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EXACT SOLUTION FOR MHD FLOW OF A GENERALIZED OLDROYD-B FLUID WITH MODIFIED DARCY'S LAW

M. Khan[1]
Department of Mathematics, Quaid-i-Azam University, 45320 Islamabad, Pakistan,

T. Hayat[2]
Department of Mathematics, Quaid-i-Azam University, 45320 Islamabad, Pakistan
and
The Abdus Salam International Centre for Theoretical Physics, Trieste, Italy

and

S. Asghar
COMSATS, Institute of Information Technology H-8, 44000 Islamabad, Pakistan.

Abstract

This paper deals with an exact solution for the magnetohydrodynamic (MHD) flow of a generalized Oldroyd-B fluid in a circular pipe. For the description of such a fluid, the fractional calculus approach has been used throughout the analysis. Based on modified Darcy's law for generalized Oldroyd-B fluid, the velocity field is calculated analytically. Several known solutions can be recovered as the limiting cases of our solution.

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[1]Corresponding author: mkhan_21@yahoo.com
[2]Junior Associate of ICTP. t_pensy@hotmail.com

1 Introduction

The analyses of the flow properties of non-Newtonian fluids are very important in the fields of fluid dynamics because of their technological application. Mechanics of non-Newtonian fluids present challenges to engineers, physicists and mathematicians. Due to the complex stress-strain relationships of non-Newtonian fluids, not many investigators have studied the flow behavior of the fluids in various flow fields. In addition, the effects of magnetic field on the non-Newtonian fluid also have great importance in engineering applications; for instance, MHD generators, plasma studies, geothermal energy excitations and in the field of aerodynamics for boundary layer control, etc. Moreover MHD flows in porous media have received wide coverage on the development of noval energy generation systems and interest in astrophysical and geophysical fluid dynamics. Due to the non-linearity of the Navier-Stokes equations and the inapplicability of the superposition principle for nonlinear partial differential equations, exact solutions are difficult to obtain and are few in number under certain conditions. For non-Newtonian fluids, such solutions are further narrowed down. Recently, Fetecau [1] and Yin and Zhu [2] have provided analytical solutions for the flow of a non-Newtonian fluid in pipe like domain.

More recently, the fractional calculus has encountered much success in the description of viscoelasticity. Specifically, rheological constitutive equations with fractional derivatives play a vital role in the description of the properties of polymer solutions and melts. The constitutive equations for generalized non-Newtonian fluids are modified from the well known fluid models by replacing the time derivative of an integer order by the so-called Riemann-Liouville fractional calculus operators. In other words, the governing equations are derived by replacing the ordinary derivatives of first, second and higher order by fractional derivatives of any noninteger order [3 − 10]. Such viscoelastic models are considered more appropriate to describe the behaviors of Xanthan gum and Sesbania gel [11].

The purpose of this paper is to discuss the MHD flow of a generalized Oldroyd-B fluid in a circular pipe. The flow is induced because of an oscillating pressure gradient. By constructing a modified expression for Darcy's law in a generalized Oldroyd-B fluid, we find the analytical solution of the problem using Fourier transform technique for the fractional calculus.

2 Mathematical model

The constitutive equations for a generalized Oldroyd-B fluid are

$$\mathbf{T} = -p\mathbf{I} + \mathbf{S}, \qquad (1 + \lambda^{\alpha} D_t^{\alpha})\,\mathbf{S} = \mu\left(1 + \theta^{\beta} D_t^{\beta}\right)\mathbf{A}_1, \tag{1}$$

where $\mathbf{T}$ is the Cauchy stress tensor, p is the pressure, $\mathbf{I}$ is the identity tensor, $\mathbf{S}$ is the extra stress tensor, μ, λ, θ are the material constants and respectively known as the viscosity coefficient, the relaxation and retardation times. $\mathbf{A}_1$ is the first Rivlin-Ericksen tensor, α and β are fractional calculus parameters such that $0 \le \alpha \le \beta \le 1$. For $\alpha > \beta$ the relaxation fraction is increasing, which is generally not responsible [12] and has requires that $\alpha \le \beta$. D_t^{β} is the Riemann-Liouville fractional derivative operator and may be defined as

