which works well for the $sl_2$-KdV hierarchy is generalizable to higher order KdV hierarchies, namely the $sl_n$-KdV hierarchies.

The basic idea of the Lax formulation consists first in considering a noncommutative integrable system which possesses the Lax representation such that the following noncommutative Moyal bracket

$$\{L, T + \partial_t\}_\theta = 0, \tag{23}$$

is equivalent to the noncommutative differential equation that we consider from the beginning and that is nonlinear in general with $\partial_t \equiv \frac{\partial}{\partial t}$.

Equation (23) and the associated pair of operators $(L, T)$ are called the Lax differential equation and the Lax pair, respectively. The differential operator $L$ defines the integrable system which we should fix from the beginning.

Note that the way with which one writes the Lax equation as in (23) is equivalent to that in (21) namely

$$\{L, T + \partial_t\}_\theta \equiv \{L, (L^k_\theta)_+ + \partial_{t_k}\}_\theta = 0, \tag{24}$$
where the operator $T$ is the analogue of $(\mathcal{L}^\frac{k}{2})_+$ describing then an operator of conformal spin $k$.

This equation, written in terms of the function $u(x,t)$, is in general a non linear differential equation belonging to the ring $\hat{\Sigma}^{(0,0)}_{k+2}$. In the present case of $sl_2$-KdV systems we have $k = 3$.

As shown in [27], the meaning of Lax representations in noncommutative spaces would be vague. However, they actually have close connections with the bi-complex method [28] leading to infinite number of conserved quantities, and the (anti)-self-dual Yang-Mills equation which is integrable in the context of twistor descriptions and ADHM constructions [29, 30].

Now, let us apply the noncommutative Lax-pair generating technique. Usually, it is a method to find a corresponding $T$-operator for a given $L$-operator. Finding the operator $T$ satisfying (23) is not an easy job in the general case. For this reason, one has to make some constraints on the operator $T$ namely:

**Ansatz for the operator $T$:**

$$T = p^n \star L^n + T',$$

(25)

where $p^n$ are momentum operators acting on arbitrary function $f(x,p)$ as shown in section 2. Note by the way that the notation $T'$ have nothing to do with the prime derivative. With the previous ansatz, the problem reduces to that for the $T'$-operator which is determined by hand so that the Lax equation should be a differential equation belonging to the ring $\hat{\Sigma}^{(0,0)}$.

The best way to understand what happens for the general case, is to focus on the following examples:

**Example 1: The $sl_2$-noncommutative KdV system.**

The $L$-operator for the noncommutative KdV equation is given, in the momentum space configuration, by

$$L = p^2 + u(x,t),$$

(26)

with

$$L \in \hat{\Sigma}^{(0,2)}_{2}/\hat{\Sigma}^{(1,1)}_{2},$$

(27)

where $\hat{\Sigma}^{(1,1)}_{2}$ is the one dimensional subspace generated by objects of type $\xi_1(x,t) \star p$ and $\xi_1(x,t)$ is an arbitrary function of conformal spin 1.

Reduced to $n = 1 = m$, for the noncommutative $sl_2$ KdV system, the ansatz (25) can be
written as follows\(^4\)

\[ T = p \star L + T'. \tag{28} \]

The operator \( T \) in this case \((k = 3)\), is shown to behave as \((L^3)_+\) with \(\partial_3 \equiv \frac{\partial}{\partial u_3}\).

Simply algebraic computations give

\[ \{L, T'\}_\theta = u'p^2 - 2\theta u''p + \frac{\dot{u}}{2\theta} + (uu' + \theta^2 u'''), \tag{29} \]

Next, our goal is to be able to extract the Lax differential equation, namely, the noncommutative KdV equation. Before that, we have to make a projection of the operator \(\{L, T'\}_\theta\) on the ring \(\hat{\Sigma}^{(0,0)}\). This projection is equivalent to cancel the effect of the terms of momentum in (29), namely the term \(u'p^2 \) and \(2\theta u''p\). To do that, we have to consider the following property:

**Ansatz for** \( T' \):

\[ T' = X \star p + Y, \tag{30} \]

where \(X\) and \(Y\) are arbitrary functions on \(u\) and its derivatives.

