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**AUTOCORRELATION BASED RECONSTRUCTION
OF TWO-DIMENSIONAL BINARY OBJECTS**

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Abstract

A method for reconstructing two-dimensional binary objects from its autocorrelation function is discussed. The objects consist of a finite set of identical elements. The reconstruction algorithm is based on the concept of *class of element pairs*, defined as the set of element pairs with the same separation vector. This concept allows to solve the redundancy introduced by the element pairs of each class. It is also shown that different objects, consisting of an equal number of elements and the same classes of pairs, provide Fraunhofer diffraction patterns with identical intensity distributions. However, the method predicts all the possible objects that produce the same Fraunhofer pattern.

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1. INTRODUCTION

A common problem in disciplines such as optics, X-ray crystallography and astronomy is the reconstruction of objects from their autocorrelation function [1-3]. Such objects can be classified as [4]:

- *amplitude-only objects* if they modify the amplitude but not the phase of an incident wavefront;
- *phase-only objects* if they modify the phase but not the amplitude and
- *amplitude and phase objects* if they modify both the amplitude and phase of the wavefront.

The complexity of the reconstruction algorithms depends on the object structure in such a way that the simplest ones concern the reconstruction of amplitude-only objects. Nowadays, the reconstruction of discrete binary objects, i.e. finite sets of identical amplitude only elements, is interesting for technological purposes. Many of the algorithms used for their reconstruction [5-8] base their ability on specific realizations of the Wiener-Kintchine theorem [9]. Indeed, they analyze the object autocorrelation obtained by Fourier transforming the intensity distribution of the Fraunhofer diffraction pattern, which is provided by the object when it is illuminated by a plane wavefront.

The performance of such algorithms can be optimized by using the concept of *class of element pairs*, as we will show below. This concept is defined as the set of element pairs with the same separation vector [10]. Both the Fraunhofer diffraction patterns and the object autocorrelations can be analyzed in terms of classes of element pairs. The efficiency of the analysis should be improved by regarding the concepts of *object and autocorrelation supports* given by Brames [7]. Indeed, limitations of reconstruction due to non-uniqueness and redundancy can be avoided. Non-uniqueness means that two or more different objects, consisting of the same number of identical elements and having the same classes of element pairs, yield Fraunhofer diffraction patterns with identical intensity distribution. Redundancy is due to the element pairs and this means that there are two or more elements in the object with identical vector separations.

2. FRAUNHOFER DIFFRACTION AND CLASSES OF ELEMENT PAIRS

Let us consider a discrete binary object, composed by N identical apertures (elements) randomly distributed on an opaque screen, but in such a way that their geometrical centres are located onto the intersections of a Cartesian grid with pitch s . This pitch should be greater than the size of the individual elements, so that they cannot overlap each other.

The transmissions of the individual elements and the whole object will be given by $h(\mathbf{r}) = 1$ within the element and equal to null outside of it, and

$$t(\mathbf{r}) = h(\mathbf{r}) \otimes \sum_{q=1}^N \delta(\mathbf{r} - \mathbf{r}_q) \quad (1)$$

respectively, with $\mathbf{r} = (x, y)$ the position vector, $\otimes$ the convolution operation and $\delta(\mathbf{r} - \mathbf{r}_q)$ the Dirac's delta function placed at $\mathbf{r} = \mathbf{r}_q$. The positions of the aperture centres can be denoted as $\mathbf{r} = (n, m) \times s$, with n and m integers.

This object is illuminated by a spatially coherent plane wave with amplitude A_0 and wavelength λ , and the Fraunhofer diffraction pattern it provides on a plane at a distance z is observed. The intensity distribution of the pattern will take the form [11]

$$I(\mathbf{u}) = (A_0/\lambda z)^2 \left| \int t(\mathbf{r}) \exp(-ik \mathbf{r} \cdot \mathbf{u}/z) d^2 r \right|^2 = I_0 |H(\mathbf{u})|^2 \sum_{q=1}^N \sum_{l=1}^N \exp[-ik (\mathbf{r}_q - \mathbf{r}_l) \cdot \mathbf{u}/z], \quad (2)$$

where $k = 2\pi/\lambda$ is the wave-number and $k \mathbf{u}/z$ is a vector, whose components are the spatial frequencies, $I_0 = (A_0/\lambda z)^2$, and $H(\mathbf{u}) = \int h(\mathbf{r}) \exp(-ik \mathbf{r} \cdot \mathbf{u}/z) d^2 r$.