$$D_t^{\beta}\left[f\left(t\right)\right]=\frac{1}{\Gamma\left(1-\beta\right)}\frac{d}{dt}\int\limits_0^t\frac{f\left(t\right)}{\left(t-z\right)^{\beta}dz},\qquad\left(0<\beta<1\right)\tag{2}$$

where $\Gamma\left(\cdot\right)$ is the Gamma function. So D_t^{β} denotes the material time derivative of fractional order. While $\alpha=\beta=1$, (1) may be simplified as simple Oldroyd-B fluid, while $\lambda=0$ the constitutive relationship describes the generalized second grade fluid. It should be noted that this model also includes the classical Newtonian fluid as a special case for $\lambda=\theta=0$, and is the fractional Maxwell fluid when $\theta=0$.

The incompressible fluid undergoes only isochoric motion and hence

$$\nabla\cdot\mathbf{V}=0.\tag{3}$$

In the above equation $\mathbf{V}$ is the velocity vector. We shall assume the velocity field and the stress of the form

$$\mathbf{V}=u\left(r,t\right)\hat{e}_z,\qquad\mathbf{S}=S\left(r,t\right)\hat{e}_z,\tag{4}$$

where $\hat{e}_z$ and u are the unit vector and velocity in the $z-$ direction, respectively. Substituting (4) into (1) and considering the initial condition

$$\mathbf{S}\left(r,0\right)=\mathbf{0},\tag{5}$$

we obtain after employing a procedure of reference [13] that $S_{rz}=S_{\theta\theta}=S_{\theta z}=S_{zz}=0$ and

$$S_{zz}+\lambda^{\alpha}\left(\frac{\partial^{\alpha}S_{zz}}{\partial t^{\alpha}}-2S_{rz}\frac{\partial u}{\partial r}\right)=-2\mu\theta^{\beta}\left(\frac{\partial u}{\partial r}\right)^2,\tag{6}$$

$$S_{rz}+\lambda^{\alpha}\frac{\partial^{\alpha}S_{rz}}{\partial t^{\alpha}}=\mu\left(1+\theta^{\beta}\frac{\partial^{\beta}}{\partial t^{\beta}}\right)\frac{\partial u}{\partial r}.\tag{7}$$

Since the flow is unsteady, the interaction terms depend upon the drag and virtual mass effect. The relationship between the pressure drop and velocity for the generalized Oldroyd-B fluid in porous media is

$$\left(1+\lambda^{\alpha}\frac{\partial^{\alpha}}{\partial t^{\alpha}}\right)\nabla p=-\frac{\mu\phi}{k}\left(1+\theta^{\beta}\frac{\partial^{\beta}}{\partial t^{\beta}}\right)\mathbf{V},\tag{8}$$

where $k\ (>0)$ and $\phi\ (0<\phi<1)$ are the (constant) permeability and porosity, respectively.

Note that (8) ignores the boundary effects on the flow and cannot be directly used to analyze the flow problem in a porous medium. Thus, the modified Darcy's law based on a local volume averaging technique will be considered in a porous medium. Considerating the balance of forces acting on a volume element of fluid, the local volume average balance of linear momentum is given by

$$\rho\frac{d\mathbf{V}}{dt}=-\nabla p+div\mathbf{S}+\mathbf{R}+\mathbf{J}\times\mathbf{B},\tag{9}$$

in which ρ is the fluid density, $\mathbf{R}$ is the Darcy resistance for the generalized Oldroyd-B fluid in porous medium, $\mathbf{J}$ is the current density, $\mathbf{B}$ is the total magnetic field so that $\mathbf{B}=\mathbf{B}_0+\mathbf{b}$, $\mathbf{b}$ is the induced magnetic field.