Next, performing straightforward computations, with \(T' = Xp - \theta X' + Y\) lead to

\[ \{L, T'\}_\theta = 2X'p^2 + (\{u, X\}_\theta - 2\theta X'' + 2Y')p + (-Xu' - \theta \{u, X'\} + \{u, Y\}) \tag{31} \]

Identifying (29) and (31), leads to the following constrain on the parameters \(X\) and \(Y\)

\[ X = \frac{1}{2}u_2^2 + a, \tag{32} \]

\[ Y = -\frac{1}{2}\theta u_2' + b, \tag{33} \]

with the following nonlinear differential equation

\[ \frac{\dot{u}}{2\theta} = \frac{3}{2}uu' - \theta^2 u'''. \tag{34} \]

where the constants \(a\) and \(b\) are to be omitted for a matter of simplicity. The last equation is nothing but the \(sl_2\) KdV equation. This noncommutative equation contains also a non linear term \(\frac{3}{2}uu'\).

We have to underline that the \(sl_2\) noncommutative KdV equation obtained through this Lax method belongs to the same class of the KdV equation derived in [25, 26] namely

\[ \dot{u} = \frac{3}{2}uu' + \theta^2 u'''. \tag{35} \]

\(^4\)One can also introduce the following definition: \( T \equiv (p \star L)_s + T' \), with the convention \((p \star L)_s \equiv \frac{(p \star L + L \star p)}{2}\) describing the symmetrized part of the operator \( p \star L \).
In fact, performing successively the following scalings \( u_2 \rightarrow -u_2 \) and \( \partial_t \rightarrow -2\theta \partial_t \) we recover exactly (35). The term \( \frac{1}{\theta} \) appearing in (34) being the coefficient of the evolution part \( \dot{u}_2 \) of the NC KdV equation can be simply shifted to one due to consistency with respect to the classical limit \( \theta_1 \sim \frac{1}{2} \).

To summarize, the momentum Lax pair operators, associated to the noncommutative \( sl_2 \)-KdV system, are explicitly given by

\[
L_{KdV} = p^2 + u_2(x,t),
\]

and

\[
T_{KdV} = p^3 + \frac{3}{2} p \star u_2(x,t) - \frac{3}{2} \theta u'_2(x,t); \tag{37}
\]

with \( T' = \frac{1}{2} p \star u_2(x,t) - \frac{3}{2} \theta u'_2(x,t) \).

Note that, the same results can be obtained by using the Gelfand-Dickey (GD) formulation based on formal (pseud) differential operators \( \partial^{\pm n} \) instead of the Moyal momentum ones.

This first example shows, among others, the consistency of the Moyal momentum formulation in describing integrable systems and the associated Lax pair generating techniques in the same way as the successful GD formulation [7, 8].

**Example 2: The Noncommutative Burgers Equation à la Moyal**

Let us apply the same noncommutative Lax techniques, presented previously, to derive the noncommutative version of the Burgers equation. Actually, our interest in this equation comes from the several important properties that are exhibited in the commutative case. Before going into applying the noncommutative Lax techniques, let us first recall some few known properties of the standard Burgers equation.

**P1:** The Burgers equation is defined on the \((1 + 1)\)-dimensional space time. In the standard pseudo-differential operator formalism, this equation is associated to the following L-operator

\[
L_{Burgers} = \partial_x + u_1(x,t) \tag{38}
\]

where the function \( u_1 \) is of conformal spin one. Using our convention notations, we can set \( L \in \hat{\Sigma}^{(0,1)} \).

**P2:** With respect to the previous L-operator, the non linear differential equation of the Burgers equation is given by

\[
\dot{u}_1 + \alpha u_1 u'_1 + \beta u''_1 = 0, \tag{39}
\]

where \( \dot{u} = \frac{\partial u}{\partial t} \) and \( u' = \frac{\partial u}{\partial x} \). The dimensions of the underlying objects are given by \([t] = -2 = -[\partial_t], [x] = -1 \) and \([u] = 1\).
**P3:** On the commutative space-time, the Burgers equation can be derived from the Navier-Stokes equation and describes real phenomena, such as the turbulence and shock waves. In this sense, the Burgers equation draws much attention amongst many integrable equations.

**P4:** It can be linearized by the Cole-Hopf transformation [31]. The linearized equation is the diffusion equation and can be solved by Fourier transformation for given boundary conditions.

