TRANSITION TO CHAOS IN EXTERNALLY MODULATED HYDRODYNAMIC SYSTEMS

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Abstract

An amplitude equation associated with externally modulated hydrodynamic systems is considered. A simple physical model to evaluate analytically the Melnikov function is proposed. The onset of chaos is studied numerically through a computation of the largest Lyapunov exponent, a construction of the bifurcation diagram, and an analysis of the phase space trajectories. Theory predicts the regions of chaotic behavior of the system in good agreement with computer calculations.

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1. Introduction

Many spatially extended systems can undergo nonequilibrium transitions that lead to inhomogeneous states exhibiting some kind of stable pattern [1] or dissipative structure [2] that subsists as long as energy is externally introduced into the system. Hydrodynamic pattern-forming systems constitute typical examples of spatially extended systems out of equilibrium, and among them, Rayleigh-Bénard convection has been usually considered as a prototypical situation [3], owing to the controllability of its experimental implementation and to the high degree of theoretical understanding about it that has been achieved over the years.

In Rayleigh-Bénard convection, a layer of fluid is subject to a vertical temperature gradient by heating it from below. Under these conditions, the fluid near the lower plate has a higher temperature, and therefore a lower density than the fluid near the upper plate. This situation is clearly unstable, because the upper layer of the fluid tends to fall and lower ones tends to rise. This buoyant tendency is balanced by dissipative effects due to viscosity, in such a way that for small temperature gradients, dissipation overcomes buoyancy and the fluid remains static, with heat being transmitted by conduction. On the other hand, as the temperature difference increases, buoyancy becomes more important and finally, the fluid initiates a convective motion, with warm fluid ascending and cold fluid descending in circular paths. Typical patterns produced close to convection onset are rolls and hexagons [1, 4]. The competition between buoyancy and dissipation is characterized by means of the dimensionless Rayleigh number. This number is proportional to the temperature difference between the plates, and is frequently used as a control parameter of the transition.

Yanagita and Kaneko [5] have introduced a coupled map lattice model which reproduces almost all phenomena known for Rayleigh-Bénard convection except those associated with the long wave-length instabilities. Simulations of the model have led to several predictions of novel phenomena: for small ratios, the formation of convective rolls, their oscillation, many routes to chaos and chaotic itinerancy are found, with increase of the Rayleigh number. Cross [6] has described work that looks at theoretical models of the chaotic states to gain some understanding of the two novel features, namely the multistability and the geometry. He first discussed a theory of dynamics of an isolated spiral or target based on the long wave-length reduced description known as phase equations. The concept of the instability of disordered states through invasive defects was then introduced, and the invasive nature of spiral and target defects was discussed based on the analysis of their dynamics.

For various systems subjected to external temperature gradients, one finds two kinds of instabilities: stationary and oscillatory [7]. For certain values of the external parameters, the stationary and oscillatory bifurcation lines intersect at a so-called codimension-two (CT) point [8]. Brand et al. [9] have given a detailed derivation of the amplitude equation near the intersection point of the stationary and oscillatory instabilities which occur in a layer of a binary fluid mixture with and without a porous medium and subjected to a vertical temperature gradient. Zielinska et al. [10] have introduced an amplitude equation for periodically modulated Rayleigh-Bénard system of binary mixture near the CT point. The phase diagram associated with this equation has been studied by numerical integration of the equation of motion.

