STABLE CHAINS AND VORTEX EQUATIONS
ON COMPLEX VECTOR BUNDLES

Xi Zhang
Department of Mathematics, Zhejiang University,
Hangzhou 310027, Zhejiang, People’s Republic of China
and
The Abdus Salam International Centre for Theoretical Physics, Trieste, Italy.

Abstract

In this paper, we study an object on almost Hermitian manifold $M$ consisting of a finite number
of $J_i$-holomorphic vector bundles $E_i$ over $M$ and homomorphisms $\phi_i : E_i \rightarrow E_{i-1}$. We call such
an object a $J$-holomorphic chain. We then prove a Hitchin-Kobayashi correspondence relating
the existence of solutions to certain chain vortex equations and an appropriate notion of stability
for the corresponding chains.

MIRAMARE – TRIESTE
July 2004

\[\text{mzhangxi@ictp.trieste.it; xizhang@zju.edu.cn}\]
1 Introduction

Let $M$ be a compact Kähler manifold and let $E$ be a holomorphic vector bundle over $M$. The classical Hitchin-Kobayashi correspondence ([8], [9], [15], [18], [20], [21]) states that a holomorphic structure is stable if and only if it is simple (i.e. it admits no non-trivial trace free infinitesimal automorphisms) and admits a Hermitian-Einstein metric, i.e. a Hermitian metric $H$ solving the Hermitian-Einstein equation:

$$\sqrt{-1} \Lambda F_H = \lambda Id_E,$$

(1.1)

where $F_H$ is the curvature of the Chern connection of metric $H$, $\Lambda$ is the contraction with the Kähler form of $M$, and $\lambda$ is a real number determined by the topology. General solutions of the Hermitian-Einstein equation correspond to polystable holomorphic structure, i.e. to bundles which are direct sum of stable bundles of the same slope.

A Hitchin-Kobayashi correspondence, where some extra structure $\Phi \in \Gamma(T^{1,0}(M) \otimes End(E))$ is added to the bundle $E$, appears in the theory of Higgs bundles. Higgs bundles were first studied by Hitchin [12] when $M$ is a compact Riemann surface, and Simpson [19] when $M$ is higher dimensional, who introduced a natural gauge equation for them, and proved a Hitchin-Kobayashi correspondence. Different to the Higgs bundles, Bradlow [4], [5] consider holomorphic vector bundles on which additional data in the form of a prescribed holomorphic global section $\phi \in \Gamma(E)$ is given, i.e. holomorphic pair $(E, \phi)$. Bradlow investigate the following vortex equation

$$\Lambda F_H - \frac{\sqrt{-1}}{2} \phi \otimes \phi^* H + \tau \frac{\sqrt{-1}}{2} Id = 0,$$

(1.2)

where $\phi^* H$ is the adjoint of $\phi$ with respect to metric $H$, and $\tau$ is a real number. This equation generalizes the Hermitian-Einstein equation (which is recovered by taking $\phi = 0$) and is the analogue of the classical Vortex equation over $R^2$. In [10], Garcia-Prada show that the vortex equations can also be obtained via dimensional reduction of the Hermitian-Einstein equations under an $SU(2)$ action on certain associated bundles on the manifolds $M \times CP^1$. The coupled vortex equations on holomorphic triple $(E_1, E_2, \phi)$, were very naturally introduced by Garcia-Prada in [10], where $\phi$ is a holomorphic morphism from $E_2$ to $E_1$. A systematic study of these equations and a proof of a Hitchin-Kobayashi type correspondence were done by Bradlow and Garcia-Prada in [6]. Furthermore, in [1], Alvare-Consul and Garcia-Prada investigated holomorphic chains - consisting of a finite number of holomorphic bundles $E_i$ and homomorphisms $\phi_i \in Hom(E_i, E_{i-1})$ - on Kähler manifold $M$. They proved a Hitchin-Kobayashi correspondence relating the existence of solutions to certain natural gauge-theoretic equations and an appropriate notion of stability for the corresponding chains. Recently, their result has been extended by Mundet i Riera [17] to more general Kähler fibration, and by Alvare-Consul and Garcia-Prada [2],[3] to quiver bundles.
In this paper, we want to discuss holomorphic chains over more general almost Hermitian manifolds. In [7], de Bartolomeis and Tian investigated the stability of complex vector bundles over almost complex manifolds, they introduced the concept of bundle almost structure \( \text{(bacs)} \) \( J \) on principal bundle, defined \( J \)-stable complex vector bundles, and proved the existence of Hermitian-Einstein metrics on \( J \)-stable complex vector bundles over a compact almost Hermitian regularized manifold. Inspired by this, we want to introduce the concept of \( J \)-holomorphic chain and discuss the \textbf{chain } \( \tau \)-vortex equations on the corresponding chains over almost Hermitian manifolds.

Let \( (M, J_M, \eta) \) be a compact \( m \)-dimensional almost Hermitian manifold. Denote \( E = (E_0, E_1, ..., E_n) \) an \((n+1)\)-tuple of complex vector bundles \( (E_i, J_i) \) of rank \( r_i \) on \( M \). We consider the principal \( GL(r_i, \mathbb{C}) \)-bundles \( C(E_i) \) of complex linear frames on \( E_i \), and assign a bundle almost complex structure \( \text{(bacs)} \) \( J_i \) on \( C(E_i) \) respectively (which we will recall in section two). By a proposition in [7] (proposition 1.3), we can see that \( \text{bacs} \) on \( C(E_i) \) are in one-to-one correspondence with linear differential operators

\[
\partial_{E_i} : \wedge^{p,q}(E_i) \longrightarrow \wedge^{p,q+1}(E_i)
\]

satisfying \( \partial \)-Leibnitz rule. We denote the set of above differential operators by \( \hat{H}(E_i) \). When \( M \) is a complex manifolds, we usually consider a holomorphic vector bundle \( E \) over \( M \), and we can define partial differentiation in the \((0,1)\) direction in a natural way, i.e. the \((0,1)\) derivative of a local holomorphic section of \( E \) is defined to be zero and the \((0,1)\) derivative of any smooth section is defined by expressing it in terms of a local holomorphic basis and using the Leibniz rule of differentiating products. There is no natural way to define partial differentiation in the \((0,1)\) direction when \( M \) equipped with an not necessarily integrable almost complex structure, this is the reason why we should assign a \( \text{(bacs)} \) on every \( C(E_i) \). And we usually denote \( J = (J_0, J_1, ..., J_n) \) is an \((n+1)\)-tuple of \( \text{bacs} \) \( J_i \) on \( C(E_i) \), and \( (E, J) = ((E_0, J_0), (E_1, J_1)), ..., (E_n, J_n)) \) is the \((n+1)\)-tuple of complex vector bundles \( E_i \) assigned with \( \text{bacs} \) \( J_i \).

Let \( \phi = (\phi_1, \phi_2, ..., \phi_n) \) be an \( n \)-tuple of homomorphisms \( \phi_i \in Hom(E_i, E_{i-1}) \) \( (1 \leq i \leq n) \). If all \( \phi_i \) satisfy:

\[
\partial_{E_i}^{\gamma \otimes E_{i-1}} \phi_i = 0,
\]

then we say that \( (E, J, \phi) \) is a \( J \)-holomorphic chain.

Let \( \partial_{E_i} \) be the element of \( \hat{H}(E_i) \) corresponding to the fixed \( \text{bacs} \) \( J_i \), and \( H_i \) be a Hermitian metric on \( E_i \). By [7], we known there exists a unique type \((1,0)\) Hermitian connection which is called the canonical Hermitian connection \( \omega_{H_i} \). Usually, \( F_{H_i} \) denotes the curvature form of the canonical Hermitian connection. We now denote the Kähler form of the base manifold by \( \eta \), and let \( \Lambda : \Omega_{M}^{1,1} \rightarrow \Omega_{M}^{0} \) be the contraction \( \Lambda(\theta) = (\theta, \eta) \). Let \( \tau = (\tau_0, \tau_1, ..., \tau_n) \in \mathbb{R}^{n+1} \), and \( H = (H_0, H_1, ..., H_n) \) be an \((n+1)\)-tuple of Hermitian metrics, where \( H_i \) is a metric on \( E_i \). We say that \( H \) satisfies the \textbf{chain } \( \tau \)-vortex equations if
\[
\sqrt{-1} \Delta F_{H_0} + \frac{1}{2} \phi_1 \circ \phi_1^H = \tau_0 \text{Id}_{E_0},
\]
\[
\sqrt{-1} \Delta F_{H_i} - \frac{1}{2} (\phi_i^*H \circ \phi_i - \phi_{i+1} \circ \phi_{i+1}^H) = \tau_i \text{Id}_{E_i},
\]
\[
\sqrt{-1} \Delta F_{H_n} - \frac{1}{2} \phi_n^*H \circ \phi_n = \tau_n \text{Id}_{E_n},
\]
(1.4)

where \(1 \leq i \leq n-1\), \(\phi_i^H\) is the adjoint of \(\phi_i\) taken with respect to \(H\). Our goal is to understand the necessary and sufficient conditions for the existence of solutions of vortex equation (1.4) on \(J\)-holomorphic chains \(C = (E, J, \phi)\). In section 4, we will give the definition of \(\text{deg}_\alpha(C)\) and \(\alpha\)-stability, where \(\alpha = (\alpha_0, \alpha_1, ..., \alpha_n) \in \mathbb{R}^{n+1}\). Our main result is the following Hitchin-Kobayashi correspondence.

**Main theorem** Assume that \(C = (E, J, \phi)\) is a \(J\)-holomorphic chain on a compact almost Hermitian regularized manifold \((M, \eta)\) (i.e., whose Kähler form \(\eta\) satisfies \(\partial \bar{\partial} \eta^{m-1} = 0\)). Let \(\tau = (\tau_0, \tau_1, ..., \tau_n) \in \mathbb{R}^{n+1}\) be such that \(\text{deg}_\tau(C) = 0\). The chain \((E, J, \phi)\) admits an \((n+1)\)-tuple \(H = (H_0, H_1, ..., H_n)\) of Hermitian metrics satisfying the chain \(\tau\)-vortex equation (1.4) if and only if it is \(\tau\)-polystable.