**P5:** The Burgers equation is completely integrable [32].

Now, we are ready to look for the noncommutative version of the Burgers equation. For that, we consider the $L$-operator of this equation in the Moyal momentum language, namely

$$ L = p + u_1(x, t). \tag{40} $$

dealing, as noticed before, to the space $\hat{\Sigma}_1^{(0,1)}$. This is a local differential operator of the generalized $n$-KdV hierarchy’s family ($n = 1$), obtained by a truncation of a noncommutative pseudo momentum operator of KP hierarchy type

$$ L = p + u_1(x, t) + u_2(x, t) \ast p^{-1} + u_3(x, t) \ast p^{-2} + ..., \tag{41} $$
of the space $\hat{\Sigma}_1^{(-\infty,1)}$. The local truncation is simply given by

$$ \hat{\Sigma}_1^{(-\infty,1)} \rightarrow \hat{\Sigma}_1^{(0,1)} \equiv [\hat{\Sigma}_1^{(-\infty,1)}]_+ \equiv \hat{\Sigma}_1^{(-\infty,1)}/\hat{\Sigma}_1^{(-\infty,-1)}, \tag{42} $$
or equivalently

$$ L_1(u_i) = p + \sum_{i=0}^{\infty} u_i \ast p^{1-i} \rightarrow p + u_1 \equiv [L_1(u_i)]_+, \tag{43} $$
where the symbol $(X)_+$ defines the local part (only positive powers of $p$) of a given pseudo operator $X$.

The noncommutative Burgers equation is said to have the Lax representation if there exists a suitable pair of operators $(L, T)$ so that the Lax equation

$$ \{p + u_1, T + \partial_t\}_\theta = 0, \tag{44} $$
reproduces the noncommutative version of the Burgers non linear differential equation. Following the same steps developed previously for the $sl_2$ noncommutative KdV systems, we consider the following ansatz for the operator $T$:

$$ T = p \ast L + T', \tag{45} $$
or

$$ T = p^2 + u_1 p + \theta u'_1 + T'. \tag{46} $$
Then, performing straightforward computations, the noncommutative Burgers Lax equation (44) reduces to
\[
\{p + u, T'\}_\theta = u'p + (uu' - \theta u'' + \frac{\dot{u}}{2}\theta)
\]  
(47)

Next, one has also to go through a constraint equation for the operator \(T'\), namely the Ansatz for \(T'\):
\[
T' = A \star p + B,
\]
(48)
where \(A\) and \(B\) are arbitrary functions for the moment. With this new ansatz for \(T'\), we have
\[
\{p + u, T'\}_\theta = (A' + \{u, A\}_\theta)p + (-Au' - \theta A'' - \theta\{u, A\}_\theta + B' + \{u, B\}_\theta)
\]  
(49)

Identifying (47) and (49) leads to the following constraints equations
\[
u' = A' + \{u, A\}_\theta
\]
(50)
and
\[
(u + A)u' + \frac{\dot{u}}{2\theta} = B' + \{u, B\}_\theta + \theta\{A', u\} + \theta(u'' - A'')
\]
(51)
A natural solution of the first constraint equation (50) is \(A = u\). This leads to a reduction of (51) to
\[
2uu' + \frac{\dot{u}}{2\theta} = B' + \{u, B\}
\]
(52)

Actually this is the noncommutative Burgers equation, which is also the projection of the Lax equation (44) to the ring of vanishing degrees in momenta namely the space \(\hat{\Sigma}_{1}^{(0, 0)}\).

Since \(\{u, \partial_t\}_\theta = -\frac{\dot{u}}{2\theta}\), a non trivial solution of the parameter \(B\) in equation (52) consists in setting \(B \equiv u^2 - \frac{\partial}{\partial t}\). But, since this nontrivial solution of \(B\) masks the noncommutative Burgers equation, it is a non desirable thing.