The amplitude formalism of the modulated Rayleigh-Bénard system of binary mixture may be described by

\[
\dot{x} = [a + f_x x^2 + \epsilon_x \cos(\omega t)]x - \frac{dV}{dx} + \epsilon_x \cos(\omega t + \phi) x
\]

\[(1)\]
with
\[ V(x) = -\frac{1}{2}b x^2 - \frac{1}{4} f_1 x^4 - \frac{1}{6} f_3 x^6, \] \tag{2}

where overdot means time derivative and \( x \) is related to the vertical component of the velocity. The parameters \( a \) and \( b \) are function of the temperature gradient and the concentration, \( \varepsilon_1 \) and \( \varepsilon_2 \) are proportional to the amplitude of modulation of the Rayleigh number. The coefficient \( f_1 \) can take both positive and negative values: \( f_1 > 0 \) corresponds to the binary mixture, while \( f_1 < 0 \) corresponds to some cases of magnetoconvection \([10,11]\). The case of binary mixture which attracts our attention in this work corresponds to \( f_1 > 0, f_2 < 0, \) and \( f_3 < 0 \).

The numerical integration of Eq. (1) has showed that, in a large part of the physically relevant parameters space, the system exhibits chaotic trajectories \([10]\). In particular, in some regions of the phase diagram, chaotic trajectories coexist with the convective phase. In a recent work, we have used numerical simulations of Eq. (1) to detect the presence of strange nonchaotic attractors in the system \([12]\).

In the present study, we are interested in global bifurcations before and after loss of stability. Basically, there are two global bifurcations which can be detected analytically by means of the Melnikov function \([10]\). They involve the transverse intersection of (stable and unstable) manifold arising from perturbation of homoclinic or heteroclinic orbits and are labeled accordingly. The reason for studying homoclinic and heteroclinic bifurcations is that they represent the starting point for a successive route to chaos. In light to this, their occurrence can be viewed as the incipient evolution towards a complicated response of the system to a simple harmonic excitation. Since these bifurcations can be detected analytically, it is possible to formulate a method to prevent theoretically chaotic dynamics.

The paper is arranged as follows. In section 2, we describe the proposed model and review briefly pertinent material of the Melnikov theory. Then we derive the bifurcation curves. We verify the analytical predictions with numerical simulations of the model in Section 3. Section 4 concludes the study.

2. **Melnikov analysis of chaotic dynamics**

In this section, a classical Melnikov’s analysis is employed to study the transition to the chaotic dynamics of the system corresponding to binary mixture. This technique is today well known, and we refer to Ref. \([8]\) for the general theory and to Ref. \([10]\) for the application to similar cases. It should however be noted that, chaos in the Melnikov’s sense means transverse intersection of a stable and unstable manifolds, which in turn assures the existence of an invariant hyperbolic Cantor set.

By setting \( \varepsilon_1 = \phi = 0, a = \varepsilon \lambda, f_2 = \varepsilon f \) and \( \varepsilon_2 = \varepsilon \mu (\varepsilon << 1) \), the equation of motion becomes:
\[ \ddot{x} - \varepsilon (\lambda + f x^2) \dot{x} - b x - f_1 x^3 - f_3 x^5 = \varepsilon \mu x \cos(\omega t) \] \tag{3}
where \( \mu \) is a physical parameter. The small parameter \( \varepsilon \) has been introduced to emphasize the smallness of excitation and damping. The nonlinear dissipating term models the viscosity of the system. Zielinska et al \([11]\) have found numerically that, in such systems it is possible to find chaotic behavior at the onset of convection that may take place via the period-doubling scenario.

We now assume that Eq. (3) characterizes the motion of a fluid particle of unit mass embedded in a substrate potential \( V(x) \) and submitted to a nonlinear damping and a parametric external force. The potential \( V(x) \), defined by Eq. (2), is a \( \Phi^6 \) potential. The dynamical properties of this potential have been analyzed in certain works \([13-15]\). Using the Melnikov theory, we derive the
analytical criteria for the occurrence of transverse intersections on the surface of homoclinic and heteroclinic orbits both for three potential and two potential well cases.