For the proof of main theorem, we use the heat flow method which is different with that Alvarez-Consul and Garcia-Prada’s in [1]. Recently, Lübke and Teleman [22] proved a very general Hitchin-Kobayashi correspondence on arbitrary compact Hermitian manifolds, but their result has no overlap with our theorem, since their result does not include the almost Hermitian case. The paper is organized as follows: in Section 2, we give some basic definitions, in section 3, we give some estimates and preliminaries which will be used in the proof of main theorem; in section 4, we introduce the definition of \(\tau\)-stability, and prove that \(\tau\)-stability is a necessary condition for the existence of Hermitian metrics satisfying chain \(\tau\)-vortex equation (1.4); in section 5, we give the proof of our main theorem.

## 2 Notation

Let \((M, J_M)\) be an \(m\)-dimensional almost complex manifold. A complex vector bundle \((E, \hat{J})\) of (complex) rank \(r\) over \(M\) is a real vector bundle \(E\) of rank \(2r\) equipped with a section \(\hat{J}\) of \(\text{End}(E)\) such that \(\hat{J}^2 = -\text{Id}_E\). We denote the principal \(GL(r, C)\)-bundle of complex linear frames on \(E\) by \(C(E)\), thus \(E\) can also be seen as an associate bundle of \(C(E)\) with standard fibre \(C^r\). Firstly, we recall the notion of bundle almost complex structure (bacs) which has been investigated by de Bartolomeis and Tian in [7].

**Definition 2.1** A bundle almost complex structure (bacs) on \(C(E)\) is an almost complex structure \(J\) on \(C(E)\) such that: (1), the bundle projection \(\pi : C(E) \to M\) is \((J, J_M)\)-holomorphic; (2), \(J\) induces the standard integrable almost complex structure \(J_S\) on the fibres; (3), \(GL(r, C)\)
acts $J$-holomorphically on $C(E)$.

$B(C(E))$ will denote the set of bacs on $C(E)$. We can define

$$T^{p,q}(C(E)) = L^{-1}(\wedge^{p,q}(E)),$$

(2.1)

where $L : T^*(C(E)) \to \wedge^*(E)$ is the standard isomorphism between tensorial $C^r$-valued forms on $C(E)$ and $E$-valued forms on $M([14])$, therefore we have

$$T^0(C(E)) = \otimes_{p+q=n} T^{p,q}(C(E)).$$

(2.2)

It is easy to check that, if a bacs is assigned on $C(E)$, then (2.2) corresponds precisely to the induced decomposition.

Let $\tilde{H}(C(E))$ be the set of all linear differential operators

$$\tilde{\partial}_{C(E)} : T^{p,q}(C(E)) \to T^{p,q+1}(C(E))$$

satisfying the following $\tilde{\partial}$-Leibnitz rule: for every $f \in C^\infty(M)$, $\alpha \in T^{p,q}(C(E))$

$$\tilde{\partial}_{C(E)}\pi^*(f)\alpha = \pi^*(\tilde{\partial}_M f) \wedge \alpha + \pi^*(f)\tilde{\partial}_{C(E)}\alpha$$

one can check that the map $J \mapsto \tilde{\partial}_J$ is a bijection between $B(C(E))$ and $\tilde{H}(C(E))$ ([7], proposition 1.3), where $\tilde{\partial}_J$ is the operator induced by $J$. On the other hand, $\tilde{H}(C(E))$ is also in one-to-one correspondence with the set $\tilde{H}(E)$ of linear differential operators $\tilde{\partial}_E : \wedge^{p,q}(E) \to \wedge^{p,q+1}(E)$, satisfying the following $\tilde{\partial}$-Leibnitz rule: for every $f \in C^\infty(M)$, $\alpha \in \wedge^{p,q}(E)$

$$\tilde{\partial}_E\pi^*(f)\alpha = \tilde{\partial}_M f \wedge \alpha + f \tilde{\partial}_E\alpha.$$

This correspondence is obviously given by $\tilde{\partial}_E = L \cdot \tilde{\partial}_{C(E)} \cdot L^{-1}$. If a bacs $J$ is assigned on $C(E)$, one can define a linear differential operator $\tilde{\partial}_E : \wedge^{p,q}(E) \to \wedge^{p,q+1}(E)$ in natural way, in fact, $\tilde{\partial}_E = L \cdot \tilde{\partial}_J \cdot L^{-1}$.

**Proposition 2.2** ([7]) The set $B(C(E))$ is in one-to-one correspondence with the set $\tilde{H}(E)$.

**Definition 2.3** Let $J \in B(C(E))$. Then a section $e$ of $E$ is said to be $J$-holomorphic if it satisfies $\tilde{\partial}_E e = 0$, where the differential operator $\tilde{\partial}_E$ is in correspondence with $J$; this is equivalence to say that, if $\xi = L^{-1}(e) \in T^0(C(E))$, then $\tilde{\partial}_J \xi = 0$.

**Definition 2.4** Assume bacs’s have been assigned on $C(E_2)$ and $C(E_1)$; a bundle morphism $\phi : E_2 \to E_1$ is said to be $J$-holomorphic if $\tilde{\partial}_{E_2} \circ \phi = 0$.

**Definition 2.5** Let $J \in B(C(E))$. Then a complex sub-bundle $E' \subset E$ is said to be a $J$-holomorphic subbundle if $\tilde{\partial}_E$ maps $\wedge^{p,q}(E')$ into $\wedge^{p,q+1}(E')$. 
Definition 2.6 Let \( J \in B(C(E)) \). A connection will be called type(1, 0), if it’s connection 1-forms on \( C(E) \) satisfies: \( \omega \in T^{1,0}(C(E), gl(r, C), ad) \).

Let \( C^1_{J}(C(E)) \) be the set of all connection 1-forms in \( C(E) \) which are of type(1, 0) with respect to \( J \). Given an \( \omega \in C^1_{J}(C(E)) \), it is easy to check that \( D_{\omega} : T^{0}(C(E)) \rightarrow T^{1}(C(E)) \) splits as \( D_{\omega} = \partial_{\omega} + \bar{\partial}_{J} \), also we have the splitting \( \nabla = \partial_{\omega} + \bar{\partial}_{E} \) of the induced exterior covariant differential operator; and the (1,1) part of curvature form is \( F^{1,1}_{\omega} = \bar{\partial}_{J}\omega \) ([7] Proposition 1.8; 1.9).

Assume a Hermitian metric \( H \) is assigned on \( E \) and let \( U_{H}(E) \) be the principal \( U(r) \)-bundle of \( H \)-unitary frames on \( E \), we have the following result:

**Proposition 2.7 ([7]; proposition 2.1)** There exists a unique connection on \( U_{H}(E) \) such that it’s connection 1-form, when extended to a connection form on \( C(E) \) is of type (1, 0) with respect to \( J \in B(C(E)) \); this connection is called the canonical Hermitian connection.

Let \( \hat{H} : C(E) \rightarrow GL(r, C) \) be defined as following: If \( u = \{ e_{1}, \ldots, e_{r}, \hat{J}e_{1}, \ldots, \hat{J}e_{r} \} \), then \( \hat{H}(u) = (H(e_{j}, e_{k}) - iH(e_{j}, \hat{J}e_{k}))_{1 \leq j, k \leq r} \). Set
\[
\omega_{H} = \hat{H}^{-1} \partial_{j}\hat{H}, \tag{2.3}
\]
it is just the canonical Hermitian connection 1-form correspondence with the metric structure \( H \). Let \( K \) be another Hermitian structure on \( E \) and let \( h = H^{-1}K \), it is easy to check that:
\[
\omega_{K} = \omega_{H} + h^{-1} \partial_{\omega_{H}} h. \tag{2.4}
\]
\[
F^{1,1}_{\omega_{K}} = F^{1,1}_{\omega_{H}} + \bar{\partial}_{E}(h^{-1} \partial_{\omega_{H}} h). \tag{2.5}
\]

We now suppose that the almost complex manifold \( M \) has a fixed Hermitian metric, with Kähler form \( \eta \). The natural operator \( \Lambda : \Omega^{1,1}_{m} \rightarrow \Omega^{0,0}_{m} \) is the contraction with \( \eta \). Choose a local real normal coordinate \( (x^{1}, \ldots, x^{2m}) \) centered at the considered point \( p_{0} \). Let
\[
J_{M}(\frac{\partial}{\partial x^{\alpha}}) = J_{\alpha}^{\beta} \frac{\partial}{\partial x^{\beta}}, \quad \alpha, \beta = 1, \ldots, 2m.
\]
By calculating directly, we have
\[
-\sqrt{-1} \Lambda \bar{\partial} \partial f = \frac{1}{2} \Delta f + \frac{1}{2} \sum J_{\alpha}^{\beta} \frac{\partial J_{\beta}^{\gamma}}{\partial x^{\alpha}} \frac{\partial f}{\partial x^{\gamma}} \tag{2.6}
\]
at the considered point \( p_{0} \). Let \( \hat{\Delta} = -2\sqrt{-1} \Lambda \bar{\partial} \partial \), and \( V = J_{M}(g^{\alpha\beta}(\nabla_{\omega_{H}} J_{M})(\frac{\partial}{\partial x^{\alpha}})) \), where \( (g^{\alpha\beta}) \) is the inverse matrix of the metric matrix in local coordinates. From the above equality, we have
\[
\hat{\Delta} f = \Delta f + \langle V, \nabla f \rangle, \tag{2.7}
\]
for any $f \in C^2(M)$. In the Kähler case, by the Kodaira identities, we know that $\bar{\Delta} = \Delta$.