Due to the volume averaging process, some information is lost, thus requiring supplementary empirical relation for the Darcy resistance [14] is known as a measure of the resistance to the flow in the bulk of the porous medium, and $\mathbf{R}$ is a measure of the flow resistance offered by the solid matrix. Thus $\mathbf{R}$ can be inferred from (8) to satisfy the following equation

$$\left(1+\lambda^{\alpha}\frac{\partial^{\alpha}}{\partial t^{\alpha}}\right)\mathbf{R}=-\frac{\mu\phi}{k}\left(1+\theta^{\beta}\frac{\partial^{\beta}}{\partial t^{\beta}}\right)\mathbf{V}. \tag{10}$$

Now neglecting the displacement currents, the Maxwell equations and the generalized Ohm's law are

$$\nabla\cdot\mathbf{B}=0,\quad \nabla\times\mathbf{B}=\mu_m\mathbf{J},\quad \nabla\times\mathbf{E}=0, \tag{11}$$

$$\mathbf{J}=\sigma\left(\mathbf{E}+\mathbf{V}\times\mathbf{B}\right), \tag{12}$$

where μ_m is the magnetic permeability, $\mathbf{E}$ is the electric field and σ is the electric conductivity.

Considering that

- the quantities ρ, μ_m and σ are all constants throughout the flow field
- the magnetic field $\mathbf{B}$ is perpendicular to the velocity field $\mathbf{V}$ and the induced magnetic field is negligible compared with the imposed magnetic field so that the magnetic Reynolds number is small [15]
- the electric field is assumed to be zero

the electromagnetic body force involved in (9) becomes

$$\mathbf{J}\times\mathbf{B}=-\sigma B_0^2\mathbf{V}, \tag{13}$$

and thus (9) along with (10) and (13) yields

$$\begin{aligned}\rho\left(1+\lambda^{\alpha}\frac{\partial^{\alpha}}{\partial t^{\alpha}}\right)\frac{d\mathbf{V}}{dt} &= -\left(1+\lambda^{\alpha}\frac{\partial^{\alpha}}{\partial t^{\alpha}}\right)\nabla p+\left(1+\lambda^{\alpha}\frac{\partial^{\alpha}}{\partial t^{\alpha}}\right)div\mathbf{S}\\ &\quad -\frac{\mu\phi}{k}\left(1+\theta^{\beta}\frac{\partial^{\beta}}{\partial t^{\beta}}\right)\mathbf{V}-\sigma B_0^2\left(1+\lambda^{\alpha}\frac{\partial^{\alpha}}{\partial t^{\alpha}}\right)\mathbf{V}.\end{aligned} \tag{14}$$

By substituting (4) and (7) into (14), the momentum equation for an incompressible electrically conducting fluid in a porous medium reduces to

$$\begin{aligned}\left(1+\lambda^{\alpha}\frac{\partial^{\alpha}}{\partial t^{\alpha}}\right)\frac{\partial u}{\partial t} &= -\frac{1}{\rho}\left(1+\lambda^{\alpha}\frac{\partial^{\alpha}}{\partial t^{\alpha}}\right)\frac{\partial p}{\partial z}+\nu\left(1+\theta^{\beta}\frac{\partial^{\beta}}{\partial t^{\beta}}\right)\left[\frac{\partial^2 u}{\partial r^2}+\frac{1}{r}\frac{\partial u}{\partial r}\right]\\ &\quad -\frac{\nu\phi}{k}\left(1+\theta^{\beta}\frac{\partial^{\beta}}{\partial t^{\beta}}\right)u-\frac{\sigma B_0^2}{\rho}\left(1+\lambda^{\alpha}\frac{\partial^{\alpha}}{\partial t^{\alpha}}\right)u,\end{aligned} \tag{15}$$

in which $\nu=\mu/\rho$ is the kinematic viscosity.