Equation (2) can be split into two terms, one for $q = l$ and the other for $q \neq l$, which correspond to:

- the superposition of the intensity contributions from the individual elements ($q = l$) and
- the superposition of the interference patterns, due to contributions from the element pairs ($q \neq l$), each one described by $\exp\{-ik (\mathbf{r}_q - \mathbf{r}_l) \cdot \mathbf{u}/z\} + \exp\{ik (\mathbf{r}_q - \mathbf{r}_l) \cdot \mathbf{u}/z\} = 2 \cos[k (\mathbf{r}_q - \mathbf{r}_l) \cdot \mathbf{u}/z]$.

Therefore, equation (2) becomes

$$I(\mathbf{u}) = I_0 |H(\mathbf{u})|^2 \left\{ N + 2 \sum_{q=1}^{N-1} \sum_{l=j+1}^N \cos[k (\mathbf{r}_q - \mathbf{r}_l) \cdot \mathbf{u}/z] \right\}. \quad (3)$$

It means that the intensity distribution of the Fraunhofer diffraction pattern results from the superposition of the intensity distributions from the Young interference patterns [4] produced by all element pairs of the object. Note that the pairs with the same separation vector $\mathbf{r}_j = \mathbf{r}_q - \mathbf{r}_l$ generate the same Young pattern, i.e. the fringes will exhibit the same frequency and orientation, independently from the specific values of the position vectors $\mathbf{r}_q$ and $\mathbf{r}_l$.

Equation (3) can be alternatively written as

$$I(\mathbf{u}) = I_0 |H(\mathbf{u})|^2 \left[N + 2 \sum_{j=1}^J p_j \cos(k \mathbf{r}_j \cdot \mathbf{u}/z) \right], \quad (4)$$

where the canonical term $I_0 |H(\mathbf{u})|^2 p_j 2 \cos(k \mathbf{r}_j \cdot \mathbf{u}/z)$ refers to the complete set of element pairs with the same separation vector $\mathbf{r}_j$, that is the j th class of element pairs. The coefficient p_j denotes the class population, i.e. the number of pairs of the class, whose value depends on the

spatial distribution of the elements in the object. Thus, an object with N identical elements will have $N(N-1)/2$ pairs, which can be gathered into J classes in such a way that

$$\sum_{j=1}^J p_j = N(N-1)/2.$$

Consequently, there are, in general, different objects with the same number of elements and the same classes of element pairs that provide identical Fraunhofer diffraction patterns, as shown in Figures 1(a), 1(b) and 1(c). Rotating or translating the object in Fig. 1(b) cannot obtain the object in Fig. 1(a). In other words, the inverse problem related to the recovery of the binary object from the Fraunhofer diffraction pattern is a multi-valued problem, i.e. it does not have in general a unique solution.

Figure 1: (a) and (b) two distinct objects with the same number of elements and the same classes of element pairs, (c) Fraunhofer diffraction pattern generated by both objects, (d) autocorrelation

3. IDENTIFYING THE CLASSES OF ELEMENT PAIRS BY MEANS OF THE AUTOCORRELATION

According to the Wiener-Kintchine theorem, the expression $\Gamma_u(\mathbf{r}) = \int I(\mathbf{u}) \exp(-ik \mathbf{r} \cdot \mathbf{u} / z) d^2u$ represents the autocorrelation of the object transmission under uniformly and spatially coherent illumination. On account of equation (4), it can be expressed as

$$\Gamma_u(\mathbf{r}) = I'_0 \Gamma_{hh}(\mathbf{r}) \otimes \left\{ N \delta(\mathbf{r}) + 2 \sum_{j=1}^J p_j [\delta(\mathbf{r} + \mathbf{r}_j) + \delta(\mathbf{r} - \mathbf{r}_j)] \right\}, \quad (5)$$

with $I'_0 = \lambda z I_0 = A_0^2 / \lambda z$ and $\Gamma_{hh}(\mathbf{r}) = \int |H(u)|^2 \exp(-ik \mathbf{r} \cdot \mathbf{u} / z) d^2u$ the autocorrelation of the transmission of the individual elements.