**Definition 2.8 (J-holomorphic chain)** A J-holomorphic chain on almost complex manifold $(M, J_M)$ is a triple $C = (E, J, \phi)$, where $E = (E_0, E_1, \ldots, E_n)$ is an $(n+1)$-tuple of complex vector bundles $(E_i, J_i)$ of rank $r_i$ on $M$, $J = (J_0, J_1, \ldots, J_n)$ is an $(n+1)$-tuple of bacs $J_i$ on $C(E_i)$, and $\phi = (\phi_1, \phi_2, \ldots, \phi_n)$ is an $n$-tuple of J-holomorphic morphisms $\phi_i \in \text{Hom}(E_i, E_{i-1})$ $(1 \leq i \leq n)$. This is represented by the following diagram:

$$C : (E_n, J_n) \xrightarrow{\phi_n} (E_{n-1}, J_{n-1}) \xrightarrow{\phi_{n-1}} \cdots \xrightarrow{\phi_1} (E_0, J_0).$$

**Definition 2.9 (chain vortex equations)** Let $C = (E, J, \phi)$ be a J-holomorphic chain over almost Hermitian manifold $(M, J_M, \eta)$, and $\tau = (\tau_0, \tau_1, \ldots, \tau_n) \in R^{n+1}$. Let $H = (H_0, H_1, \ldots, H_n)$ be an $n+1$-tuple of Hermitian metrics, where $H_i$ is a metric on $E_i$. We say that $H$ satisfies the chain $\tau$-vortex equations if

$$\sqrt{-1} \Delta F_{H_0} + \frac{1}{2} \phi_1 \circ \phi_1^H = \tau_0 \text{Id}_{E_0},$$

$$\sqrt{-1} \Delta F_{H_i} - \frac{1}{2} (\phi_i^H \circ \phi_i - \phi_{i+1} \circ \phi_{i+1}^H) = \tau_i \text{Id}_{E_i},$$

$$\sqrt{-1} \Delta F_{H_n} - \frac{1}{2} \phi_n^H \circ \phi_n = \tau_n \text{Id}_{E_n},$$

where $1 \leq i \leq n-1$, $\phi_i^H$ is the adjoint of $\phi_i$ taken with respect to $H$.

### 3 Some preliminaries on chain vortex equations

Given a J-holomorphic chain $C = (E, J, \phi)$ on almost Hermitian manifold $(M, J_M, \eta)$, the main purpose of this paper is to find an $n+1$-tuple of Hermitian metrics $H = (H_0, H_1, \ldots, H_n)$ satisfying the chain $\tau$-vortex equations (2.9). Let $K = (K_0, K_1, \ldots, K_n)$ be the initial $n+1$-tuple of Hermitian metrics on chain $C$. Consider a family of tuples of Hermitian metrics $H(t) = (H_0(t), H_1(t), \ldots, H_n(t))$ on $C$ with initial metric $H(0) = K$. And denote $h(t) = (h_0(t), h_1(t), \ldots, h_n(t))$ be an $n+1$-tuple of endomorphisms $h_i = K_i^{-1}H_i$. When there is no confusion, we will omit the parameter $t$ and simply write $H$, $h$ for $H(t)$, $h(t)$. We consider the following heat equations of (2.9)

$$H_0^{-1} \frac{\partial H_0}{\partial t} = -2(\sqrt{-1} \Delta F_{H_0} + \frac{1}{2} \phi_1 \circ \phi_1^H - \tau_0 \text{Id}_{E_0}),$$

$$H_i^{-1} \frac{\partial H_i}{\partial t} = -2(\sqrt{-1} \Delta F_{H_i} - \frac{1}{2} (\phi_i^H \circ \phi_i - \phi_{i+1} \circ \phi_{i+1}^H) - \tau_i \text{Id}_{E_i}),$$

$$H_n^{-1} \frac{\partial H_n}{\partial t} = -2(\sqrt{-1} \Delta F_{H_n} - \frac{1}{2} \phi_n^H \circ \phi_n - \tau_n \text{Id}_{E_n}).$$


where 1 ≤ i ≤ n − 1. It is completely equivalent to the following evolution equations

\[
\begin{align*}
\frac{\partial u_{i}}{\partial t} &= -2\sqrt{-\Lambda} \bar{\beta}_{E_{0}} \partial K_{0} h_{0} + 2\sqrt{-\Lambda} (\bar{\beta}_{E_{0}} h_{0}^{-1} \partial K_{0} \partial h_{0}) - 2\sqrt{-\Lambda} h_{0} \Lambda F_{K_{0}} + 2\tau_{0} h_{0} \\
& \quad - h_{0} \phi_{1} h_{1}^{-1} \phi_{1}^{*} K h_{0}, \\
\frac{\partial u_{i}}{\partial t} &= -2\sqrt{-\Lambda} \bar{\beta}_{E_{i}} \partial K_{i} h_{i} + 2\sqrt{-\Lambda} (\bar{\beta}_{E_{i}} h_{i}^{-1} \partial K_{i} \partial h_{i}) - 2\sqrt{-\Lambda} h_{i} \Lambda F_{K_{i}} + 2\tau_{i} h_{i} \\
& \quad + \phi_{i}^{*} K h_{i-1} \phi_{i} - h_{i} \phi_{i+1} h_{i+1}^{-1} \phi_{i+1}^{*} K h_{i}, \\
\frac{\partial u_{n}}{\partial t} &= -2\sqrt{-\Lambda} \bar{\beta}_{E_{n}} \partial K_{n} h_{n} + 2\sqrt{-\Lambda} (\bar{\beta}_{E_{n}} h_{n}^{-1} \partial K_{n} \partial h_{n}) - 2\sqrt{-\Lambda} h_{n} \Lambda F_{K_{n}} + 2\tau_{n} h_{n} \\
& \quad + \phi_{n}^{*} K h_{n-1} \phi_{n}
\end{align*}
\] (3.2)

where we have used the formula (2.5) and the identities

\[
\phi_{i}^{*} K = h_{i}^{-1} \phi_{i}^{*} K h_{i-1}.
\] (3.3)

We know that the above equations are a nonlinear parabolic system, as in [8], h_{i}(t) are self adjoint with respect to H_i for t > 0 since h_{i}(0) = Id_{E_{i}}. We denote

\[
\Theta^2 = \left|\sqrt{-\Lambda} F_{H_{0}} + \frac{1}{2} \phi_{1} \circ \phi_{1}^{*H} - \tau_{0} Id_{E_{0}}\right|_{H_{0}}^2 + \left|\sqrt{-\Lambda} F_{H_{i}} - \frac{1}{2} \phi_{n} \circ \phi_{n} - \tau_{n} Id_{E_{n}}\right|_{H_{n}}^2
\]

(3.4)

and

\[
\Phi^2 = \sum_{i=1}^{n} |\phi_{i}|_{H_{i}}^2.
\] (3.5)

**Proposition 3.1** Let H(t) = (H_{0}(t), H_{1}(t), ..., H_{n}(t)) be a solution of heat flow (3.1), then

\[
(\bar{\Delta} - \frac{\partial}{\partial t}) \Theta^2 \geq 0.
\] (3.6)

and

\[
(\bar{\Delta} - \frac{\partial}{\partial t}) \sum_{i=0}^{n} T \theta_{i} = 0
\] (3.7)

where, we denote \theta_{i} = \sqrt{-\Lambda} F_{H_{i}} - \frac{1}{2} (\phi_{i} \circ \phi_{i} - \phi_{i+1} \circ \phi_{i+1}^{*}) - \tau_{i} Id_{E_{i}}.

**Proof.** By calculating directly, we have

\[
\frac{\partial \theta_{i}}{\partial t} = \sqrt{-\Lambda} \bar{\beta}_{E_{i}} (\partial H_{i} (h_{i}^{-1} \frac{\partial h_{i}}{\partial t}) + \frac{1}{2} h_{i}^{-1} \frac{\partial h_{i}}{\partial t} \phi_{i}^{*H} \phi_{i} - \frac{1}{2} \phi_{i}^{*H} h_{i-1}^{-1} \frac{\partial h_{i}}{\partial t} \phi_{i}
\] (3.8)

and

\[
\bar{\Delta} |\theta_{i}|_{H_{i}}^2 = 2Re(-2\sqrt{-\Lambda} \bar{\beta}_{E_{i}} \partial H_{i} \theta_{i} \lor \theta_{i} H_{i}) + Re(2\sqrt{-\Lambda} F_{H_{i}}^{1,1} \theta_{i} \lor H_{i})
\]

\[
+ 2|\partial H_{i} \theta_{i}|_{H_{i}}^2 + 2|\bar{\beta}_{E_{i}} \theta_{i}|_{H_{i}}^2.
\]

Using the above formulas, we have

\[
(\bar{\Delta} - \frac{\partial}{\partial t}) \Theta^2
\]

\[
= 2 \sum_{i=0}^{n} |\nabla \theta_{i}|_{H_{i}}^2 + \sum_{i=0}^{n} (|\phi_{i} \theta_{i}|^2 + |\theta_{i} \phi_{i}^{*H}|^2 + |\theta_{i} \phi_{i+1}|^2 + |\phi_{i+1}^{*}|^2)
\]

\[
- 2 \Theta_{-1} \phi_{i} \theta_{i} \phi_{i} \theta_{i} > - 2 \Theta_{i+1} \phi_{i+1} \phi_{i+1}^{*} \theta_{i} >
\]

\[
= 2 \sum_{i=0}^{n} |\nabla \theta_{i}|_{H_{i}}^2 + \sum_{i=0}^{n} (|\phi_{i} \theta_{i}|^2 - 2 \Theta_{i+1} \phi_{i+1} \phi_{i+1}^{*} \theta_{i} + |\theta_{i+1} \phi_{i+1}^{*H}|^2)
\]

\[
+ \sum_{i=0}^{n} (|\phi_{i} \theta_{i}|^2 - 2 \Theta_{i-1} \phi_{i} \theta_{i} + |\theta_{i-1} \phi_{i}|^2)
\]

\[
\geq 0.
\]

The formula (3.7) can be deduce from (3.8) directly.
Proposition 3.2 Let $H(t) = (H_0(t), H_1(t), \ldots, H_n(t))$ be a solution of heat flow (3.1), then there exist a positive constant $C_1$ depending only on chain $(E, J, \phi)$ such that

$$(\bar{\Delta} - \frac{\partial}{\partial t}) \Phi^2 \geq 2 \sum_{i=1}^{n} |\partial_H \phi_i|^2 + C_1 \Phi^4 - \max_{1 \leq i \leq n} \{|\tau_i - \tau_{i-1}|\} \Phi^2. \quad (3.9)$$