In fact, one can see that when parameter $b < f_1^2 / 4 f_3$, the potential $V(x)$ admits three wells, with two maxima. The minima and maxima of $V(x)$ correspond to the center (C1, C2 and C3) and saddle-point respectively (see Fig. 1a). In Fig. 1b, plotted for $f_1=1$, $f_3=-5$, and $b=-0.045$, the phase portrait shows two symmetric saddles which are connected by a heteroclinic loop which surrounds the stable trivial solution $x=0$. The saddles also have homoclinic orbits which encircle two other centers. These centers appear for large amplitudes. Then, there are two saddles and two different mechanisms of manifold intersections, ensuing from perturbation of homoclinic and heteroclinic orbits. When the parameter $b \geq 0$, the potential function is a symmetric double-well potential with a barrier whose maximum occurs at $x=0$ (see Fig. 2a). In this case, the system has a hyperbolic fixed point at the origin ($x=0$). Emerging from this point in forward and reverse time are the unperturbed system’s homoclinic orbits. Fig. 2b presents the phase plane obtained for $f_1=0.5$, $f_3=-0.05$ and $b=1$. There are two symmetric centers (C1 and C2) which are divided by the trivial solution which becomes a saddle after the bifurcation. The homoclinic orbit acts as impermeable separatrix: a motion of the unperturbed system that starts inside a region enclosed by a separatrix remains confined to this region for all time; that is, the motion never escapes from the potential well associated with that system. The homoclinic orbit can be viewed as intersection of the system’s stable and unstable manifolds with a plane section ($x, v=x$). For the unperturbed system, the stable and unstable manifolds coincide. The perturbation causes the stable and unstable manifolds to separate. In the latter case, on the other hand, there is only one saddle and chaos may arise only as a consequence of perturbation of homoclinic orbit.

In the case where $b < f_1^2 / 4 f_3$, there are five equilibrium points for the unperturbed system ($\varepsilon=0$) given by

$$\begin{align*}
-x_2, & \quad -x_1, & \quad 0, & \quad x_1, & \quad x_2

d\text{with}

x_1 &= \frac{-f_1 + \sqrt{f_1^2 - 4bf_3}}{2f_3} \quad (4.b)

d\text{and}

x_2 &= \frac{-f_1 - \sqrt{f_1^2 - 4bf_3}}{2f_3} \quad (4.c)
\end{align*}$$

where $0<x_1<x_2$. $x_1$ and $x_2$ are well defined quantities because in the case considered $b<0$, $f_1>0$ and $f_3<0$.

It is now possible to compute the heteroclinic and homoclinic orbits. If we define the parameters

$$\begin{align*}
\theta &= \frac{x_2}{x_1} = \frac{f_1 + \sqrt{f_1^2 - 4bf_3}}{f_1 - \sqrt{f_1^2 - 4bf_3}}, \quad (5.a)

g &= x_1^2 \sqrt{-2f_3(\theta^2 - 1)}, \quad (5.b)

\beta &= \frac{5-3\theta^2}{3\theta^2 - 1}, \quad (5.c)
\end{align*}$$

the heteroclinic orbit is given by:
\[ x_{\text{het}}(t) = \pm x_1 \sqrt{2} \frac{\sinh(\gamma t/2)}{[-\beta + \sinh(\gamma t)]^{1/2}}, \]  

\[ x_{\text{hom}}(t) = \pm x_1 \sqrt{2} \frac{\cosh(\gamma t/2)}{[\beta + \cosh(\gamma t)]^{1/2}}. \]  

and the homoclinic orbit:

The Melnikov functions for the system (3) are:

\[ M(t_0) = 2\lambda \int_0^\infty x_h^2(t) \, dt + 2\int_0^\infty x_h^2(t) \, dt \sin(\omega t_0) \int_0^\infty x_h(t) \sin(\omega t) \, dt, \]

where the suffix ‘‘h’’ stands for ‘‘het’’ and ‘‘hom’’.