Remarking also that \([B] = 2\), we use these dimensional arguments and set
\[
B = \xi u' + \eta u^2
\]
(53)
where \(\xi\) and \(\eta\) are arbitrary coefficient numbers. Injecting this expression into (52) gives the final expression of the noncommutative Burgers equation namely
\[
\frac{\dot{u}}{2\theta} + 2(\xi - 1)uu' + \eta u'' = 0
\]
(54)
whose Lax pair in the noncommutative Moyal momentum formalism are explicitly given by
\[
L_{Burgers} = p + u_1(x, t)
\]
(55)
and
\[
T_{Burgers} = p^2 + 2u_1(x, t)p + \eta u_1^2(x, t) + \xi u_1'(x, t)
\]
(56)
4 Concluding Remarks

To summarize the main lines of the present work, let us present some important remarks:

1. We have presented a systematic study of noncommutative generating Lax pair operators in the Moyal momentum framework. The essential result deals with the derivation of the noncommutative KdV and Burgers systems.

2. Concerning the noncommutative derived KdV system, this is an integrable model due to the existence of a noncommutative Lax pair operators \((L, T)\). This existence is an important indication of integrability, but we guess that the realistic source of integrability of this model is the underlying conformal symmetry, shown to play a similar role as in the commutative case.

3. Concerning the \(\theta\) parameter appearing in the noncommutative KdV equation, we already mentioned before that its classical limit is not the same as the standard one corresponding usually in taking \(\theta_{\text{limit}} = 0\). This is because the KdV hierarchy systems, in general, deal with the conformal symmetry and its \(W_s\)-extensions describing non trivial Lie algebras. Recall that in the language of 2d-conformal field theory, the above mentioned currents \(W_s\) are taken in general as primary satisfying the OPE [2]

\[
T(z)W_s(\omega) = \frac{s}{(z-\omega)^2}W_s(\omega) + \frac{W'_s(\omega)}{(z-\omega)}.
\]

As we are interested in the \(\theta\)-deformation case, we have to add that the spaces \(\Sigma_{k(0,0)}\) are \(\theta\)-depending and the corresponding \(W_{\theta}\)-algebra is shown to exhibit new properties related to the \(\theta\) parameter and reduces to the standard \(W\)-algebra once some special limits on the \(\theta\) parameters are performed.

As an example, consider for instance the \(W^2_{\theta}\)-algebra [23, 25] generalizing the Zamolodchikov algebra [8, 9]. The conserved currents of this extended algebra are shown to take the following form

\[
\begin{align*}
    w_2 &= u_2 \\
    w_3 &= u_3 - \theta u'_2
\end{align*}
\]

which coincides with the standard case once \(\theta = \frac{1}{2}\).

4. Concerning the noncommutative Burgers equation (54) that we consider in the second example, it’s also an integrable equation whose Lax pair operators are explicitly derived (55-56). Note for instance that the Burgers Lax operator \(L_{\text{Burgers}} = p + u_1(x, t)\) is a momentum operator belonging to the space \(\hat{\Sigma}_1^{(0,1)}\).

5. As a first checking of integrability for the noncommutative Burgers system, we proceeded to an explicit derivation of the Lax pair operators \((L, T)\) giving the following differential equation

\[
2uu' + \frac{u}{2\theta} = B' + [u, B].
\]

The idea is to solve this equation in terms of the coefficient parameter \(B\) such that it can reduce to the non linear Burgers equation belonging to the space \(\hat{\Sigma}_3^{(0,0)}\). Solving this equation gives explicitly the requested Lax operator. Presently, we have two possible solutions:
The first solution:

\[ B_1 = u^2 - \partial_t, \]  

(59)
defines the parameter \( B \) in terms of time derivative and is not suitable for us because it masks the nonlinear differential equation that we are looking for.

The second solution:

\[ B_2 = \xi u' + \eta u^2 \]  

(60)
which is consistent with dimensional arguments maps us to the noncommutative Burgers equation (54).

6. We should also underline that the importance of this study comes also from the fact that the results obtained in the framework of Moyal momentum are similar to those obtained by using the Gelfand-Dickey pseudo operators approach.

7. Finally, a significant question is to know if there is a possibility to establish a correspondence between the two Systems. The reason is that for the noncommutative KdV model, the problem of integrability does not arise in the same way as it does for the noncommutative Burgers equation.

The first one is mapped to conformal field theory through the Liouville model. This is in fact a strong indication of integrability in favour of noncommutative KdV equation which could help to understand more the noncommutative Burgers system.

Such a correspondence between the two systems, if it can be realized, might bring new insights towards understanding much more their integrability. This and other questions will be done in a future work.

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