HELLY-TYPE THEOREMS FOR ROUGHLY CONVEXLIKE SETS

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Abstract

For a given positive real number $\gamma$, a subset $M$ of an $n$-dimensional Euclidean space is said to be roughly convexlike (with the roughness degree $\gamma$) if $x_0, x_1 \in M$ and $\|x_1 - x_0\| > \gamma$ imply $]x_0, x_1[ \cap M \neq \emptyset$. In this paper, we present Helly-type theorems for such sets then solve an open question about sets of constant width raised by Buchman & Valentine and Sallee [Croft, Falconer and Guy, Unsolved Problems in Geometry, Springer-Verlag New York, Inc. 1991, pp. 131-132].

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1. Introduction

Let $\mathbb{E}^n$ denote the $n$-dimensional Euclidean space. Helly’s famous theorem states

$$
\text{Let } F \text{ be a collection of convex sets in } \mathbb{E}^n, \text{ finite or all members of } F \text{ compact. Then if every } n + 1 \text{ sets have at least one point in common, then all of } F \text{ have at least one intersecting point.}
$$

([8] and [15]). A large number of its applications can be seen in [3], [5], [9] and [16]. The following Klee’s theorem is given in [5] and [9].

$$
\text{Let } F \text{ be a collection of at least } n + 1 \text{ convex sets in } \mathbb{E}^n, M \text{ a convex set in } \mathbb{E}^n \text{ and } F \text{ finite or } M \text{ and all members of } F \text{ compact. If every } n + 1 \text{ sets of } F \text{ intersect some translate of } M \text{ simultaneously, then all members of this family intersect some translate of } M \text{ simultaneously.}
$$

This result is also valid if the word “intersect” is replaced by “contain” or “are contained in”. Klee’s theorem leads to the open question raised by Buchman & Valentine and Sallee (see [3], [4] and [16]):

"What conditions are required to ensure that if every $n + 1$ of a family of compact convex sets in $\mathbb{E}^n$ intersect a set of constant width $d$ simultaneously, then all members of this family intersect a set of constant width $d$ simultaneously?"

In this paper, Helly-type theorems for roughly convexlike sets are given (Theorems 2.1-2.2) and then the open question above is solved (Corollary 2.1 and Example 2.1).

Before starting the analysis, we recall some definitions and properties. The self-Jung constant of a normed space $X$ is defined by

$$
J_s(X) := \sup \left\{ \frac{2 \inf_{x \in \text{conv} M} \sup_{y \in M} \|x - y\|}{\sup_{y, z \in M} \|y - z\|} : M \subset X \text{ is bounded, non-empty, and non-singleton} \right\},
$$

where $\text{conv} M$ denotes the convex hull of $M$ (i.e., the smallest convex set that includes $M$). It is shown in [1] that $J_s(X) \leq \frac{2n}{n+1}$ if $\dim X = n$.

For $x_0, x_1 \in X$, set

$$
[x_0, x_1] := \{(1 - \lambda)x_0 + \lambda x_1 : \lambda \in [0, 1]\}
$$

$$
| x_0, x_1 | := [x_0, x_1] \setminus \{x_0, x_1\}.
$$

For a given positive real number $\gamma$, a set $M \subset X$ is said to be roughly convexlike (with the roughness degree $\gamma$) if $x_0, x_1 \in M$ and $\|x_1 - x_0\| > \gamma$ imply $|x_0, x_1| \cap M \neq \emptyset$ ([14]). Note that $\delta$-convexity ([2]) and outer $\gamma$-convexity ([14]) are special kinds of rough convexlikeness. The nonconvexity measure of a closed and roughly convexlike set in concrete normed spaces is estimated by the self-Jung constant. Theorem 3.1 [12] says that if $M \subset X$ is closed and roughly
convexlike (with the roughness degree $\gamma$) and $x \in \text{conv}M \setminus M$, then there exists $z \in M$ such that $\|x - z\| \leq \frac{1}{2} J_s(X) \gamma$. Since $J_s(E^n) = \sqrt{\frac{2n}{n+1}}$ (see [7] and [10]) we conclude that

**Proposition 1.1:** Suppose $M \subset E^n$ is closed and roughly convexlike (with the roughness degree $\gamma$). Then

$$\forall x \in \text{conv}M \setminus M \exists z \in M : \|x - z\| \leq \gamma \sqrt{\frac{n}{2(n+1)}}.$$ 

A set of constant width is a bounded closed convex set for which every two parallel support hyperplanes are at the same distance apart. A ball $B(x, R)$ denotes the closed solid ball in $X$ of radius $R$ about the centre $x$. $B(x, R)$ is called the circum-ball of a set $M$ if $M \subset B(x, R)$ and $R$ is the smallest possible (see [6]).