**Proof.** By calculating directly, we have

$$(\bar{\Delta} - \frac{\partial}{\partial t}) |\phi_1|^2_H = 2|\partial_H \phi_1|^2_H + 2|\phi_1 \phi_1^H|^2 - \langle \phi_2 \phi_2^H, \phi_1^H \phi_1 \rangle + 2(\tau_1 - \tau_0)|\phi_1|^2,$$

$$(\bar{\Delta} - \frac{\partial}{\partial t}) |\phi_i|^2_H = 2|\partial_H \phi_i|^2_H + 2|\phi_i \phi_i^H|^2 - \langle \phi_{i+1} \phi_{i+1}^H, \phi_i^H \phi_i \rangle - \langle \phi_i \phi_i^H, \phi_i^H \phi_{i-1} \rangle + 2(\tau_i - \tau_{i-1})|\phi_i|^2,$$

$$(\bar{\Delta} - \frac{\partial}{\partial t}) |\phi_n|^2_H = 2|\partial_H \phi_n|^2_H + 2|\phi_n \phi_n^H|^2 - \langle \phi_n \phi_n^H, \phi_n^H \phi_{n-1} \rangle + 2(\tau_n - \tau_{n-1})|\phi_n|^2, \quad (3.10)$$

where we have used $\partial_{E_i \otimes E_{i-1}} \phi_i = 0$, and equations (3.2). On the other hand, one can easily check that:

$$|\phi_i^H \phi_i|^2_{H_i} = |\phi_i^H \phi_i|^2_{H_{i-1}}, \quad (3.11)$$

and

$$|\phi_i \phi_i^H|^2_{H_{i-1}} \geq \frac{1}{\tau_i}|\phi_i|^4_{H_i}. \quad (3.12)$$

From above equalities we have

$$(\bar{\Delta} - \frac{\partial}{\partial t}) \Phi^2 \geq 2 \sum_{i=1}^{n} |\partial_H \phi_i|^2 + \frac{1}{2(2i)} (\sum_{i=1}^{n} |\phi_i \phi_i^H|^2) - \max_{1 \leq i \leq n} \{|\tau_i - \tau_{i-1}|\} \Phi^2 \geq 2 \sum_{i=1}^{n} |\partial_H \phi_i|^2 + C_1 \Phi^4 - \max_{1 \leq i \leq n} \{|\tau_i - \tau_{i-1}|\} \Phi^2. \quad (3.13)$$

Next, we will introduce the Donaldson’s “distance” on the space of Hermitian metrics as follows.

**Definition 3.3** For any two Hermitian metrics $H, K$ on a vector bundle $E$ set

$$\sigma(H, K) = Tr H^{-1} K + Tr K^{-1} H - 2\text{rank}E. \quad (3.14)$$

It is obvious that $\sigma(H, K) \geq 0$ with equality if and only if $H = K$. The function $\sigma$ is not quite a metric but it serves almost equally well in our problem. In particular, a sequence of metrics $H_t$ converges to $H$ in the usual $C^0$ topology if and only if $\sup_M \sigma(H_t, H) \to 0$. 

9
Let $H = (H_0, H_1, ..., H_n)$ and $K = (K_0, K_1, ..., K_n)$ are two tuples of Hermitian metrics on chain $(E, J, \phi)$. We define the Donaldson's distance of two tuples as the following:

$$\sigma(H, K) = \sum_{i=0}^{n} \sigma(H_i, K_i).$$

(3.15)

Denoting $h = (h_0, h_1, ..., h_n)$, where $h_i = K_i^{-1}H_i$; applying $-\sqrt{-1}\Lambda$ to (2.5) and taking the trace in the bundle $E_i$, we have

$$Tr(\sqrt{-1}h_i(\Lambda F_{H_i}^{1,1} - \Lambda F_{K_i}^{1,1})) = -\frac{1}{2} \bar{\Delta} Trh_i + Tr(-\sqrt{-1}\Lambda \partial E_i h_i h_i^{-1} \partial K_i h_i).$$

(3.16)

Let two tuples $H(t), K(t)$ are two solutions of Heat flow (3.1). Using the above formula, we have

$$(\bar{\Delta} - \frac{\partial}{\partial t})(\sum_{i=0}^{n} Trh_i(t)) = 2 \sum_{i=0}^{n} Tr(\sqrt{-1}\Lambda \bar{\partial} E_i h_i h_i^{-1} \partial K_i h_i)$$

$$+ \sum_{i=1}^{n} Tr(h_i \bar{\partial}^* E_i K_i h_i - h_i \partial E_i h_i h_i^{-1} \partial K_i h_i)$$

$$+ Tr(h_0 \bar{\partial}^* E_i K_i h_0 - h_0 \partial E_i h_0 h_0^{-1} \partial K_i h_0)$$

$$= 2 \sum_{i=0}^{n} Tr(\sqrt{-1}\Lambda \bar{\partial} E_i h_i h_i^{-1} \partial K_i h_i)$$

$$+ \sum_{i=0}^{n} Tr(h_i \bar{\partial}^* E_i K_i h_i - 2h_i \partial E_i h_i h_i^{-1} \partial K_i h_i) + h_i \partial E_i h_i h_i^{-1} \partial K_i h_i).$$

(3.17)

In the similar way, we have

$$(\bar{\Delta} - \frac{\partial}{\partial t})(\sum_{i=0}^{n} Trh_i^{-1}(t)) = 2 \sum_{i=0}^{n} Tr(\sqrt{-1}\Lambda \bar{\partial} E_i h_i^{-1} \partial K_i h_i)$$

$$+ \sum_{i=0}^{n} Tr(h_i^{-1} \partial E_i h_i h_i^{-1} \partial K_i h_i)$$

$$+ Tr(h_0^{-1} \partial E_i h_0 h_0^{-1} \partial K_i h_0)$$

$$= 2 \sum_{i=0}^{n} Tr(\sqrt{-1}\Lambda \bar{\partial} E_i h_i h_i^{-1} \partial K_i h_i)$$

$$+ \sum_{i=0}^{n} Tr(h_i^{-1} \partial E_i h_i h_i^{-1} \partial K_i h_i - 2h_i \partial E_i h_i h_i^{-1} \partial K_i h_i) + h_i \partial E_i h_i h_i^{-1} \partial K_i h_i).$$

(3.18)

On the other hand, it is not hard to check that

$$Tr(h_i \bar{\partial}^* E_i K_i h_i - 2h_i \partial E_i h_i h_i^{-1} \partial K_i h_i) \geq 0,$$

(3.19)

and

$$Tr(h_i \partial E_i h_i \partial K_i h_i) \geq 0,$$

(3.20)

Using the above formula and the facts([8], [20])

$$Tr(\sqrt{-1}\Lambda \bar{\partial} E_i h_i h_i^{-1} \partial K_i h_i) \geq 0, \quad Tr(\sqrt{-1}\Lambda \partial E_i h_i h_i^{-1} \partial K_i h_i) \geq 0,$$

(3.21)

we have proved the following proposition.

**Proposition 3.4** Let two $n+1$-tuples $H(t), K(t)$ are two solutions of the Heat flow (3.1) then

$$(\bar{\Delta} - \frac{\partial}{\partial t})\sigma(H(t), K(t)) \geq 0.$$

(3.22)

**Corollary 3.5** Let $H$ and $K$ are two tuples of Hermitian metrics satisfying the chain $\tau$-vortex equation (2.9), then:

$$\bar{\Delta}\sigma(H, K) \geq 0.$$

(3.23)
**Proposition 3.6** Let the $n+1$-tuple $\mathbf{H}(x,t)$ be a solution of the heat flow (3.1) with the initial tuple $\mathbf{K}$, then

\[
(\tilde{\Delta} - \frac{\partial}{\partial t}) \log \left\{ \sum_{i=0}^{n} (\text{Tr}(K_{i}^{-1}H_{i}) + \text{Tr}(H_{i}^{-1}K_{i})) \right\} 
\geq -\left( \sum_{i=0}^{n} |2\sqrt{-1}\Lambda F_{K_{i}}^{1,1} - (\phi_{i}^* K \phi_{i} - \phi_{i+1} \phi_{i+1}^* K) - \tau_{i} \text{Id}_{E_{i}}|_{K_{i}} \right).
\]  

(3.24)

**Proof.** Let $h_{i} = K_{i}^{-1}H_{i}$, applying (3.1) and (3.16), we have

\[
(\tilde{\Delta} - \frac{\partial}{\partial t}) \text{Tr} h_{i} = \text{Tr} \left( 2\sqrt{-1}h_{i} \Lambda F_{K_{i}} - \frac{1}{2}(\phi_{i}^* K h_{i-1} \phi_{i} - h_{i} \phi_{i+1} h_{i+1}^{-1} \phi_{i+1}^* K h_{i}) - \tau_{i} \right) h_{i}
\]

\[
+ 2\text{Tr}(\sqrt{-1} \Lambda \partial_{E_{i}} h_{i} h_{i}^{-1} \partial_{K_{i}} h_{i}).
\]

(3.25)

and

\[
(\tilde{\Delta} - \frac{\partial}{\partial t}) \text{Tr} h_{i}^{-1} = \text{Tr} \left( 2\sqrt{-1}h_{i}^{-1} \Lambda F_{K_{i}} - \frac{1}{2}(h_{i}^{-1} \phi_{i}^* K h_{i-1} \phi_{i}^{-1} - \phi_{i+1} h_{i+1}^{-1} \phi_{i+1}^* K) - \tau_{i} h_{i}^{-1} \right).
\]

(3.26)

Direct calculation shows that ([20])

\[
2(\text{Tr} h_{i})^{-1} \text{Tr} \left( \sqrt{-1} \Lambda \partial_{E_{i}} h_{i} h_{i}^{-1} \partial_{K_{i}} h_{i} \right) - (\text{Tr} h_{i})^{-2} |\nabla \text{Tr} h_{i}|^{2} \geq 0,
\]

\[
2(\text{Tr} h_{i}^{-1})^{-1} \text{Tr} \left( \sqrt{-1} \Lambda \partial_{E_{i}} h_{i}^{-1} \partial_{K_{i}} h_{i}^{-1} \right) - (\text{Tr} h_{i}^{-1})^{-2} |\nabla \text{Tr} h_{i}^{-1}|^{2} \geq 0.
\]

(3.27)

For simplicity, we denote that $a = \sum_{i=0}^{n} (\text{Tr} h_{i} + \text{Tr} h_{i}^{-1})$. From the above inequalities, it is easy to check

\[
a \{ \sum_{i=0}^{n} \text{Tr} \left( 2\sqrt{-1}h_{i}^{-1} \partial_{K_{i}} h_{i} - 2\sqrt{-1} \Lambda \partial_{E_{i}} h_{i}^{-1} h_{i} \partial_{K_{i}} h_{i}^{-1} \right) \}
\geq | \sum_{i=0}^{n} (\nabla \text{Tr} h_{i} + \nabla \text{Tr} h_{i}^{-1}) |^{2}.
\]

(3.28)