Equation (5) points out that the normalized Fourier spectrum $\Gamma_u(\mathbf{r}) / I'_0$ consists of a set of peaks, centred at the positions $\mathbf{r}_j$ determined by the separation vectors of the classes of element pairs, whose shapes are determined by the autocorrelation $\Gamma_{hh}(\mathbf{r})$. The height of the peak at the coordinate origin is N , whereas the height of the peaks located at the positions $\mathbf{r} = \pm \mathbf{r}_j$ is p_j .

In summary, the j th peak reveals the presence of the class of element pairs with separation vector $\mathbf{r}_j$, and its height determines the class population p_j . So, the autocorrelation obtained by Fourier transforming the intensity distribution of the Fraunhofer diffraction pattern allows us to identify all the classes of element pairs that conform the object.

Now, this interpretation of the autocorrelation allows to design an algorithm for obtaining all the solutions of the inverse problem i.e. all the binary objects with the same autocorrelation. In this sense, the reconstruction task can be exhaustively performed [see Figure 1(d) that shows the autocorrelation of both objects in Figures 1(a) and 1(b)].

4. OBJECT SUPPORT AND AUTOCORRELATION SUPPORT

The determination of the support of all the objects compatible to the support of a given autocorrelation is the first step toward the complete solution of the inverse problem. To perform it, we will apply the definitions of both the object and the autocorrelation supports according to Brames [7]. So, two sets of points located onto the intersections of a Cartesian grid with pitch s will be used to represent the positions of both the object elements and the autocorrelation peaks respectively. The transmission of the object elements and the heights of the autocorrelation peaks will be denoted by means of two corresponding matrices.

The object support

Following the above reasoning and according to equation (1), the object transmission can be written as a discrete distribution of Dirac's delta functions of unitary amplitude, located at the crossings of the Cartesian grid, that is,

$$O \equiv \sum_{q=1}^N (1) \delta(\mathbf{r} - \mathbf{r}_q). \quad (6)$$

Let us suppose that the sizes of the object along the grid axes are $n_o \times s$ and $m_o \times s$ respectively, with n_o and m_o positive integers. Therefore, the object will be contained within a rectangle of area $(n_o \times m_o) s^2$.

Figure 2: Object in Figure 1(a). (a) Object $\tilde{O}$ according to equation (7). (b) and (c) representations of the object in terms of the set of points R_o and the matrix $[\tilde{O}]$

As denoted in Figure 2(a), we fix the coordinate origin at the upper left corner of the rectangle and take the axes along its sides as positive definite. Thus, the position vectors for Dirac's delta functions of the object will take the form $\mathbf{r}_q = \mathbf{r}_{(n,m)} = (n, m) \times s$, with $0 \leq m \leq m_o$ and $0 \leq n \leq n_o$. Then, equation (6) becomes

$$\tilde{O} \equiv \sum_{n=0}^{n_o} \sum_{m=0}^{m_o} o_{(m+1,n+1)} \delta(\mathbf{r} - \mathbf{r}_{(n,m)}), \quad (7)$$

where

$$o_{(m+1,n+1)} = 0, \text{ if there is not an element at } \mathbf{r}_{(n,m)} = (n, m) \times s, \text{ and}$$

$$o_{(m+1,n+1)} = 1, \text{ if there is an element at } \mathbf{r}_{(n,m)} = (n, m) \times s.$$

So, we represent the object by the matrix $[\tilde{O}] = \{o_{(m+1, n+1)}\}$, with $(m_o + 1 \times n_o + 1)$ elements, and equivalently by the set of points $R_o = \{\mathbf{r}_{(n, m)}\}$ distributed over a rectangle of area $(n_o \times m_o) s^2$. Figure 2(a) shows the object in Figure 1(a) according to equation (7). Figures 2(b) and 2(c) are the representations in terms of the set of points R_o and the matrix $[\tilde{O}]$ respectively.