3 Exact solution

Consider the motion of an incompressible electrically conducting non-Newtonian fluid in the presence of a magnetic field acting along the radius of a circular pipe. We consider the flow as axially symmetric and fully developed in a porous medium. The motion starts due to a pressure gradient. We select a cylindrical polar coordinate system with $z-$axis in the direction of motion. Under the above mentioned assumption, the unsteady flow of the generalized Oldroyd-B fluid can be described by (15) with the no-slip boundary condition given as

$$u\left(a,t\right)=0, \tag{16}$$

where a is the radius of the cylinder.

Let the pressure gradient oscillate with frequency ω_0 and amplitude P_0, i.e.

$$\frac{\partial p}{\partial z}=P_0 e^{i\omega_0 t}. \tag{17}$$

Let us introduce the dimensionless variable

$$u^*=\frac{u}{u_0},\qquad r^*=\frac{r}{a},\qquad t^*=\frac{\nu t}{a^2},\qquad \omega_0^*=\frac{\omega_0 a^2}{\nu}, \tag{18}$$

where u_0 is the reference velocity.

Using (17) and (18), the dimensionless governing equation and boundary condition are obtained as follows (for brevity the dimensionless mark "$*$" are omitted)

$$\begin{aligned}\left(1+\lambda_1^{\alpha}\frac{\partial^{\alpha}}{\partial t^{\alpha}}\right)\frac{\partial u}{\partial t} &= -P_0\left(1+\lambda_1^{\alpha}\frac{\partial^{\alpha}}{\partial t^{\alpha}}\right)e^{i\omega_0 t}+\nu\left(1+\lambda_2^{\beta}\frac{\partial^{\beta}}{\partial t^{\beta}}\right)\left[\frac{\partial^2 u}{\partial r^2}+\frac{1}{r}\frac{\partial u}{\partial r}\right]\\ &\quad -\frac{1}{K}\left(1+\lambda_2^{\beta}\frac{\partial^{\beta}}{\partial t^{\beta}}\right)u-M^2\left(1+\lambda_1^{\alpha}\frac{\partial^{\alpha}}{\partial t^{\alpha}}\right)u,\end{aligned} \tag{19}$$

$$u\left(1,t\right)=0, \tag{20}$$

where

$$\lambda_1=\frac{\nu\lambda}{a^2},\qquad \lambda_2=\frac{\nu\theta}{a^2},\qquad P_0^*=\frac{P_0 a^2}{\mu u_0},\qquad M^2=\frac{\sigma B_0^2}{(\mu/a^2)},\qquad \frac{1}{K}=\frac{\phi}{(k/a^2)}. \tag{21}$$

In order to solve (19) and (20), we define the temporal Fourier transform pair as

$$U\left(r,\omega\right)=\int_{-\infty}^{\infty}u\left(r,t\right)e^{-i\omega t}dt, \tag{22}$$

$$u\left(r,t\right)=\frac{1}{2\pi}\int_{-\infty}^{\infty}U\left(r,\omega\right)e^{i\omega t}d\omega, \tag{23}$$

and the Fourier transform for the fractional derivative is given by [7]

$$\int_{-\infty}^{\infty}D_t^{\beta}\left[u\left(r,t\right)\right]e^{-i\omega t}dt=\left(i\omega\right)^{\beta}U\left(r,\omega\right), \tag{24}$$

where

$$(i\omega)^{\beta} = |\omega|^{\beta} e^{i\beta\pi/2\mathrm{sign}\omega} = |\omega|^{\beta}\left(\cos\frac{\beta\pi}{2} + i\mathrm{sign}\omega\sin\frac{\beta\pi}{2}\right),$$

and sign is the signum function.

