STABILITY OF EXCESS DEMAND FUNCTIONS WITH RESPECT TO A STRONG VERSION OF WALD’S AXIOM

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Abstract

In this paper, a strong version of Wald’s Axiom of excess demand functions $Z : P \subset \mathbb{R}^n_+ \rightarrow \mathbb{R}^n$ is introduced, namely “there exists $\sigma > 0$ such that $p, q \in P, q^T Z(p) - \delta \leq 0, |\delta| < \sigma$, and $Z(q) \neq Z(p)$ imply $p^T Z(q) + \delta > 0$”. We show that $Z$ satisfies the strong version of Wald’s Axiom iff $-Z$ is a $s$-quasimonotone function introduced in [1]. Consequently, an excess demand function $Z$ satisfies the strong version of Wald’s Axiom iff $-Z$ is stable with respect to the pseudomonotonicity (i.e., there exists $\epsilon > 0$ such that $-Z + a$ fulfills the pseudomonotonicity for all $a \in \mathbb{R}^n$ satisfying $\|a\| < \epsilon$). Some properties on the measure of the strong version of Wald’s Axiom of excess demand functions are presented.
1. Introduction

Consider excess demand functions as they occur in the theory of general economic equilibrium (see [2] and [9]). Wald’s Axiom of excess demand functions was first formulated in 1936 as a sufficient condition for uniqueness of equilibrium in a general equilibrium model. Some weak and strong versions of Wald’s Axiom were originally introduced first by Samuelson [10] in 1947 and then by Houthakker ([4]) in 1953. Since then, weak and strong versions of Wald’s Axiom (Weak Axiom of Revealed Preference, respectively) for excess demand functions (individual demand functions, respectively) have been investigated by many economists (see [3], [6], [12]). Moreover, it is well-known that the pseudomonotonicity of excess demand functions guarantees at least a convex equilibrium set. John ([7]) showed that a demand function which yields a pseudomonotone excess demand if it is combined with an arbitrary supply function (in particular, a constant function) is necessarily monotone.

In this paper, a strong version of Wald’s Axiom (denoted in short SWA) is introduced (see below) and we show that an excess demand function \( Z : P \subset \mathbb{R}^{n}_{>0} \to \mathbb{R}^n \) satisfies SWA iff \( -Z \) is a \( s \)-quasimonotone function introduced in [1] (Proposition 2.1). Consequently, an excess demand function \( Z \) satisfies SWA iff \( -Z \) is stable with respect to the pseudomonotonicity (Corollary 2.1). Moreover, we show that a demand function which yields a pseudomonotone excess demand if it is combined with a sufficiently small supply function is necessarily \( s \)-quasimonotone (Corollary 2.2) and a characterization of SWA by the Jacobian matrix \( J_Z \) is given (Proposition 2.2). Finally, some properties on the SWA measure of excess demand functions are presented (Propositions 3.1 and Corollary 3.1).

Before starting the analysis, we recall some definitions and properties (see [1]-[5] and [9]). Denote by \( T \) the matrix transposition and by \( \mathbb{R}^{n}_{>0} \) the subset of \( \mathbb{R}^n \) with all positive coordinates. In an \( n \)-good competitive economy, \( Z : P \subset \mathbb{R}^{n}_{>0} \to \mathbb{R}^n \) is said to be an excess demand function if it is homogeneous in \( p \in P \), i.e., \( Z(\lambda p) = Z(p), \lambda > 0 \) and satisfies Walras’ Law, \( p^T Z(p) = 0 \).

The following property is called Wald’s Axiom (denoted in short WA):

\[
(\text{WA}) \quad p, q \in P; \quad p^T Z(q) \leq 0 \text{ and } q^T Z(p) \leq 0 \implies Z(q) = Z(p).
\]

We introduce the strong version SWA of WA as follows

\[
(\text{SWA}) \quad \exists \sigma > 0 \text{ such that } \begin{cases} p, q \in P, & |\delta| < \sigma \smallskip \quad Z(q) \neq Z(p), q^T Z(p) - \delta \leq 0 \end{cases} \implies p^T Z(q) + \delta > 0.
\]

By setting \( \delta = 0 \) in the definition of SWA, one sees that SWA implies WA.

Recall that \( Z \) is pseudomonotone if

\[
(q - p)^T Z(p) \geq 0 \quad \implies \quad (q - p)^T Z(q) \geq 0
\]

for all \( p, q \in P \) (see [5]).
For each \(a \in \mathbb{R}^n\), let \(Z + a\) denote a function satisfying \((Z + a)(p) = Z(p) + a\) for all \(p \in P\). A function \(Z\) is said to be stable w.r.t. the property A if there exists \(\epsilon > 0\) such that \(Z + a\) has the property A for all \(a \in \mathbb{R}^n\) satisfying \(\|a\| < \epsilon\). Then, pseudomonotone functions are not stable w.r.t. their characterizations and a new kind of generalized monotone functions were introduced, namely \(s\)-quasimonotone functions. Recall that \(Z\) is \(s\)-quasimonotone if there exists \(\sigma > 0\) such that

\[
p, q \in P; |\delta| < \sigma, \frac{(q - p)^T}{\|q - p\|} Z(p) - \delta \geq 0 \Rightarrow \frac{(q - p)^T}{\|q - p\|} Z(q) - \delta \geq 0
\]

([1]). Clearly, a monotone function is \(s\)-quasimonotone and a \(s\)-quasimonotone function is pseudomonotone. Many basic properties of \(s\)-quasimonotone functions were already investigated in [1]. For example, the stability of \(s\)-quasimonotone functions are shown as follows.