Now, the object support S_o is defined as the convex region with the smallest area that contains completely the object represented by the set of points R_o . A region is convex if all the points of the straight segments that join two arbitrary points of the region belong to the region. Therefore, the object support will be delimited by a convex polygon with N_o sides and the same number of vertices, denoted as Pol_o . The vertices will be points of R_o determined by position vectors

$$\mathbf{r}_k^{(o)}, \quad k = 1, 2, \dots, N_o, \quad (8)$$

ordered in counter clockwise. Then, we define the k th side of the polygon as

$$lo_k : \mathbf{o}_k = \eta \mathbf{r}_{k-1}^{(o)} + (1 - \eta) \mathbf{r}_k^{(o)}, \quad 0 \leq \eta \leq 1. \quad (9)$$

Consequently, the union of all sides lo_k will give the polygon

$$\text{Pol}_o = \cup lo_k, \quad k = 1, 2, \dots, N_o. \quad (10)$$

The autocorrelation support

On account of equation (5), the autocorrelation of the object transmission will be given by

$$A \equiv \left\{ N \delta(\mathbf{r}) + 2 \sum_{j=1}^J p_j [\delta(\mathbf{r} + \mathbf{r}_j) + \delta(\mathbf{r} - \mathbf{r}_j)] \right\}. \quad (11)$$

Equation (11) describes a distribution of Dirac's delta functions, which is symmetric with respect to the coordinate origin. The amplitudes will be N for the peak at the origin and p_j for the other peaks. Furthermore, the autocorrelation sizes along the grid axes are $(2n_o) \times s$ and $(2m_o) \times s$ respectively, so that it will be contained within a rectangle of area $4(n_o \times m_o) s^2$.

Once again let us fix the coordinate origin at the upper left corner of the rectangle and let us consider the semi-axes along its sides as positive definite. Thus, the symmetry centre of the autocorrelation will be located at $\mathbf{r}_o = \mathbf{r}_{(n_o, m_o)} = (n_o, m_o) \times s$, and equation (11) becomes

$$\tilde{A} \equiv \left\{ \sum_{l=0}^{2n_o} \sum_{k=0}^{2m_o} a_{(k+1, l+1)} \delta(\mathbf{r} - \mathbf{r}_{(l, k)}) \right\}, \quad (12)$$

with $\mathbf{r}_{(l,k)} = (l, k) \times s$ and

$a_{(k+1,l+1)} = 0$ if there is not a class of element pairs with separation vector $\mathbf{r}_{(l,k)} - \mathbf{r}_O$.

$a_{(k+1,l+1)} = N$ for $(k,l) = (n_O, m_O)$, and

$a_{(k+1,l+1)} = p_j$, if there is a class of element pairs with separation vector $\mathbf{r}_j = \mathbf{r}_{(l,k)} - \mathbf{r}_O$,

that is, $\mathbf{r}_j = \mathbf{r}_{(l,k)} - \mathbf{r}_{(n_O, m_O)} = (l - n_O, k - m_O) \times s$.

According to the symmetry of the autocorrelation, the vectors $\pm \mathbf{r}_j$ denote the same class of element pairs. Consequently, the autocorrelation will be redundant, i.e. all the required information for completely determining the classes of element pairs will be contained within a half of it.

So, we represent the autocorrelation by the matrix $[\tilde{A}] = \{a_{(k+1,l+1)}\}$, with $(2m_O + 1 \times 2n_O + 1)$ elements, and equivalently by the set of points $R_A = \{\mathbf{r}_{(l,k)}\}$ distributed over a rectangle of area $4(n_O \times m_O) s^2$. Figure 3(a) shows the autocorrelation of the object transmission in Figure 1(a). Figures 3(b) and 3(c) show the corresponding set of points R_A and the matrix $[\tilde{A}]$ respectively.

Figure 3: Autocorrelation of the objects in Figures 1(a) and 1(b). (a) Autocorrelation $\tilde{A}$, according to equation (12). (b) and (c) representations of the autocorrelation in terms of the set of points R_A and the matrix $[\tilde{A}]$

The centre of symmetry of this autocorrelation will be located at $\mathbf{r}_{(n_O, m_O)} = \mathbf{r}_{(3,3)}$, and there is a peak of amplitude $a_{(k+1,l+1)} = a_{25} = 1$ at the position $\mathbf{r}_{(l,k)} = \mathbf{r}_{(4,1)}$, which reveals the presence of a class with only one element pair, whose separation vector is $\mathbf{r}_j = \mathbf{r}_{(4,1)} - \mathbf{r}_{(3,3)} = (1, -2) \times s$.