Then, we have

\[
(\tilde{\Delta} - \frac{\partial}{\partial t}) \log \left\{ \sum_{i=0}^{n} (\text{Tr} h_{i} + \text{Tr} h_{i}^{-1}) \right\}
\geq a^{-1} \{ \sum_{i=0}^{n} (\nabla \text{Tr} h_{i} + \nabla \text{Tr} h_{i}^{-1}) |^{2} \}
\geq a^{-1} \{ \sum_{i=0}^{n} \text{Tr} \left( 2\sqrt{-1}h_{i}^{-1} \Lambda F_{K_{i}} - (\phi_{i}^* K \phi_{i} - \phi_{i+1} \phi_{i+1}^* K) - 2\tau_{i} \text{Id}_{E_{i}} \right) \}
\]

\[
+ a^{-1} \{ \sum_{i=0}^{n} \text{Tr} \left( h_{i} \phi_{i + 1} h_{i+1}^{-1} \phi_{i+1}^* K - h_{i} \phi_{i+1} h_{i+1}^{-1} \phi_{i+1}^* K - h_{i} \phi_{i+1} \phi_{i+1}^* K \right) \}
\]

\[
+ a^{-1} \{ \sum_{i=0}^{n} \text{Tr} \left( 2\sqrt{-1} \Lambda \partial_{E_{i}} h_{i}^{-1} h_{i} \partial_{K_{i}} h_{i}^{-1} \right) \}
\geq - \sum_{i=0}^{n} \{ 2\sqrt{-1} \Lambda F_{K_{i}}^{1,1} - (\phi_{i}^* K \phi_{i} - \phi_{i+1} \phi_{i+1}^* K) - 2\tau_{i} \text{Id}_{E_{i}} |_{K_{i}} \}
\]

where we have used formula (3.28) and the same argument as in (3.17).

\[\square\]

Using (3.16), (3.28), and arguing as in the above proposition, we have

**Proposition 3.7** Let $\mathbf{H}$ and $\mathbf{K}$ are two $n+1$-tuples of Hermitian metrics, then

\[
\tilde{\Delta} \log \left\{ \sum_{i=0}^{n} (\text{Tr}(K_{i}^{-1}H_{i}) + \text{Tr}(H_{i}^{-1}K_{i})) \right\}
\geq -\left( \sum_{i=0}^{n} |2\sqrt{-1}\Lambda F_{K_{i}}^{1,1} - (\phi_{i}^* K \phi_{i} - \phi_{i+1} \phi_{i+1}^* K) - \tau_{i} \text{Id}_{E_{i}}|_{K_{i}} \right)
\]

\[-(\sum_{i=0}^{n} |2\sqrt{-1}\Lambda F_{H_{i}}^{1,1} - (\phi_{i}^* H \phi_{i} - \phi_{i+1} \phi_{i+1}^* H) - \tau_{i} \text{Id}_{E_{i}}|_{H_{i}}).\]

(3.29)
**Corollary 3.8** Let $H$ be an $n+1$-tuple of Hermitian metrics satisfying the chain $\tau$-vortex equation (2.9), and $K$ be a fixed tuple of Hermitian metrics, then

\[
\bar{\nabla} \log \left\{ \sum_{i=0}^{n} (\text{Tr}(K_i^{-1}H_i) + \text{Tr}(H_i^{-1}K_i)) \right\} \\
\geq - \left( \sum_{i=0}^{n} |2\sqrt{-1}\Lambda^{1,1}_{K_i} - (\phi^* K_i \phi_i - \phi_{i+1} \phi^{*K}_{i+1}) - \tau_i \text{Id}_{E_i}|K_i) \right)
\]  \hspace{1cm} (3.30)

At the end of this section, we use the Moser-iteration to deduce the following mean-value inequality which will be used in the proof of main theorem. The major geometric-analytic property of $M$ which we are going to use is the Sobolev inequality on the geodesic ball $B_R$. Namely, for any $\psi \in C^\infty_0(B(R))$, there exists a constant $C_s$ only dependent on the geometry of $M$ around $B(R)$ such that

\[
C_s \left( \int_{B(R)} \psi \frac{4m}{2m-2} \right)^{\frac{2m-2}{2m}} \leq \int_{B(R)} |\nabla \psi|^2.
\]  \hspace{1cm} (3.31)

**Theorem 3.9** Suppose that nonnegative function $f$ satisfies

\[
\bar{\nabla} f \geq -B_1 f,
\]  \hspace{1cm} (3.32)

where $B_1$ is a positive constant. Let $p > 0$, then there exist constant $B_2$ depending only on $B_1$, $p$ and $M$ such that

\[
\sup_{B(R)} f \leq B_2 \left( \int_{B(R)} f^p \right)^{\frac{1}{p}}.
\]  \hspace{1cm} (3.33)

**Proof.** Setting $0 < r_2 < r_1 \leq R$, and let $\varphi$ be the cut-off function

\[
\varphi(x) = \begin{cases} 
1; & x \in B(r_2) \\
0; & x \in B(R) \setminus B(r_1)
\end{cases}
\]  \hspace{1cm} (3.34)

$0 \leq \varphi(x) \leq 1$ and $|\nabla \varphi| \leq 2(r_1 - r_2)^{-1}$.

Let $q \geq p > 1$. Multiplying with $f^{q-1}\varphi^2$ on both side of (3.32) and integrating by parts we have

\[
(q - 1) \int_{B(R)} f^{q-2}\varphi^2 |\nabla f|^2 \leq -2 \int_{B(R)} \langle \nabla \varphi, \nabla f \rangle f^{q-1}\varphi + \int_{B(R)} \langle V, \nabla f \rangle f^{q-1}\varphi^2 + B_1 \int_{B(R)} f^q \varphi^2.
\]  \hspace{1cm} (3.35)

Using Schwartz inequality and Young inequality, we have

\[
\int_{B(R)} |\nabla (f^{\frac{q}{2}} \varphi)|^2 \leq \frac{q}{q - 2} \int_{B(R)} (|V|^2 + B_1)f^q \varphi^2 + \int_{B(R)} f^q |\nabla \varphi|^2.
\]  \hspace{1cm} (3.36)
Applying the Sobolev inequality (3.31) to $f^2_2 \varphi$, we get
\[
\left( \int_{B_{r_2}} f^q_2 \frac{2m}{2m-2} \right)^{\frac{2m-2}{2m}} \leq C(M, p, B_1, |V|)(1 + (r_1 - r_2)^{-2}) \int_{B(r_1)} f^q_1.
\] (3.37)

Then, by the standard Moser-iteration argument we deduce (3.33) for $p > 2$. On the other hand a general argument in [16] shows that $p > 0$ case follows from $p > 2$.

\[ \square \]

**Corollary 3.10** If nonnegative function $f$ satisfies
\[
\tilde{\Delta} f \geq -B_3
\] (3.38)
then there exists positive constants $B_4$, $B_5$ depending only on $M$ and $B_3$ such that
\[
\|f\|_\infty \leq B_4(\|f\|_1 + B_5)
\] (3.39)

**Proof.** Let $f' = f + B_3$, then we have $\tilde{\Delta} f' \geq -f'$. Applying the mean value inequality (3.33) to $f'$, we can easily conclude the inequality (3.39).

\[ \square \]

4 Stability of $J$-holomorphic chains

Let $(M, J_M, \eta)$ be a compact $m$-dimensional almost Hermitian manifold whose Kähler form $\eta$ satisfies $\partial_M \bar{\partial}_M \eta^{m-1} = 0$, and let $(E, J, \phi)$ be a $J$-holomorphic chain over $M$ as in (2.8). Let $H_i$ be a Hermitian metric on $E_i$, then we define the degree of $E_i$ as follow:
\[
\deg(E_i) = \frac{\sqrt{-1}}{2\pi} \int_M (Tr \Lambda F_{H_i}^{1,1}) \eta^{\lfloor m \rfloor},
\] (4.1)
where $\eta^{\lfloor m \rfloor} = \frac{1}{m!} \eta^m$. From the condition on the Kähler form, we known that the above definition is independent of Hermitian metrics on the $E_i$. Let $E_0' \subset E_i$ be a complex subbundle, using the Hermitian Codazzi-Mainardi equation, we have the following proposition ([7], proposition 2.4):

**Proposition 4.1** Let $(E_i, J_i, H_i)$ be a Hermitian bundle with a fixed bacs $J_i$, let $E'_i \subset E_i$ be a complex sub-bundle. Then the following facts are equivalent:
(1), $E'_i$ is a $J_i$-holomorphic sub-bundle;
(2), the orthogonal projection $\pi_i : E_i \rightarrow E'_i$ satisfies
\[
(Id - \pi_i) \circ \bar{\partial} E_i \otimes E_i \pi_i = 0.
\] (4.2)
For further consideration, let us introduce the following class of objects $F(E_i, J_i)$ ([7]): $E'_i \in F(E_i, J_i)$ if and only if

1. there exists a closed subset $\Sigma_i \subset M$ with $H_{2m-4}(\Sigma_i) < +\infty$, such that $E'_i |_{M \setminus \Sigma_i}$ is a $J_i$-holomorphic sub-bundle of $E_i |_{M \setminus \Sigma_i}$;

2. for any $x \in \Sigma_i$, and any local $J_M$-holomorphic curve $C$ through $x$ not contained in $\Sigma_i$, $E'_i |_{C - \{x\}}$ extends to $C$ as sub-bundle.

where $H_s$ denote the $s$-dimensional Hausdorff measure. If $E'_i \in F(E_i, J_i)$, we will call $E'_i$ be a weakly $J_i$-holomorphic sub-bundle of $E_i$, and $\Sigma_i$ be the singular set. On the other hand, when $E'_i \in F(E_i, J_i)$, it is easy to see that the corresponding section $\pi_i : E_i \to E'_i$ of $E'_i \otimes E_i$ is in $L^2(End(E_i))$. So it is possible to define the degree of $E'_i$ as follow ([7]):

$$deg(E'_i) := \frac{1}{2\pi} \int_M (|\partial E'_i \otimes E_i \pi_i|^2 |\eta|^m),$$

(4.3)

and the slope, $\mu(E'_i)$, is defined

$$\mu(E'_i) = \frac{deg(E'_i)}{rank E'_i},$$

(4.4)

where $H_i$ is any Hermitian metric on $E_i$. By Codazzi-Mainardi equations, if $E'_i$ is regular, it is easy to check that this definition coincides with the one given in (4.1).