**Proposition 1.1** ([1]) The following are equivalent

a) \(Z\) is \(s\)-quasimonotone;
b) \(Z\) is stable w.r.t. the \(s\)-quasimonotonicity;
c) \(Z\) is stable w.r.t. the pseudomonotonicity.

**Proposition 1.2** (see [7]) \(Z : P \subset \mathbb{R}^n \rightarrow \mathbb{R}\) is monotone iff \(Z + a\) is pseudomonotone for all \(a \in \mathbb{R}^n\).

2. Stability of excess demand functions

First, the relationship between \(s\)-quasimonotonicity of \(-Z\) and the SWA of \(Z\) is given as follows.

**Proposition 2.1**

a) Suppose that \(\text{diam} P := \sup\{\|q - p\| : p, q \in P\} < \infty\). Then SWA of \(Z\) implies the \(s\)-quasimonotonicity of \(-Z\),
b) Suppose that \(\|q - p\| \geq 1\) for all distinct \(p, q \in P\). Then the \(s\)-quasimonotonicity of \(-Z\) implies SWA of \(Z\).

**Proof:** a) Suppose that \(Z\) satisfies SWA with some \(\sigma > 0\). Set \(\sigma_1 := \sigma / \text{diam} P\) and suppose that

\[
|\delta_1| < \sigma_1, p, q \in P, \frac{(q - p)^T}{\|q - p\|} Z(p) - \delta_1 \leq 0.
\]

The case \(Z(q) \neq Z(p)\) remains to be considered. By Walras’ Law,

\[
\frac{(q - p)^T}{\|q - p\|} Z(p) - \delta_1 \leq 0
\]

is equivalent to \(q^T Z(p) - \delta_1 \|q - p\| \leq 0\). Set \(\delta := \delta_1 \|q - p\|\) then \(|\delta| < \sigma\). By SWA, we conclude that \(p^T Z(q) + \delta > 0\). It follows from Walras’ Law that

\[
\frac{(q - p)^T}{\|q - p\|} Z(q) - \delta_1 = \frac{-p^T Z(q) + \delta}{\|q - p\|} < 0.
\]
Thus, \(-Z\) is \(s\)-quasimonotone.

b) Suppose that \(-Z\) is \(s\)-quasimonotone with some \(\sigma > 0\) and

\[ p, q \in P, \ q^T Z(p) - \delta \leq 0, \ |\delta| < \sigma, \text{ and } Z(q) \neq Z(p). \]

It follows from Walras’ Law that

\[ \frac{(q - p)^T}{\|q - p\|} Z(p) - \frac{\delta}{\|q - p\|} \leq 0. \]

Set \(\delta_1 = \delta/\|q - p\|\) then \(|\delta_1| < |\delta|\) since \(\|q - p\| \geq 1\). By the \(s\)-quasimonotonicity of \(-Z\), we conclude that

\[ \frac{(q - p)^T}{\|q - p\|} Z(q) - \frac{\delta}{\|q - p\|} \leq 0. \]

Applying Walras’ Law, we get \(p^T Z(q) + \delta \geq 0\).

Now we are in position to show that \(p^T Z(q) + \delta > 0\). Assume the contrary, that \(p^T Z(q) + \delta = 0\). Applying Walras’ Law, we get

\[ \frac{(p - q)^T}{\|q - p\|} Z(q) + \frac{\delta}{\|q - p\|} = 0. \]

By the \(s\)-quasimonotonicity of \(-Z\), we conclude that

\[ \frac{(p - q)^T}{\|q - p\|} Z(p) + \frac{\delta}{\|q - p\|} \leq 0. \]

By Walras’ Law, we get \(-q^T Z(p) + \delta \leq 0\). This combining with \(q^T Z(p) - \delta \leq 0\) implies that \(q^T Z(p) = \delta\). Hence, \(q^T Z(p) = -p^T Z(q)\). Again, by Walras’ Law, we have \((q - p)^T Z(p) = (q - p)^T Z(q)\), a contradiction. 

Propositions 1.1 and 2.1 yield the stability of excess demand functions w.r.t. the pseudomonotonicity.

**Corollary 2.1** Suppose that \(\text{diam} P < \infty\) and \(\|q - p\| \geq 1\) for all distinct \(p, q \in P\). Then, \(Z\) satisfies SWA iff \(-Z\) is stable w.r.t. the pseudomonotonicity.