In a similar way as before, the autocorrelation support S_A is defined as the convex region with the smallest area, which completely contains the autocorrelation represented by the set of points R_A . This support will be delimited by the convex polygon Pol_A , with N_A sides and the same number of vertices, which are determined by the position vectors

$$\mathbf{r}_k^{(A)}, \quad k = 1, 2, \dots, N_A \quad (13)$$

All the vertices of Pol_A will be points of the set R_A with an associated amplitude of 1 in the matrix $[\tilde{A}]$. The sides of this polygon will be defined as

$$la_k : \mathbf{a}_k = \eta \mathbf{r}_{k-1}^{(A)} + (1 - \eta) \mathbf{r}_k^{(A)}, \quad 0 \leq \eta \leq 1. \quad (14)$$

Consequently, the polygon will be defined as

$$\text{Pol}_A = \cup la_k, \quad k = 1, 2, \dots, N_A. \quad (15)$$

Figure 4: (a) Convex polygon Pol_O that delimits the object support S_O , (b) convex polygon Pol_A that delimits the autocorrelation support S_A

Figures 4(a) and 4(b) show the convex polygons that delimit the supports S_O and S_A of the object in Figure 1(a) and its autocorrelation respectively.

Determination of the object support from the autocorrelation support

Following Brames [7], it is possible to determine all object supports that provide the same autocorrelation support if the object is discrete [equation (6)]. Because of the redundancy of the

autocorrelation, Pol_O can be obtained from only a half of Pol_A , i.e. $\frac{1}{2}\text{Pol}_A$. Following Brames' method, $\frac{1}{2}\text{Pol}_A$ can be obtained by an appropriate union of the sides of Pol_O , i.e. some sides of $\frac{1}{2}\text{Pol}_A$ can result from the union of two parallel sides of Pol_O . The lengths of the joined segments satisfy the condition $s\sqrt{m^2 + n^2}$, with m and n integers. This procedure yields new convex polygons that determine the supports of all objects, which provide the same autocorrelation support.

Figure 5: Object supports obtained from the autocorrelation support according to Brames' method

Figure 5 shows all object supports obtained from the autocorrelation in Figure 4(b). So, Pol_{O1} delimits the object support of Figure 1(a), Pol_{O2} the object support of Figure 1(b) and Pol_{O3} the

support of other objects that can solve the inverse problem too. While Pol_{o1} is straightforwardly obtained by joining the sides of $\frac{1}{2}\text{Pol}_A$, Pol_{o2} and Pol_{o3} are obtained by splitting some sides of $\frac{1}{2}\text{Pol}_A$, and then joining the new segments and the remainder sides of $\frac{1}{2}\text{Pol}_A$.

5. THE RECONSTRUCTION ALGORITHM

Object supports corresponding to solutions for the inverse problem

Brames' method determines all object supports compatible with a specific autocorrelation support. However, only a subset of them will provide the same autocorrelation function $[\tilde{A}]$. To identify it we compare the classes of element pairs corresponding to a set of points $R_o^{(P)}$, named *the proving object*, with the classes of element pairs in the autocorrelation.

$R_o^{(P)}$ is distributed over a rectangular area of size $(n_o \times m_o)s^2$ according to Brames' method, and $[\tilde{O}]^{(P)}$ denotes the $(m_o + 1 \times n_o + 1)$ binary matrix associated to it. The autocorrelation of this matrix, denoted by $[\tilde{A}]^{(P)}$, will be effectively compared with the $(2m_o + 1 \times 2n_o + 1)$ matrix $[\tilde{A}]$. The central element of $[\tilde{A}]$ gives the total number of points of the proving object, i.e. $a_{(m_o+1, n_o+1)} = N$.