**Definition 4.2** Let $C = (E, J, \phi)$ be a $J$-holomorphic chain, as in (2.8)

(1) A weakly $J$-holomorphic sub-chain of $C$ is a chain

$$C' : E'_n \xrightarrow{\phi'_n} E'_{n-1} \xrightarrow{\phi'_{n-1}} \cdots \xrightarrow{\phi'_0} E'_0,$$

(4.5)

such that $E'_i$ is a weakly $J_i$-holomorphic sub-bundle of $E_i$ with singular set $\Sigma_i$, and $\phi_i \circ f_i |_{M \setminus \Sigma_i \cup \Sigma_{i-1}} = f_{i-1} \circ \phi'_i |_{M \setminus \Sigma_i \cup \Sigma_{i-1}}$ for $1 \leq i \leq n$, where $f_i : E'_i \to E_i$ are the inclusion morphisms. When $\bigcup_{i=0}^n \Sigma_i = \emptyset$, we call $C'$ be a $J$-holomorphic sub-chain of $C$.

(2) The $J$-holomorphic sub-chain $C' \hookrightarrow C$ is called proper if $0 < \sum_{i=0}^n rank E'_i < \sum_{i=0}^n rank E_i$.

(3) The $J$-holomorphic chain $C$ is called decomposable if it can be written as a direct sum $C = C^1 \oplus C^2$ of $J$-holomorphic sub-chains with $C^1 \neq C$, $C^2 \neq C$. Otherwise, $C$ is called indecomposable.

(4) The $J$-holomorphic chain $C$ is called simple if its only $J$-holomorphic endomorphisms are the multiples $\lambda Id_C$ of the identity endomorphism.

**Definition 4.3** Let $C = (E, J, \phi)$ be a $J$-holomorphic chain, as in (2.8). Let $\alpha = (\alpha_0, \alpha_1, ..., \alpha_n) \in R^{n+1}$, and $H = (H_0, H_1, ..., H_n)$ be an $n + 1$-tuple of hermitian metrics on the chain $C$. 

14
The $\alpha$-degree and $\alpha$-slope of a weakly J-holomorphic sub-chain $C'$ are defined by

$$
\text{deg}_\alpha C' = \sum_{i=0}^{n} \text{deg} E'_i - \frac{\text{Vol}(M, \eta)}{2\pi} \sum_{i=0}^{n} \alpha_i \text{rank} E'_i, \quad \mu_\alpha(C') = \frac{\text{deg}_\alpha C'}{\sum_{i=0}^{n} \text{rank} E'_i},
$$

(4.6)

respectively. We say that the J-holomorphic chain $C$ is $\alpha$-(semi) stable if for all proper weakly J-holomorphic sub-chain $C_0 \hookrightarrow C$, $\mu_\alpha(C') < (\leq) \mu_\alpha(C)$. A direct sum of $\alpha$-stable J-holomorphic chains, all of them with the same $\alpha$-slope, is called $\alpha$-polystable.

Suppose that the chain $C$ has an $(n+1)$-tuple $H$ of Hermitian metrics satisfying the chain $\tau$-vortex equations (2.9), then taking traces in (2.9), integrating over $(M, \eta)$, and summing for $0 \leq i \leq n$, one sees that the $\tau$-parameters are constrained by the relation $\sum_{i=0}^{n} \text{deg} E_i = \frac{\text{Vol}(M, \eta)}{2\pi} \tau_i \text{rank} E_i$. This equation can also be written as

$$
\text{deg}_\tau(C) = 0.
$$

(4.7)

This means that there are only $n$ independent parameters among $\tau_0, ..., \tau_n$. In the proof of the Hitchin-Kobayashi correspondence, we shall take $\tau$ satisfying (4.7) for the equations, for the stability condition, it will be convenient to use $\alpha = (\alpha_0, ..., \alpha_n)$, define by

$$
\alpha_i = \tau_i - \tau_0,
$$

(4.8)

so that $\alpha_0 = 0$ and the independent parameters are $\alpha_1, ..., \alpha_n$. By the definition, we have

$$
\mu_\alpha(C) = \mu_\tau(C) + \frac{\text{Vol}(M, \eta)}{2\pi} \tau_0,
$$

(4.9)

hence the stability condition does not change under global translations of the parameter vector. So, $\tau$-(semi) stability is equivalent to $\alpha$-(semi) stability.

Next, we will show that the $\tau$-stability is the necessary condition for the existence of solutions of the chain $\tau$-vortex equation (2.9). In fact, we prove the following theorem.

**Theorem 4.4** Let $(M, J_M, \eta)$ be a compact $m$-dimensional almost Hermitian manifold whose Kähler form $\eta$ satisfies $\partial_M \bar{\partial}_M \eta^{n-1} = 0$, and $C = (E, J, \phi)$ be a J-holomorphic chain over $M$ as in (2.8). Let $\tau = (\tau_0, ..., \tau_n) \in \mathbb{R}^{n+1}$ be such that $\text{deg}_\tau(C) = 0$. Suppose that the chain $C$ admits an $(n+1)$-tuple $H = (H_0, ..., H_n)$ of hermitian metrics satisfying the chain $\tau$-vortex equations, then the chain $C$ must be $\tau$-polystable.

**Proof:** First of all, let $\alpha = (\alpha_0, ..., \alpha_n)$ be defined by (4.8). Recall that $\tau$-stability is equivalent to $\alpha$-stability. We can assume that $C$ is indecomposable, then we only need to prove that it is $\alpha$-stable. Let $C' = (E'_0, E'_1, ..., E'_n)$ be a proper weakly J-holomorphic sub-chain, and $\pi_i$ be the section of $E^* \otimes E$ associated to the weakly $J_i$-holomorphic sub-bundle $E'_i \hookrightarrow (E_i, H_i)$. Using (2.8), we have
Let the chain of metrics which satisfies the chain vortex equation (2.9). Let $(C^0, E, J, \phi)$ be a compact $m$-dimensional almost Hermitian manifold whose kähler form $\eta$ satisfies $\partial_M \bar{\partial}_M \eta^{m-1} = 0$, and $C = (E, J, \phi)$ be a $J$-holomorphic chain as in (2.8). Let $K = (K_0, K_1, \ldots, K_n)$ be the initial $n+1$-tuple of Hermitian metrics on the chain $C$, then we consider the evolution equation (3.1), where the parameter vector $\tau = (\tau_0, \ldots, \tau_n)$ satisfies (4.7). First of all, we will prove that the above equations have long-time solution $H(t)$; next, under the assumption of $\tau$-stability, we will show that the solution $H(t)$ converges to an $n+1$-tuple $H(\infty)$ of Hermitian metrics which we need. The main points in the argument are similar with that in [7].

From formula (3.2), we known that the evolution equations which we considered is a nonlinear strictly parabolic system, so standard parabolic theory gives the short-time existence.

**Proposition 5.1** For sufficiently small $\epsilon > 0$, the system (3.1) has a smooth solution $H(t) = (H_0(t), \ldots, H_n(t))$ defined for $0 \leq t < \epsilon$. 

\begin{align}
\deg(E'_i) &= \frac{1}{2\pi} \int_M (\sqrt{-1}T^\tau \partial_M \Lambda E^{1,1}_{H_i} - |\bar{\partial}_E \pi_i|^2) \eta^{[m]} \\
&= \frac{1}{2\pi} \int_M \{ \frac{1}{2} \text{Tr}(\pi_i(\phi_i^H \phi_i - \phi_{i+1} \phi_{i+1}^H)) - |\bar{\partial}_E \pi_i|^2 \} \eta^{[m]} + \frac{1}{2\pi} \text{rank}E'_i \text{Vol}(M),
\end{align}

Let $\phi'_i = \pi_{i-1} \circ \phi_i \circ \pi_i$, $\phi''_i = \pi^\perp_{i-1} \circ \phi_i \circ \pi^\perp_i$, $\phi^\perp_i = \pi_{i-1} \circ \phi_i \circ \pi^\perp_i$, where $\pi^\perp_i = \text{Id} - \pi_i$. Then a straightforward computation shows that

$$
\sum_{i=0}^n \int_M \frac{1}{2} \text{Tr}(\pi_i(\phi_i^* \phi_i - \phi_{i+1} \phi_{i+1}^H)) = - \sum_{i=0}^n \int_M |\phi_i^\perp|^2_H,
$$

where $\phi_i^\perp$ is the adjoint of $\phi_i^\perp$ with respect to metrics $H$. Therefore,

$$
\mu_\alpha(C') = \mu_\alpha(C) - \frac{1}{2\pi \sum_{i=0}^n \text{rank}E'_i} \sum_{i=0}^n \int_M |\bar{\partial}_E \pi_i|^2 + |\phi_i^\perp|^2_H.
$$

Consequently, it follows that $\mu_\alpha(C') < \mu_\alpha(C)$, or $\int_M |\bar{\partial}_E \pi_i|^2_H$ and $\int_M |\phi_i^\perp|^2_H$ are all zero. In the latter case, we know that $\pi_i$ satisfying $D_H \pi_i = 0$, so $\pi_i$ is global regular and $E_i$ splits $E_i = E_i^\perp \otimes E_i^\perp$ $J$-holomorphically. On the other hand, $\phi_i^\perp \equiv 0$ for all $0 \leq i \leq n$, we know that the chain $C$ is a direct sum $C = C' \oplus C^\perp$ of $J$-holomorphic sub-chain, where

$$
C' : E_n^\perp \xrightarrow{\phi_n} E_{n-1} \xrightarrow{\phi_{n-1}} \cdots \xrightarrow{\phi_1} E_0',
$$

$$
C^\perp : E_n^\perp \xrightarrow{\phi_n} E_{n-1} \xrightarrow{\phi_{n-1}} \cdots \xrightarrow{\phi_1} E_0^\perp.
$$

This contradicts the indecomposibility of the chain $C$. 

\[ \square \]
Let $H(t)$ be a solution of the evolution equation (3.1), and $h_i = K_i^{-1}H_i$, for $0 \leq i \leq n$. Then

$$\frac{\partial}{\partial t}(\log Tr h_i) = \left| \frac{\mathrm{Tr}(\phi_i)}{\mathrm{Tr} h_i} \right|,$$

$$= 2|\mathrm{Tr}h_i(\sqrt{-1}AF_{H_i} - \frac{1}{2}(\phi_i^*H \circ \phi_i - \phi_{i+1} \circ \phi_{i+1}^H) - \tau_i \mathrm{Id}_{E_i})|,$$

$$\leq 2|\sqrt{-1}AF_{H_i} - \frac{1}{2}(\phi_i^*H \circ \phi_i - \phi_{i+1} \circ \phi_{i+1}^H) - \tau_i \mathrm{Id}_{E_i})|,$$  \hfill (5.1)

and

$$\frac{\partial}{\partial t}(\log Tr h_i^{-1}) \leq 2|\sqrt{-1}AF_{H_i} - \frac{1}{2}(\phi_i^*H \circ \phi_i - \phi_{i+1} \circ \phi_{i+1}^H) - \tau_i \mathrm{Id}_{E_i}|.$$

(5.2)

**Theorem 5.2** Suppose that a smooth solution $H(t)$ to the evolution equations (3.1) is defined for $0 \leq t < T$. Then $H(t)$ converges in $C^0$-topology to some $(n+1)$-tuple $H(T)$ of continuous non-degenerate metrics as $t \to T$.