**Proposition 1.2** ([6] and [11]): For a given set of constant width $d$ in $E^n$, there is a circum-ball of this set with radius $R$ satisfying

$$\frac{1}{2} d \leq R \leq d \sqrt{\frac{n}{2(n+1)}}. \quad (1)$$

Both these limits are attained, the first for a ball and the second for any set of constant width $d$ which contains a regular simplex of diameter $d$. When $n = 2$ this class reduces essentially to one set, the Reuleaux triangle of side $d$ (formed from three equal circular arcs of radius $d$).

2. Results

The first Helly-type theorem for roughly convexlike sets is stated as follows.

**Theorem 2.1:** Let $\mathcal{F}$ be a collection of closed and roughly convexlike sets (with the same roughness degree $\gamma$) in $E^n$, $\mathcal{F}$ finite or all members of $\mathcal{F}$ compact. If the $n+1$ convex hulls corresponding to every $n+1$ sets of $\mathcal{F}$ have at least one point in common, then all the members of $\mathcal{F}$ have at least one ball with radius $\gamma \sqrt{\frac{n}{2(n+1)}}$ in their intersection.

**Proof:** Due to Helly’s Theorem, all the members of $\mathcal{F}^*$ (family of convex hulls of roughly convexlike sets given in $\mathcal{F}$) have at least one point in common, say $x$. By Proposition 1.1, the ball with centre $x$ and radius $\gamma \sqrt{\frac{n}{2(n+1)}}$ intersects all the members of $\mathcal{F}$. \qed

Note that the number $\gamma \sqrt{\frac{n}{2(n+1)}}$ in Theorem 2.1 is the best possible in $E^n$. To see it, consider the equilateral triangle $\triangle abc$ of side $\gamma$ with centre $x$ in $E^2$. Choose $a_1, b_1$ and $c_1$ respectively on small circular arcs $bc, ca$ and $ab$ respectively of the circum-circle of $\triangle abc$ such that $\triangle a_1b_1c_1$ is equilateral. Then

$$\mathcal{F} := \{\{a, b, c\}; \{a_1, b_1, c_1\}; \{a_1, b, c_1\}; \{a_1, b, c\}\}$$

is the collection of compact and roughly convexlike sets (with the same roughness degree $\gamma$) and 3 convex hulls corresponding to every 3 sets of $\mathcal{F}$ have at least one point in common (e.g.

$$\text{conv}\{a, b, c\} \cap \text{conv}\{a_1, b_1, c_1\} \cap \text{conv}\{a_1, b, c\} \neq \emptyset.$$
Clearly, the circum-ball of equilateral triangle $\triangle abc$ (with centre $x$ and radius $\gamma \sqrt{\frac{n}{2(n+1)}}$) intersects all sets of $\mathcal{F}$ and it is a smallest one which intersects every member of $\mathcal{F}$. Moreover, this example also indicates that the number $\gamma \sqrt{\frac{n}{2(n+1)}}$ in Proposition 1.1 is the best possible in $IE^n$.

**Theorem 2.2:** Let $\mathcal{F}$ be a collection of closed and roughly convexlike sets (with the same roughness degree $\gamma$) in $IE^n$, $\mathcal{F}$ finite or all members of $\mathcal{F}$ compact. If the $n+1$ convex hulls corresponding to every $n+1$ sets of $\mathcal{F}$ intersect a set of constant width $d$ simultaneously, then all sets of this family intersect simultaneously a set of constant width $w$ satisfying

$$w \leq \gamma \sqrt{\frac{2n}{n+1}} + d.$$  

(2)

**Proof:** Consider $n+1$ arbitrary members of the family $\mathcal{F}$, say $M_{i_0}, M_{i_1}, \ldots, M_{i_n}$. Then their convex hulls $\text{conv} M_{i_0}, \text{conv} M_{i_1}, \ldots, \text{conv} M_{i_n}$ intersect some set $M$ of constant width $d$ simultaneously. By Proposition 1.2, there is a circum-ball $B(p, R_{i_0}, R_{i_1}, \ldots, R_{i_n})$ of $M$ satisfying (1) with some $p \in IE^n$. It follows that $\text{conv} M_{i_0}, \text{conv} M_{i_1}, \ldots, \text{conv} M_{i_n}$ intersect $B(p, R_{i_0}, R_{i_1}, \ldots, R_{i_n})$ simultaneously. Denote the supremum of numbers $R_{i_0}, R_{i_1}, \ldots, R_{i_n}$ by $R$. Then $R$ satisfies (1). According to Helly’s theorem, for the family of convex sets

$$\mathcal{F}_B := \{ \text{conv} M + B(0, R) : M \in \mathcal{F} \},$$

we conclude that all members of $\mathcal{F}_B$ have at least one point in common, say $q$. It follows from Proposition 1.1 that the members of $\mathcal{F}$ have the ball with center $q$ and radius $\gamma \sqrt{\frac{n}{2(n+1)}} + R$, in intersecting. This ball can be taken to be a set $K$ of constant width

$$w = 2 \left( \gamma \sqrt{\frac{n}{2(n+1)}} + R \right) = \gamma \sqrt{\frac{2n}{n+1}} + 2R.$$  