Now, we determine the initial N_o points of $R_o^{(P)}$, placed at the vertices of the polygon Pol_o that delimits the object support S_o [see equation (8)]. Consequently, the matrix $[\tilde{O}]^{(P)}$ will initially have N_o elements $o_{(m+1, n+1)}^{(P)} = 1$, which correspond to those points. Its remaining elements will be equal to null.

We use the matrix difference

$$[D] = \{d_{(k+1, l+1)}\} = [A] - [\tilde{A}]^{(P)}. \quad (16)$$

to compare the matrices $[\tilde{A}]$ and $[\tilde{A}]^{(P)}$. If $[D]$ satisfies the condition

$$d_{(k+1, l+1)} \geq 0 \quad (17)$$

for all values of k and l , then all the classes of element pairs of $R_o^{(P)}$ are included in $[\tilde{A}]^{(P)}$ too. Consequently, S_o will be the support of a solution object. If $d_{(k+1, l+1)} < 0$ holds for any value of k and l , $[\tilde{O}]^{(P)}$ will contain classes that do not appear in $[\tilde{A}]$. Therefore, the corresponding object support will be not a solution for the inverse problem.

Thereafter, the remaining $(N - N_o)$ points for the solution object can be iteratively determined, assuring the achievement of condition (17) in each step. This procedure will be shown in detail in the following section. The proving object $[\tilde{O}]^{(P)}$, determined at the end of the iteration, will be a solution to the inverse problem if $d_{(k+1, l+1)} = 0$ for all values of k and l .

Figure 6: (a) Polygons Pol_{O1}, Pol_{O2} and Pol_{O3} and sets of points $R_{O1}^{(P)}$, $R_{O2}^{(P)}$ and $R_{O3}^{(P)}$ corresponding to the object support in Figure 5. (b) matrices $[\tilde{O}]_1^{(P)}$, $[\tilde{O}]_2^{(P)}$ and $[\tilde{O}]_3^{(P)}$ associated to the sets of dots in Figure 6(a).

(c) difference matrices $[D]_1$, $[D]_2$ and $[D]_3$

Figure 6(a) shows the polygons Pol_{O1}, Pol_{O2} and Pol_{O3} and the set of points $R_{O1}^{(P)}$, $R_{O2}^{(P)}$ and $R_{O3}^{(P)}$, placed at their vertices. Figure 6(b) shows the corresponding matrices $[\tilde{O}]_1^{(P)}$, $[\tilde{O}]_2^{(P)}$ and $[\tilde{O}]_3^{(P)}$ and Fig. 6(c) the difference matrices $[D]_1$, $[D]_2$ and $[D]_3$.

It is apparent that $[D]_1$ and $[D]_2$ satisfy the condition (17), in such a way that S_{o1} and S_{o2} should be supports of object solutions for the inverse problem (Fig. 5). However, $[D]_3$ does not satisfy this condition, and S_{o3} will be not a support for an object solution. In fact, the elements $(d_{31})_3 = -2$ and $(a_{31}) = 0$ (Fig. 3(c)) point out that $[\tilde{O}]_3^{(P)}$ contains a class of element pairs with separation vector $\mathbf{r}_j = \mathbf{r}_{(0,2)} - \mathbf{r}_{(3,3)} = (-3, -1) \times s$ and population $p_j = 2$, which does not appear in the autocorrelation $[\tilde{A}]$. Furthermore, because of elements $(d_{36})_3 = -1$ and $(a_{36}) = 1$ there is a class of element pairs with separation vector $\mathbf{r}_j = \mathbf{r}_{(5,2)} - \mathbf{r}_{(3,3)} = (2, -1) \times s$ in both $[\tilde{O}]_3^{(P)}$ and $[\tilde{A}]$, but its population will be $p_j = 2$ in $[\tilde{O}]_3^{(P)}$ and it takes the value $p_j = 1$ in $[\tilde{A}]$. Therefore, $[\tilde{O}]_3^{(P)}$ will be not a solution for the inverse problem and its support S_{o3} must be discarded.

Reconstructing the object

Once the solution supports are identified, the positions of the $(N - N_o)$ remaining points should be determined, beginning with the points along each side of Pol_o . The intermediate points on the polygon sides will be placed at the intersections of the Cartesian grid corresponding to the domain of $R_o^{(P)}$. Points can be distributed along a side of length $s\sqrt{m^2 + n^2}$, $\gamma - 1$, with γ the maximum common divisor for the integers m and n .