The Smale-Birkhoff theorem states that the necessary condition for the occurrence of chaos is that, the Melnikov function induced by the perturbation has simple zeros [16]. For the considered system, this condition is the Melnikov inequality:

\[
\frac{a \gamma}{\epsilon_2} \leq \left[ 4\pi \frac{\omega}{\gamma} \frac{1-\beta}{1+\beta} \sinh \left( \frac{\omega}{\gamma} \arccos \beta \right) \right] \times \left[ S_1 \right] = R_{\text{hom}}(\frac{\omega}{\gamma}, \beta) \]  

\[
\frac{a \gamma}{\epsilon_2} \leq \left[ 4\pi \frac{\omega}{\gamma} \frac{1-\beta}{1+\beta} \sinh \left( \frac{\omega}{\gamma} \arccos(-\beta) \right) \right] \times \left[ S_2 \right] = R_{\text{het}}(\frac{\omega}{\gamma}, \beta) \]  

The main conclusion which follows from equations (8) is that, transversal homoclinic (or heteroclinic) points arise if the ratio of the damping constant ‘‘a’’ and the amplitude of the modulation of the Rayleigh number \( \epsilon_2 \) is less than a critical value \( R_{\text{h}} \). This critical value turns out to be dependent on the frequency \( \omega \) of the modulation. Eqs. (8) and Fig.3 give the region in the parameters space where the chaos may occur. Fig.3 shows up the variation of \( R_{\text{h}} \) according to the frequency \( \omega \). In each case, the critical curves in Fig.3 separate the non-chaotic zone (above) from the possibly chaotic region (below). All the critical curves have the classical bell shape which gives a single chaotic resonance [18]. They decrease exponentially fast to zero when the frequency of the modulation increases. This means that, for sufficiently small periods of modulation, the system is stable. At the same time, the critical curves approach zero when the frequency tends to zero, so that the system is also chaotically protected from very large periods of the modulation of the Rayleigh number. The graphs of Fig.3 have been plotted for \( f_1=1, f_2=-5 \) and \( b=-0.045 \). Fig.3a is obtained for \( f_2=0 \). In this first case, \( R_{\text{hom}}>R_{\text{het}} \) for every frequency. This means that for small values of the amplitude \( \epsilon_2 \) no
intersections occur [region III of Fig.3a]. When $\varepsilon_2$ crosses its first critical value, homoclinic bifurcation can take place [region II of Fig.3a]. After the second global bifurcation [region I of Fig.3a], the region where the chaotic effects are observable enlarges. When $\varepsilon_2$ further increases, a transition to a chaotic attractor is feasible. The global bifurcations can therefore be considered as the starting point for this route to completely chaotic dynamics.

Fig.3b is obtained for $f_2=-0.02$ and $a=-0.01$. One can observe the influence of the non-linear viscosity term $f_2$ in the dynamics. Firstly, the maximum value of $R_h$ decreases. In this case, no qualitative difference exists with respect of the case of Fig.3a; on the other hand, a different succession of events occurs when the amplitude $\varepsilon_2$ increases. Secondly, we can notice that, when the non-linear viscosity term is present, the range of the area below (chaotic region) the critical curve reduces for the benefit of the area above. Thus to ensure the stability of the system it is desirable to move the parameter $f_2$ closer to -1.

When the parameter $b \geq 0$, chaos may occur as a consequence of homoclinic bifurcation. The homoclinic solution of the unperturbed system is:

$$x_{\text{hom}}(t) = \pm 2 x_1 \sqrt{\frac{\theta^2 \beta}{1 - \theta^2}} \frac{1}{\sqrt{\beta + \cosh(\gamma t)}},$$  

where

$$\gamma = 2 b^2 \theta \sqrt{f_3}, \quad \beta = \frac{(1 - \theta^2) \sqrt{3}}{(3 \theta^2 + 1)(3 + \theta^3)}, \quad \theta = x_2/x_1.$$  