(3)

Therefore, the members of $\mathcal{F}$ have at least $K$, in intersecting. By (1) and (3), the inequality (2) is satisfied. $\square$

Theorem 2.2 yields a result dealing with Salle and Buchman & Valentine’s question as follows

**Corollary 2.1:** Let $\mathcal{F}$ be a collection of closed and convex sets in $IE^n$, $\mathcal{F}$ finite or all members of $\mathcal{F}$ compact. If every $n+1$ sets of $\mathcal{F}$ intersect a set of constant width $d$ simultaneously, then all sets of this family intersect simultaneously a set of constant width $w$ satisfying

$$d \leq w \leq d \sqrt{\frac{2n}{n+1}}.$$  

(4)

**Proof:** Take an arbitrary positive real number $\gamma$. Then $\mathcal{F}$ is a collection of closed and roughly convexlike sets with the roughness degree $\gamma$. Hence, by Theorem 2.2, (2) holds true for all $\gamma > 0$. It follows that all sets of $\mathcal{F}$ intersect simultaneously a set of constant width $w$ satisfying

$$d < w \leq d \sqrt{\frac{2n}{n+1}}.$$
If every \( n + 1 \) sets of \( F \) intersect a set of constant width \( d \) simultaneously which is a closed ball with diameter \( d \), then by Klee’s Theorem, all sets of \( F \) intersect simultaneously a some transfer of this ball. Since a transfer of this ball is a set of constant width \( w = d \), the left limit in (4) holds true.

If every \( n + 1 \) sets of \( F \) intersect a set of constant width \( d \) simultaneously and this set contains a regular simplex of diameter \( d \) then the radius of it’s circum-ball is \( R = d \frac{n}{2(n + 1)} \) (see p. 125 [6]). By Klee’s Theorem, all sets of \( F \) intersect simultaneously a some transfer of this circum-ball. Since a transfer of this ball is a set of constant width \( w = 2R \), the right limit in (4) holds true.

Finally, we show an example in \( E^2 \) to illustrate that although every 3 of a family of compact convex sets intersect a Reuleaux triangle of side \( d \) simultaneously, there does not exist any set of constant width \( d \) which intersects all members of this family simultaneously and there is a set of constant width \( w = 2d\sqrt{\frac{1}{3}} \) which intersects all members of this family simultaneously.

**Example 2.1:** Suppose that \( \triangle abc \) is an equilateral triangle of side \( d \) in \( E^2 \). Denote the middle points of small circular arcs \( ab, bc, \) and \( ca \) of the circum-circle of \( \triangle abc \) by \( q, r \) and \( t \), respectively. Denote the point symmetrical about \( b \) (the line \( ab \), respectively) with the centre of the \( \triangle abc \) (with \( c \), respectively) by \( g \) (by \( h \), respectively). Denote the intersects of lines \( bq \) and \( at, bq \) and \( rc \), and, \( rc \) and \( at \) by \( p, m \) and \( n \), respectively (see Figure 1). Set

\[
F := \{ [b, m]; [c, r]; [a, t]; q \}.
\]

Then every 3 sets of \( F \) intersect a set of constant width \( d \) simultaneously. Indeed, the sets \([b, m]; [c, r]; [a, t]\) intersect Reuleaux triangle \( abc \), the sets \([b, m]; [c, r]; q\) intersect Reuleaux triangle \( gqr \), the sets \([b, m]; [a, t]; q\) intersect Reuleaux triangle \( abh \) and the sets \([c, r]; [a, t]; q\) intersect Reuleaux triangle \( qrt \). Then there is no set of constant width \( d \) intersecting all sets of \( F \) (assume the contrary, that some set \( M \) of constant width \( d \) intersects all sets of \( F \).
Since diam$M = d$ and since the distance between $q$ and $[c, r]$ ($[b, m]$ and $[a, t]$, respectively) is $\|q - r\| = d$ ($\|a - b\| = d$, respectively), we conclude that $q, r \in M$ ($a, b \in M$, respectively). It follows that $d = \text{diam}M \geq \|a - r\| = 2d/\sqrt{3}$, a contradiction. In this case, the circum-ball of equilateral triangle $\triangle abc$ intersects all sets of $\mathcal{F}$ and it is a set of constant width $w = 2d\sqrt{\frac{2}{3}}$.

4. Concluding remarks

In [13], a nonconvexity measure of a bounded set estimated by the self-Jung constant was presented. Using this result, we can propose a Helly-type theorem for bounded sets and some applications. This problem and computational aspects of Helly-type theorems mentioned above will be the subjects of another paper.

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