Then, we start at an arbitrary vertex of the polygon and run along the sides. The direction is not relevant. A point should be placed on the first side that satisfies the condition $\gamma - 1 > 0$ and the corresponding element in the matrix $[\tilde{O}]^{(P)}$ should be set to 1. It yields new matrices $[\tilde{A}]^{(P)}$ and $[D]$. The new point will belong to the solution if $[D]$ satisfies the condition (17). Otherwise, the point must be discarded. The procedure should be repeated at the next point.

By verifying the difference matrix is possible, sometimes, to determine if a point belongs to the solution before placing it. Let us consider, for example, the polygons in Figure 5 and the corresponding difference matrices in Figure 6. The matrix element $(d_{31})_1 = 0$ means that there are no pairs with separation vector $\mathbf{r}_j = (-3, -1) \times s$. Consequently, any point should appear along the side l_{o5} of the polygon Pol_{o1} . A similar situation occurs with the polygon Pol_{o2} .

Once all points along the sides of the solution polygon are placed, condition (17) should be verified. If $d_{(k+1, l+1)} > 0$ holds, further points should be placed within the polygon area, i.e. at some of the n intersections of the Cartesian grid enclosed by the polygon. The number of internal points to be placed will be

$$N_i = [N - (\text{number of points on } \text{Pol}_o \text{ sides})] \leq n.$$

The following are the steps of this procedure:

- Placing a point in the array $R_o^{(P)}$ at the position corresponding to the first upper-left intersection inside the polygon,
- modifying the matrix $[\tilde{O}]^{(P)}$,
- determining the autocorrelation $[\tilde{A}]^{(P)}$,
- evaluating the difference matrix $[D]$ and
- verifying the condition (17).

If this condition is satisfied, the procedure will be repeated at the following intersection along the grid row. Otherwise, the modifications of both $R_o^{(P)}$ and $[\tilde{O}]^{(P)}$ should be removed and the next intersection of the grid should be considered. The grid intersections will be scanned from left to right and from up to down until the condition $d_{(k+1, l+1)} = 0$ be fulfilled. The array of points $R_o^{(P)}$ determined under this condition will represent the reconstructed object.

However, if $d_{(k+1, l+1)} > 0$ after the complete scanning of the grid intersections inside the polygon, the internal points should be redistributed. The same procedure will be applied after changing the starting point: now the first point will be placed at the next position to the first upper-left intersection grid inside the polygon. So, all the solutions will be accurately determined.

To illustrate the performance of the above reconstruction algorithm, let us reconstruct, step-by-step (Figure 7), a binary object whose matrix $[\tilde{A}]$ is

$$\begin{pmatrix} 1 & 0 & 1 & 0 & 0 & 2 & 0 & 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 3 & 1 & 1 & 2 & 0 & 2 & 1 & 0 \\ 0 & 1 & 2 & 0 & 3 & 0 & 2 & 3 & 0 & 1 & 0 \\ 2 & 0 & 2 & 2 & 1 & 9 & 1 & 2 & 2 & 0 & 2 \\ 0 & 1 & 0 & 3 & 2 & 0 & 3 & 0 & 2 & 1 & 0 \\ 0 & 1 & 2 & 0 & 2 & 1 & 1 & 3 & 0 & 1 & 0 \\ 1 & 0 & 0 & 1 & 0 & 2 & 0 & 0 & 1 & 0 & 1 \end{pmatrix}$$

According to the value of the middle matrix element, 9 elements compose the object. Furthermore, the matrix size indicates that the object is contained within a region of area $(5 \times 3)s^2$.

The column on the left in Figure 7 shows the proving object $R_o^{(P)}$, the column in the middle shows the (4×6) associated matrix $[\tilde{O}]^{(P)}$, which is corresponding to the (7×11) autocorrelation matrix $[\tilde{A}]$, and the column on the right shows the difference matrix.