**Proof:** Given $\epsilon > 0$, by continuity at $t = 0$ we can find a $\delta$ such that

$$\sup_M \sigma(H(t), H(t')) < \epsilon,$$

for $0 < t, t' < \delta$. Then Proposition 3.4 and the Maximum principle imply that

$$\sup_M \sigma(H(t), H(t')) < \epsilon,$$

for all $t, t' > T - \delta$. This implies that the $H_i(t)$ are a uniformly Cauchy sequence and converge to a continuous limiting metric $H_i(T)$, for every $0 \leq i \leq n$. By proposition 3.1, we known that $|\sqrt{-1}AF_{H_i} - \frac{1}{2}(\phi_i^*H \circ \phi_i - \phi_{i+1} \circ \phi_{i+1}^H) - \tau_i \mathrm{Id}_{E_i})|$ are bounded uniformly. Using formula (5.1) and (5.2), one can conclude that $\sigma(H_i(t), K_i)$ are bounded uniformly, therefore $H_i(T)$ is a non-degenerate metric.

Arguing like that in [8; Lemma 19] or [13; Lemma 4.3.2], one can easily prove the following lemma.

**Lemma 5.3** Let $H(t)$, $0 \leq t < T$, be any one-parameter family of Hermitian metrics on complex vector bundle $E$ over almost Hermitian manifold $M$. If $H(t)$ converges in the $C^0$ topology to some continuous metric $H(T)$ as $t \to T$, and if $\sup_M |AF_{H}^{1,1}|$ is bounded uniformly in $t$, then $H(t)$ are bounded in $C^{1,\alpha}$ (for $0 < \alpha < 1$) and also bounded in $L^p$ (for any $1 < p < \infty$) uniformly in $t$.

**Theorem 5.4** Given any initial tuple $K$ of hermitian metrics, then the evolution equation (3.1) has a unique solution $H(t)$ which exists for $0 \leq t < \infty$. 

17
Proposition 5.1 guarantees that a solution exists for a short time. Suppose that the solution \( H(t) \) exists for \( 0 \leq t < T \). By theorem 5.2, \( H(t) \) converges in \( C^0 \)-topology to a \( n + 1 \)-tuple \( H(T) \) of non-degenerate continuous limit metrics as \( t \to T \). We have known that
\[
|\sqrt{-1}\Lambda F_{H_i} - \frac{1}{2}(\phi_i^x H \circ \phi_i - \phi_{i+1} \circ \phi_i^x H) - \tau_i Id_{E_i}|_{H_i}
\]
is bounded independently of \( t \). Moreover, from proposition 3.2, we have
\[
(\tilde{\Delta} - \frac{\partial}{\partial t})\Phi^2 \geq 2 \sum_{i=1}^n |\partial_H \phi_i|^2 + C_1 \Phi^4 - \max_{1 \leq i \leq n} \{ |\tau_i - \tau_{i-1}| \}\Phi^2
\]
where \( \Phi^2 = \sum_{i=1}^n |\phi_i|^2_{H_i} \). Assume that \( \Phi^2 \) attains its maximum on \( M \times [0, T) \) at the point \((x_0, t_0)\) with \( 0 < t_0 < T \), \( x_0 \in M \). If \( \Phi^2(x_0, t_0) > \frac{\max_{1 \leq i \leq n} \{ |\tau_i - \tau_{i-1}| \}}{C_1} \), then
\[
(\tilde{\Delta} - \frac{\partial}{\partial t})\Phi^2 \geq 0,
\]
on a neighborhood of \((x_0, t_0)\). This contradicts the maximum principle of the heat operator. So we have
\[
\Phi^2 \leq \max\{ \sup_M \sum_{i=1}^n |\phi_i|^2_K, \frac{\max_{1 \leq i \leq n} \{ |\tau_i - \tau_{i-1}| \}}{C_1} \}. \tag{5.3}
\]
Moreover, \( \sup_M |\sqrt{-1}\Lambda F_{H_i^{1,1}_{K_i}}|_{K_i} \) is bounded independently of \( t \), for every \( 0 \leq i \leq n \). Hence by lemma 5.3, \( H_i(t) \) are bounded in the \( C^1 \)-topology and also bounded in \( L^p \) (for any \( 1 < p < \infty \)) uniformly in \( t \). Since the evolution equation (3.2) is quadratic in the first derivative of \( h_i \) we can apply Hamilton’s method [11] to deduce that \( H_i(t) \to H_i(T) \) in \( C^\infty \), for every \( 0 \leq i \leq n \), and the solution can be continued past \( T \). Then the evolution equation (3.1) has a solution \( H(t) \) define for all times.

On the other hand, suppose that \( H'(t) \) is another solution of equation (3.1) with the same initial tuple \( K \) of hermitian metrics, from proposition 3.4, we have
\[
(\tilde{\Delta} - \frac{\partial}{\partial t})\sigma(H(t), H'(t)) \geq 0,
\]
and \( \sigma(H, H')|_{t=0} = 0 \). By the maximum principle, we have
\[
\sigma(H(t), H'(t)) \equiv 0, \text{i.e.} H(t) \equiv H'(t).
\]
So we have proved the uniqueness of the solution.

Next, we will use the stability to deduce that the solution \( H(t) \) of above the evolution equations must converges to an \( n + 1 \)-tuple \( H(\infty) \) metric which we need. For the further discussion, we shall introduce the following machineries. Let \( M_D(K_i, H_i) \) be the Donaldson Lagrangian ([8], [7]) of two hermitian metrics \( K_i, H_i \), and we define the modified Donaldson Lagrangian \( M_{\phi, \alpha} \) of two \((n + 1)\)-tuples of Hermitian metrics as following
\[
M_{\phi, \alpha}(K, H) = \sum_{i=0}^n M_D(K_i, H_i) + \sum_{i=1}^n \int_M (|\phi_i|^2_H - |\phi_i|^2_K) - 2 \sum_{i=0}^n \int_M \alpha_i Tr(\log(K_i^{-1}H_i)) \tag{5.4}
\]
where $K = (K_0, \ldots, K_n)$, $H = (H_0, \ldots, H_n)$, $K_i$ and $H_i$ are hermitian metrics on bundle $E_i$, and $\alpha = (\alpha_0, \ldots, \alpha_n) \in \mathbb{R}^{n+1}$.

For the reader’s convenience, we recall some notation in [19] and [7]. Let $K$ be a fixed Hermitian metric on the bundle $E$, denote

$$S_K(E) = \{ s \in \Omega^0(M, \text{End}(E)) | s^*(K) = s \}.$$

Given $\rho \in C^\infty(R, R)$ and $s \in S_K(E)$. We define $\rho(s)$ as follows. At each point $x$ on $M$, chose $\{e_i\}_{i=1}^r$ be an unitary basis with respect to metric $K$, such that $s(e_i) = \delta_i e_i$. Set:

$$\rho(s)(e_i) = \rho(\delta_i) e_i.$$

Given $\Psi \in C^\infty(R \times R, \rho \in \Omega^0(M, \text{End}(E))$. In a similar way, we define $\Psi[s](p)$ as follows. Let $\{e^*_i\}_{i=1}^r$ be the dual basis for $\{e_i\}_{i=1}^r$, then $p \in \Omega^0(M, \text{End}(E))$ can be written

$$p = \sum p_{ij} e^*_i \otimes e_j.$$

Set:

$$\Psi[s](p) = \sum \Psi(\delta_i, \delta_j) p_{ij} e^*_i \otimes e_j.$$

In fact, the Donaldson’s Lagrangian is defined as follows

$$M_D(K, H) = 2 \int_M \langle \log(K^{-1}H), \sqrt{-1} \Lambda F_{K^{-1}}^{1,1} \rangle_K + 2 \int_M \langle \log(K^{-1}H), \sqrt{-1} \Lambda \overline{\partial} E(\varphi \langle \log(K^{-1}H) \rangle \langle \partial_K \log(K^{-1}H) \rangle) \rangle_K,$$

where $\varphi(x, y) = \frac{e^{y-x}+(x-y)^{-1}}{(x-y)^2}$.

**Lemma 5.5** (1) Let $H^1$, $H^2$, $H^3$ be three $n + 1$-tuples of hermitian metrics on chain $C = (E, J, \phi)$, and $\alpha = (\alpha_0, \ldots, \alpha_n) \in \mathbb{R}^{n+1}$, then

$$M_{\phi, \alpha}(H^1, H^3) = M_{\phi, \alpha}(H^1, H^2) + M_{\phi, \alpha}(H^2, H^3). \quad (5.5)$$

(2) Let $H(t)$ be a family of $n + 1$-tuples of hermitian metrics on chain $C$, then

$$\frac{d}{dt} M_{\phi, \alpha}(H(0), H(t)) = \int_M \langle H_0^{-1} \frac{dH_0}{dt}, \sqrt{-1} \Lambda F_{H_0}^{1,1} + \phi_1 \phi^*_H - 2\alpha_0 \text{Id}_{E_0} \rangle_{H_0} + \sum_{i=1}^{n-1} \int_M \langle H_i^{-1} \frac{dH_i}{dt}, \sqrt{-1} \Lambda F_{H_i}^{1,1} - \phi^*_i \phi_i - \phi_{i+1} \phi^*_{i+1} \rangle_{H_i} - 2\alpha_i \text{Id}_{E_i} \rangle_{H_i} + \int_M \langle H_n^{-1} \frac{dH_n}{dt}, \sqrt{-1} \Lambda F_{H_n}^{1,1} - \phi^*_n \phi_n - 2\alpha_n \text{Id}_{E_n} \rangle_{H_n}. \quad (5.6)$$

**Proof.** (1) Formula (5.5) can be deduced directly by the properties of Donaldson’s Lagrangian and the definition of modified Donaldson Lagrangian (5.4).