Transforming (19) and (20) , we get

$$\frac{d^2U}{dr^2} + \frac{1}{r}\frac{dU}{dr} + \xi^2 U = P_0\left[\frac{1+\lambda_1^{\alpha}(i\omega_0)^{\alpha}}{1+\lambda_2^{\beta}(i\omega)^{\beta}}\right]\delta(\omega-\omega_0), \tag{25}$$

$$U(1,\omega) = 0, \tag{26}$$

where

$$\begin{aligned}\xi^2 &= -\left\{\frac{1}{K} + \left(M^2 + i\omega\right)\left[\frac{1+\lambda_1^{\alpha}(i\omega)^{\alpha}}{1+\lambda_2^{\beta}(i\omega)^{\beta}}\right]\right\}, \\ &= -\left\{\frac{1}{K} + \left(M^2 + i\omega\right)\left[\frac{1+\lambda_1^{\alpha}|\omega|^{\alpha}\left(\cos\frac{\alpha\pi}{2} + i\mathrm{sign}\omega\sin\frac{\alpha\pi}{2}\right)}{1+\lambda_2^{\beta}|\omega|^{\beta}\left(\cos\frac{\beta\pi}{2} + i\mathrm{sign}\omega\sin\frac{\beta\pi}{2}\right)}\right]\right\},\end{aligned}$$

and $\delta(\cdot)$ is the dirac delta function.

Solving (25) subject to (26), we get the velocity field in the frequency domain

$$U(r,\omega) = -\frac{P_0\left(1+\lambda_1^{\alpha}(i\omega_0)^{\alpha}\right)\delta(\omega-\omega_0)}{\frac{1}{K}\left(1+\lambda_2^{\beta}(i\omega)^{\beta}\right) + \left(M^2+i\omega\right)\left(1+\lambda_1^{\alpha}(i\omega)^{\alpha}\right)}\left\{1 - \frac{J_0(\xi r)}{J_0(\xi)}\right\}, \tag{27}$$

where $J_0(\cdot)$ is the zeroth-order Bessel function.

Applying the inverse Fourier transform to (27) and using the property of delta function, we can get the solution of the velocity in the time domain

$$u(r,t) = \frac{P_0}{\xi_0^2}\frac{1+\lambda_1^{\alpha}(i\omega)^{\alpha}}{1+\lambda_2^{\beta}(i\omega)^{\beta}}\left\{1 - \frac{J_0(\xi_0 r)}{J_0(\xi_0)}\right\}e^{i\omega_0 t}, \tag{28}$$

where

$$\xi_0 = \xi|_{\omega=\omega_0}.$$

We also introduce enhancement A_u defined as the amplitude to dimensionless velocity on the axis of the tube in (28) . It can be obtained as follows

$$A_u = \left|\frac{P_0}{\xi_0^2}\frac{1+\lambda_1^{\alpha}(i\omega)^{\alpha}}{1+\lambda_2^{\beta}(i\omega)^{\beta}}\left(1 - \frac{1}{J_0(\xi_0)}\right)\right|. \tag{29}$$

4 Results and discussion

The differential equation (15) subject to (16) is solved analytically to obtain velocity field. An analysis is made for two kinds of fluids: an Oldroyd-B fluid (when $\alpha = \beta = 1$) and the generalized Oldroyd-B fluid (when $0 < \alpha < \beta < 1$). Of interest are the effects of the magnetic parameter M and the permeability of porous medium K.