Let us consider continuous market demand and supply functions \(D, S : P \rightarrow \mathbb{R}_\geq^n\). \(Z := D - S\) is the excess demand function. John ([7]) showed that a demand function which yields a pseudomonotone excess demand function \(Z\) if it is combined with an arbitrary supply function (in particular, a constant function) is necessarily monotone. There arises a question: What kind of a demand function is it if the assumption “arbitrary supply function” is replaced by the assumption “supply function with sufficiently small norm”? By Proposition 1.1, we get directly the following result.

**Corollary 2.2** A demand function which yields a pseudomonotone excess demand function if it is combined with a supply function (in particular, a constant function) with sufficiently small norm, is necessarily \(s\)-quasimonotone.
The characterization of Wald’s Axiom by the Jacobian matrix $J_Z$ shown by John under the regular condition of excess demand functions is somewhat more delicate (Theorem 2 [3] and Corollary 2 [8]). Without the regular condition of excess demand functions, we get a characterization of SWA by the Jacobian matrix $J_Z$.

**Proposition 2.2** Let $Z : P \to \mathbb{R}^n$ be differentiable on the open convex set $P \subset \mathbb{R}^n$, $\text{diam} P < \infty$ and $\|q - p\| \geq 1$ for all distinct $p, q \in P$. Then, the followings are equivalent

a) $Z$ satisfies SWA;

b) Jacobian matrix $J_Z$ of the function $Z$ satisfies

$$\exists \sigma > 0 \text{ s.t. } v \in \mathbb{R}^n, p \in P, \left| \frac{v^T}{\|v\|} Z(p) \right| < \sigma \Rightarrow v^T J_Z(p) v \leq 0. \quad (2)$$

**Proof:** It follows directly from Propositions 1.1, 2.1 and Theorem 3.3 [1]. Moreover, the numbers $\sigma$ which occur in the definition of the $s$-quasimonotonicity of $-Z$ and in (2) can be chosen to be the same. \qed

3. **SWA measure of excess demand functions**

We now turn to the definition of SWA. The supremum of the set of numbers $\sigma$ given in SWA is called the SWA measure of excess demand function $Z$ and is denoted by $s_Z$. We are in position to find the SWA measure of an excess demand function. Under the assumption “$\text{diam} P < \infty$ and $\|q - p\| \geq 1$ for all distinct $p, q \in P$”, we can see from the proof of Proposition 2.1 that

$$\frac{s_Z}{\text{diam} P} \leq s^*_Z := \sup \{ \sigma > 0 : -Z \text{ satisfies (1)} \} \leq s_Z. \quad (3)$$

On the other hand, it follows from Proposition 2.1 b) and the proof of Proposition 2.2 that

$$s^*_Z = \sup \{ \sigma : Z \text{ satisfies (2)} \}. \quad (4)$$

It is of interest to know if $s_Z = +\infty$ implies that $-Z$ is monotone. By Propositions 1.1-1.2 and 2.1 we conclude that $s^*_Z = +\infty$ iff $-Z$ is monotone. Therefore, (3) yields

**Proposition 3.1** Suppose that $\text{diam} P < \infty$ and $\|q - p\| \geq 1$ for all distinct $p, q \in P$. Then, $s_Z = +\infty$ iff $-Z$ is monotone.

It follows from Propositions 2.3 and 3.1 the following

**Corollary 3.1** Suppose that $P$ is open and convex, $\text{diam} P < \infty$, $\|q - p\| \geq 1$ for all distinct $p, q \in P$ and $Z$ is differentiable and satisfies SWA. If $-Z$ is not monotone, then the set of all $\sigma > 0$ given in (2) is bounded above.

**Proof:** Assume the contrary that the set of all $\sigma > 0$ given in (2) is not bounded above. It follows from (4) that $s^*_Z = +\infty$ and therefore, $-Z$ must be monotone, a contradiction. \qed

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4. Concluding remarks

We have seen that SWA is a strong version of WA and guarantees both uniqueness of equilibrium in a general equilibrium model and stability w.r.t. the pseudomonotonicity of excess demand functions. Hence, it is very important to estimate the SWA measure $s_Z$ of an excess demand function $Z$. It can be shown that if $P \subset \mathbb{R}^1$, then based on Propositions 2.2, 3.1 and Corollary 3.1, an algorithm for finding $s_Z^*$ is given. Therefore, by (3), $s_Z$ is estimated. The problem to estimate $s_Z$ in case $P \subset \mathbb{R}^n$ with $n > 1$ is still open.

Through the present paper, under the condition SWA, the homogeneity of excess demand functions can be omitted and the stability investigation is done under the Walras’ Law. In fact, a weak version of Walras’ Law, namely, “$p^T Z(p) \leq 0$ for all $p \in P$”, was used by Debreu and Smale (see p. 82 [2] and p. 338-340 [11]). We can continue the stability investigation of excess demand functions under this weak version of Walras’ Law and the stability investigation w.r.t. Weak Axiom of Revealed Preference for individual demand functions in a parallel fashion. These problems will be the subjects of another paper.

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References