Note that $x_2>x_1$, and $x_2$ is the abscissa of the stable equilibrium point of the unperturbed system, $x_1$ is the modulus of the complex root of the equation $b + f_1 x_1^2 + f_3 x_1^4 = 0$. The Melnikov function is again given by Eq.(7) and the necessary condition for the occurrence of chaos becomes:

$$\frac{a \beta}{\varepsilon_2} \leq \frac{\sinh \left(\frac{\pi \omega}{\gamma} \arccos \beta\right)}{\sinh \left(\frac{\pi \omega}{\gamma}\right)} \times [S]^{-1},$$  

with

$$S = \frac{\beta}{8} \left[1 + \frac{2}{\beta \sqrt{1 - \beta^2}} \arctan \left(\frac{1 - \beta}{\sqrt{1 + \beta}}\right)\right] + \frac{f_1 (2 + \beta^2)}{12 a (1 - \beta^2)} \left[1 - \frac{6 \beta}{(2 + \beta^2) \sqrt{1 - \beta^2}} \arctan \left(\frac{1 - \beta}{\sqrt{1 + \beta}}\right)\right].$$  

This latter condition is similar to Eq. (8.a), so that all the observation regarding that function also hold in this case. The main difference is that here we have only one critical curve which separates the chaotic and the non-chaotic regions. This scenario is substantially similar to those studied in [10-11,17]. Consequently, this case does not require special attention.

In the case where the parameter $b$ is such as $f_1^2 \leq 4bf_3$ with $b<0$, the potential $V(x)$ admits only one well without relative maximum. Then, the system is structurally stable. The trivial solution $x=0$ is no longer saddle, so the Melnikov method cannot be applied.

3. Results of numerical simulations

In this section, we report the results of numerical studies of some nonlinear features of the modulated Rayleigh-Bénard system of binary mixture. To see how good are the predictions of the
above Melnikov approach on the transition of a dynamics system from a periodic to a chaotic state, we proceed by using numerical simulations of Eq.(1). To integrate this nonlinear system, we use the fourth-order Runge-Kutta algorithm. The time step is $\Delta t = T/1000$, where $T$ is the period of the modulation of the Rayleigh number. The calculations are performed using reals in extended mode. The integration time is greater than $t = 10^4$. Only the case where the potential $V(x)$ admits three wells is treated because, as mentioned above, it is the most remarkable. We restrict ourselves to the numerical values of the physical parameters of Eq.(1) close to those used in [10]. Then we choose $f_1 = 1$, $f_2 = -0.02$, $f_3 = -5$, $b = -0.045$, $a = -0.01$, $\omega = 0.3$, the amplitude of the modulation $\varepsilon_2$ is varied as a control parameter.

To express the sensitive dependence on the initial conditions, the greatest Lyapunov exponent is evaluated. To study the dynamic behavior of this system, we construct its bifurcation diagram and its phase portrait. To display conveniently the results, we present separately the analysis of the Lyapunov exponent, the bifurcation diagram and the phase space trajectories.

The Lyapunov exponents quantify the exponential divergence or convergence of initially nearby trajectories. Several methods for calculating these exponents have been developed. In this study, we use the symplectic method to evaluate the Lyapunov exponents of our system [19]. Symplectic methods have been applied with success to classical dynamical problems [9,12]. These methods take advantage of the Hamiltonian structure of many systems and avoid the renormalization and reorthogonalisation in the numerical computation of the exponents [19]. We use the terms chaotic and regular behavior in conformity with the definition that a random process in which the most probable value of the greatest Lyapunov exponent is positive (nonpositive), is called a chaotic (regular) process [19]. The plot of the Lyapunov exponent versus control parameter $\varepsilon_2$ is shown in Fig. 4 with $\varepsilon_2$ changes from 0 to 0.1. If $\varepsilon_2$ is small, the Lyapunov exponent is negative, so Eq.(1) does not show sensitive dependence on the initial conditions. When $\varepsilon_2$ is increasing up to about 0.0620, the Lyapunov exponent changes suddenly from negative to positive values and the behavior of Eq.(1) is chaotic.