$R_o^{(P)}$

$$[O]^{(P)}$$

$[D]$

$$\begin{vmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{vmatrix}$$

1	0	1	0	0	2	0	1	0	0	1
0	1	0	3	1	1	2	0	2	1	0
0	1	2	0	3	0	2	3	0	1	0
2	0	2	2	1	9	1	2	2	0	2
0	1	0	3	2	0	3	0	2	1	0
0	1	2	0	2	1	1	3	0	1	0
1	0	0	1	0	2	0	0	1	0	1

a

$$\begin{vmatrix} 1 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 1 \end{vmatrix}$$

0	0	1	0	0	0	0	1	0	0	0
0	1	0	3	1	1	2	0	2	1	0
0	1	2	0	3	0	2	3	0	1	0
0	0	2	2	1	5	1	2	2	0	0
0	1	0	3	2	0	3	0	2	1	0
0	1	2	0	2	1	1	3	0	1	0
0	0	0	1	0	0	0	0	1	0	0

b

$$\begin{vmatrix} 1 & 0 & 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 1 \end{vmatrix}$$

[illegible]

c

$$\begin{vmatrix} 1 & 0 & 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 1 \end{vmatrix}$$

[illegible]

d

e

f

g

h

i

j

Figure 7: Illustrating, step-by-step, a complete reconstruction of a binary object. The column on the left shows the proving object $R_o^{(P)}$, the column at the middle shows the associated matrix $[\tilde{O}]^{(P)}$, which is corresponding to the autocorrelation matrix $[\tilde{A}]$, and the column on the right shows the difference matrix.

The reconstruction steps are described as follows:

- a.** $[\tilde{O}]^{(P)}$ is initialized by setting all its elements to zero. Thus, the difference matrix will be equal to the autocorrelation matrix $[\tilde{A}]$.
- b.** The size of the rectangular autocorrelation support is $(10 \times 6)s^2$, and then there is a unique object support of size $(5 \times 3)s^2$. Consequently, points should be set at the four corners of the polygon Pol_o (for the sake of clarity, we do not plot neither the polygon nor the grid at the left column, but just the points), and the corresponding elements of $[\tilde{O}]^{(P)}$ should be set to 1. Then, the autocorrelation of $[\tilde{O}]^{(P)}$ is determined and the difference matrix is evaluated. Because the result satisfies the condition (17), we continue to the next step.
- c.** Now, we look for intermediate points on the rectangle sides. There is only one that satisfies the condition (17). Then, we seek for points within the rectangle by starting at the second row of the grid.

- d.** The first internal point will be corresponding to the element $o_{(2,2)}^{(P)} = 1$.
- e.** The evaluation of the difference matrix deals with the next point, corresponding to the element $o_{(2,4)}^{(P)} = 1$. The second row does not contain further points, so we continue along the third row.
- f.** The first point on this row is corresponding to the element $o_{(3,3)}^{(P)} = 1$. Because it is not possible to set more elements in this row to 1, the procedure should finish. However, the internal distribution of points is not correct because the difference matrix is not a null matrix. Consequently, a new distribution must be found by initialising $[\tilde{O}]^{(P)}$ as in **c** and starting at the element $o_{(2,4)}^{(P)} = 1$ instead of $o_{(2,2)}^{(P)}$.
- g. to j.** The elements $o_{(3,2)}^{(P)}$, $o_{(3,3)}^{(P)}$ and $o_{(3,5)}^{(P)}$ are set to 1. Because the final difference matrix is a null matrix, the reconstruction finishes successfully and the reconstructed object will be represented by the corresponding array $R_o^{(P)}$.

The purpose of this paper has been to show a simple method for reconstructing binary objects assuming that the input signal (the Fraunhofer diffraction pattern) is free of the influence of noise. However, in practice this input signal involves some noise. Therefore, this method can be applied whenever the maximum amplitude of the noise spectrum is much less than the minimum amplitude of the autocorrelation peaks. In this case, by image processing methods it is possible to identify the autocorrelation peaks from the spectrum noise peaks.

CONCLUSIONS

A very simple method for reconstructing binary objects composed of identical elements has been proposed. This method is based on the concept of class of element pairs, defined as the set of element pairs with the same separation vector, and the concepts of object support and autocorrelation support. By interpreting the autocorrelation function as a set of classes of element pairs, it is possible to obtain all the object solutions for the retrieval problem. The method has been demonstrated with some examples.

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