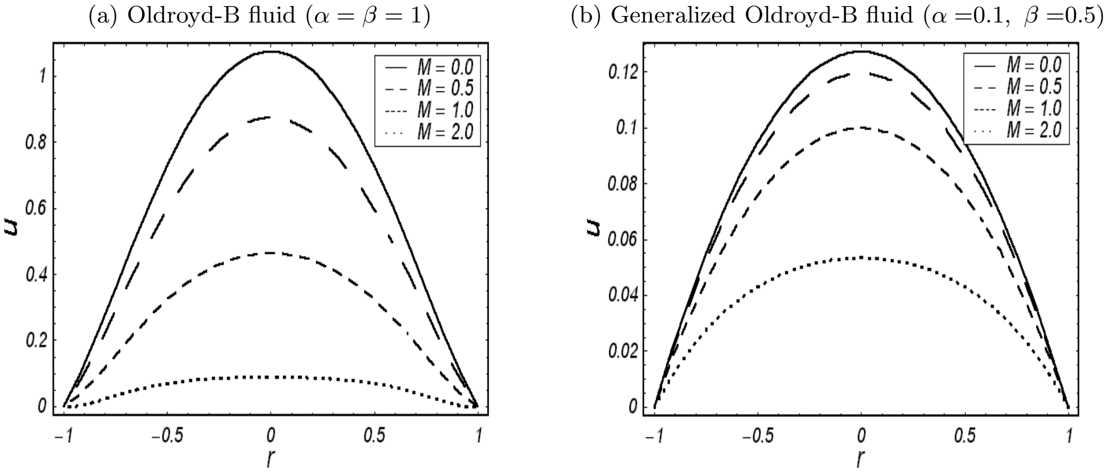

Figure 1: Profiles of velocity $u\,(r,t)$ for various values of magnetic parameter M when $\lambda_1 = 10,\ \lambda_2 = 1,\ \omega_0 = 1.5,\ P_0 = -1,\ t = 1$ and $K = 1$ are fixed.

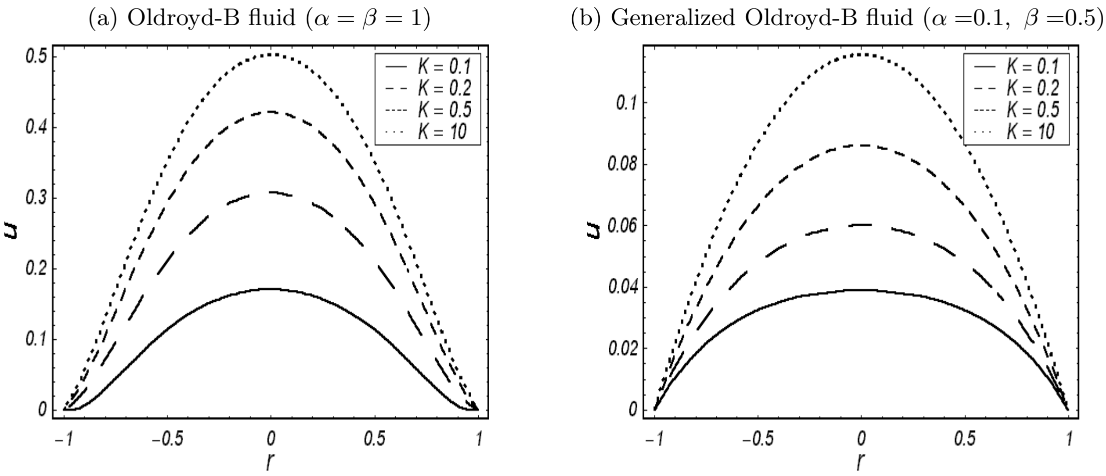

Figure 2: Profiles of velocity $u\,(r,t)$ for various values of permeability parameter K when $\lambda_1 = 10,\ \lambda_2 = 1,\ \omega_0 = 1.5,\ P_0 = -1,\ t = 1$ and $M = 1$ are fixed.

To see the effects of magnetic parameter M on the velocity profile, we have plotted u against r in figure 1. Panel a shows the effect of M for an Oldroyd-B fluid and panel b for generalized Oldroyd-B fluid. From these figures, it is noted that an increase in magnetic parameter M reduces the velocity profile monotonically due to the effect of the magnetic force against the direction of the flow. It is also observed that the velocity profiles for an Oldroyd-B fluid are much larger than those for the generalized Oldroyd-B fluid.

The influence of the permeability of the porous medium K on the flow is illustrated in figure 2. As expected from the governing equation (15) the increase in the permeability of the porous medium K yields an effect opposite to that of the magnetic parameter M.

5 Concluding remarks

A generalized MHD Oldroyd-B fluid model based on modified Darcy's law is first developed here using the fractional calculus approach. Using the Fourier transform of fractional calculus, the expression for velocity field is constructed. The developed fluid model is appropriate to describe the behaviors of Xanthan gum and Sesbania gel. The presented analysis is more general and the results of several unattempted problems (e.g., generalized Maxwell and second grade fluids) can be recovered as special cases.

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