Looking at the results obtained from the analysis of the dynamical system by the Melnikov’s technique, we have also constructed the bifurcation diagram. The bifurcation diagram shows the coordinate $x$ of the system in the Poincaré cross-section versus $\varepsilon_2$, which has been increased in small steps. An interesting question related to the problem of chaos is the way it appears in the system. We have drawn in Fig.5 the bifurcation diagram showing transition to chaos while the amplitude of the modulation of the Rayleigh number $\varepsilon_2$ increases. It follows from our numerical studies that the transition to chaos is abrupt. When $\varepsilon_2$ is increasing up to its critical value ($\varepsilon_2 = 0.0620$), the dynamic becomes chaotic. The ranges of chaotic behavior and windows of nonchaotic behavior, after the critical value of the control parameter, are illustrated in Fig.5.

The phase space presents the global behavior of the system. For this purpose, we have plotted in Fig.6 the phase space trajectories of Eq.(1) for different values of the control parameter. These values are chosen for the sequence presented in Fig.4. The initial point is chosen to be a centre of the unperturbed motion. Figs.6(a) – 6(f) present the phase space trajectories obtained under the same initial conditions. Figs.6(a) – 6(d) are plotted for $\varepsilon_2 = 0.038$, 0.040, 0.063 and 0.065 respectively. One can notice that by increasing $\varepsilon_2$, we find a serie of period-doubling bifurcations [Figs.6(a) and 6(b)], leading to chaotic trajectories for sufficiently large value of $\varepsilon_2$ [Figs.6(c) and 6(d)]. In the region of stable limit cycle, the Lyapunov exponent is negative. The sharp maxima in this region correspond to the point of period-doubling. In Figs.6(e) and 6(f), plotted for $\varepsilon_2 = 0.065$, we change the value of the parameter $f_2$ which becomes $f_2 = -0.1$ and $f_2 = -0.5$ for Figs.6(e) and 6(f) respectively. In this case, when the parameter $f_2$ tends to -1, the system which initially presented a complicated behavior becomes coherent.
4. Conclusion

In this paper, we have analyzed the transition to chaos in some hydrodynamic systems in presence of periodic modulation of the Rayleigh number. We have used a well-established mathematical technique to analyze the chaotic dynamics of the system. The occurrence of the homoclinic and heteroclinic global bifurcations is detected analytically. They represent the starting point for the successive route to full chaotic dynamics, so that they permit to obtain a safety criterion to prevent chaos. The transition from the periodic to the chaotic region is thus studied with the help of the Lyapunov exponent, the bifurcation diagram and the phase space analysis. In order to compare these numerical results to analytical predictions, we have chosen as parameter of bifurcation the amplitude of the modulation of the Rayleigh number. The theoretical predictions are confirmed by numerical simulations. In [10], the authors have shown numerically that for such systems, the chaotic region for negative values of the parameter $a$ can be found in cases where $\epsilon_2$ is larger than or comparable to $a$ and $b$. Our results are in good agreement with this earlier study. It is therefore clear that, the Melnikov method is an excellent tool, which permits to evaluate analytically the bifurcation curves of a dynamic system.

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References

**Fig. 1**: (a) Graph of the potential $V(x)$ with three wells and (b) the equivalent phase portrait.
Fig. 2: (a) The symmetric double-well potential $V(x)$ and (b) the equivalent phase portrait.
Fig. 3: Graph of the critical ratio $R_{h_{\text{sm}}}$ and $R_{h_{\text{et}}}$ vs the frequency of the modulation $\omega$ for (a) $f_2=0.0$ and (b) $f_2=-0.02$. 
Fig. 4: Lyapunov exponent of Eq. (1) vs control parameter $\epsilon_2$.

Fig. 5: Bifurcation diagram vs control parameter $\epsilon_2$. 
Fig. 6: Phase portrait showing transition from [(a) – (b)] periodic and quasiperiodic motion to [(c) – (d)] chaotic behavior, and [(e) – (f)] the contribution of the nonlinear viscosity term $f_2$. 