(2) Using formula (5.5), we only need to compute $\frac{d}{dt} M_{\phi, \alpha}(H(0), H(t))$ for $t = 0$. It is easy to check that
\[
\frac{d}{dt} M_{\phi, \alpha}(H(0), H(t))|_{t=t_0} = \frac{d}{dt} M_{\phi, \alpha}(H(0), H(t_0 + t))|_{t=0} = \frac{d}{dt} M_{\phi_0, \alpha}(H(t_0), H(t_0 + t))|_{t=0},
\]

\[
= \int_M \langle H_0^{-1} \frac{d}{dt} H_0, 2\sqrt{-\Lambda} F_{H_0}^{1,1} \rangle + \phi^*_1 H - 2\alpha_0 Id_{E_0} \rangle_{H_0}|_{t=t_0} + \sum^n_{i=1} \int_M \langle H_1^{-1} \frac{d}{dt} H_1, 2\sqrt{-\Lambda} F_{H_1}^{1,1} \rangle - (\phi^*_1 \phi - \phi_1 + \phi^*_1 H_i) - 2\alpha_i Id_{E_i} \rangle_{H_i}|_{t=t_0} + \int_M \langle H_n^{-1} \frac{d}{dt} H_n, 2\sqrt{-\Lambda} F_{H_n}^{1,1} \rangle - \phi^*_n H_n - 2\alpha_n Id_{E_n} \rangle_{H_n}|_{t=t_0}. \]

For the further argument, we need the following proposition.

**Proposition 5.6 ([7]; Theorem 0.2)** Let \((M, J_M, g), (N, J_N, h)\) be two almost Hermitian manifolds with \(\text{dim}_R M = 2m\), and assume there exists a bounded closed 2-form \(\alpha\) on \(N\) such that \(\alpha^{1,1} > 0\) uniformly. Let \(\sigma : M \rightarrow N\) be a \(L^1\)-weakly \((J_M, J_N)\)-holomorphic map. Then there exists a closed subset \(\Sigma \subset M\) with \(H_{2m-4}(\Sigma) < +\infty\), such that \(\sigma\) is smooth on \(M \setminus \Sigma\); moreover, for any \(x \in \Sigma\), any local \(J_M\)-holomorphic curve \(C\) through \(x\) not contained in \(\Sigma\), \(\sigma|_{C_{\{x\}}}\) extends smoothly to \(C\).

**Proof of the main theorem:** Let \(H(t) = (H_0(t), ..., H_n(t))\) be a solution of equation (3.1) with initial tuple \(K\), and \(h(t) = (h_0(t), ..., h_n(t))\) where \(h_i = K_i^{-1} H_i = \exp(s_i)\). From proposition 3.7, we have

\[
\Delta \log \{\sum^n_{i=0} (Tr(h_i) + Tr(h_i^{-1}))\} \geq - (\sum^n_{i=0} |2\sqrt{-\Lambda} F_{K_i}^{1,1} \rangle - (\phi^*_1 K \phi_i - \phi_1^* K_i) - 2\tau_i Id_{E_i}|_{K_i}) \]

\[
- (\sum^n_{i=0} |2\sqrt{-\Lambda} F_{H_i}^{1,1} \rangle - (\phi^*_1 H_i \phi_i - \phi_1^* H_i) - 2\tau_i Id_{E_i}|_{H_i}). \tag{5.7}
\]

By proposition 3.1, we know that \(\sup M |2\sqrt{-\Lambda} F_{H_i}^{1,1} \rangle - (\phi^*_1 H_i \phi_i - \phi_1^* H_i) - \tau_i Id_{E_i}|_{H_i}\) is bounded independently of \(t\). Using Corollary 3.10, there exists two constants \(B_5\) and \(B_6\) such that

\[
\|\log \{\sum^n_{i=0} (Tr(h_i) + Tr(h_i^{-1}))\}\|_{\infty} \leq B_5 (\int_M \log \{\sum^n_{i=0} (Tr(h_i) + Tr(h_i^{-1}))\} + B_6). \tag{5.8}
\]

On the other hand, one can check that

\[
\log \left\{ \frac{1}{\sum^n_{i=0} r_i} \sum^n_{i=0} (Tr h_i + Tr h_i^{-1}) \right\} \leq \sum^n_{i=0} s_i|_{K_i} = \sum^n_{i=0} s_i|_{H_i} \leq \left( \sum^n_{i=0} r_i^2 \right) \log \sum^n_{i=0} (Tr h_i + Tr h_i^{-1}) \tag{5.9}
\]

where \(r_i = \text{rank}E_i\). So there exist constants \(B_7 > 0, B_8 > 0\) such that, for every \(t \in [0, +\infty)\), we have:

\[
\sum^n_{i=0} \|s_i(t)\|_{\infty} \leq B_7 + B_8 (\sum^n_{i=0} \|s_i(t)\|_1). \tag{5.10}
\]

Now, there are two possibilities:

(1) There exists constant \(B_9 > 0\) such that, for every \(t \in [0, +\infty)\),

\[
\sum^n_{i=0} \|s_i(t)\|_{\infty} < B_9.
\]
\[ (2), \lim_{t \to -\infty} \left( \sum_{i=0}^{n} \| s_i(t) \|_1 \right) = +\infty. \]

Assume we are in case (1). Using the condition \( \partial_M \bar{\partial}_M \eta^{m-1} = 0 \), it is not hard to check that

\[
\begin{align*}
\int_M (s_i, \sqrt{-1} \Lambda \bar{\partial}_E (\varphi [s_i] (\partial_H s_i))) H_i \eta^{[m]} \\
= \int_M (\Psi [s_i] (\bar{\partial}_E s_i), \bar{\partial}_E s_i) H_i \eta^{[m]} - \sqrt{-1} \int_M T r s_i H_i^{-1} \varphi [s_i] (\partial_H s_i)^T H_i \wedge \eta^{m-1} \\
= \int_M (\Psi [s_i] (\bar{\partial}_E s_i), \bar{\partial}_E s_i) H_i \eta^{[m]} - \frac{1}{2} \sqrt{-1} \int_M (\bar{\partial} (T r s_i^2)) \wedge \eta^{m-1} 
\end{align*}
\]

where \( \Psi(x, y) = \varphi(y, x) = \frac{e^{\sqrt{-1}(x-y)}}{(x-y)^4} \). By formula (5.4), we have

\[
M_{\phi, \tau}(K, H) \geq - \sum_{i=0}^{n} \int_M |s_i| |2 \sqrt{-1} \Lambda F_{H_i}^{1,1} - 2 \tau_i Id_H| \eta^{[m]} + 2 \sum_{i=0}^{n} \int_M (\Psi [s_i] (\bar{\partial}_E s_i), \bar{\partial}_E s_i) H_i \eta^{[m]} + \sum_{i=0}^{n} \int_M (|\phi_i|^2 H_i - |\phi_i|^2 K_i). \tag{5.11} \]

From \( \sum_{i=0}^{n} \| s_i(t) \|_\infty < B_9 \) for every \( t \in [0, +\infty) \), it follows that \( \Psi \geq B_9 \) on the range of the \( s_i(t) \)'s; so that

\[
\int_M (\Psi [s_i] (\bar{\partial}_E s_i), \bar{\partial}_E s_i) H_i \eta^{[m]} \geq 2 \tau_i \| \bar{\partial}_E s_i \|_2^2, \tag{5.12} \]

for every \( 0 \leq i \leq n \). On the other hand, from theorem 3.2, we known that \( \sum_{i=1}^{n} |\phi_i|^2 |H(t)| \) is bounded uniformly. Therefore, there exists \( B_{11} > 0 \) such that, for every \( t \in [0, +\infty) \)

\[
M_{\phi, \tau}(K, H(t)) \geq -B_{11}. \tag{5.13} \]

From (5.6), we have

\[
\frac{d}{dt} M_{\phi, \tau}(K, H(t)) = - \int_M \sum_{i=0}^{n} |2 \sqrt{-1} \Lambda F_{H_i}^{1,1} - (\phi_i^{*H} \phi_i - \phi_{i+1}^{*H} \phi_{i+1}) - 2 \tau_i Id_{E_i} | H_i. \tag{5.14} \]

By (5.11), (5.12), (5.14), we known that \( \| \bar{\partial}_E s_i \|_2 \) and also \( \| \bar{\partial}_E h_i \|_2 \) are uniformly bounded. Thus, there exits a subsequences \( t_j \to +\infty \), such that \( h_i(t_j) \) weakly converges to \( h_i (\infty) \) in \( L^2 \), for every \( 0 \leq i \leq n \). By (5.13) and (5.14), we known that \( \sum_{i=0}^{n} |2 \sqrt{-1} \Lambda F_{H_i}^{1,1} - (\phi_i^{*H} \phi_i - \phi_{i+1}^{*H} \phi_{i+1}) - 2 \tau_i Id_{E_i} | H_i (t_j) \) weakly converges to 0 in \( L^2 \). Then, the standard elliptic regularity implies that \( h_i (\infty) \) is smooth and \( H_i (\infty) = K_i h_i (\infty) \) satisfies the chain \( \tau \) vortex equations (2.9).

By conformal transformations, we can assume that the initial \( n+1 \)-tuple of Hermitian metrics \( K = (K_0, ..., K_n) \) satisfies:

\[
\sum_{i=0}^{n} Tr (\sqrt{-1} \Lambda F_{K_i}^{1,1} - \frac{1}{2} (\phi_i^{*K} \phi_i - \phi_{i+1}^{*K} \phi_{i+1}) - \tau_i Id_{E_i}) = 0. \tag{5.15} \]

Assume, from now on, we are in case (2). In particular, we can choose a sequence \( \{ t_j \} \) such that: \( t_j \to \infty \) and \( \sum_{i=0}^{n} |s(t_j)|_1 \to \infty \). Set \( E = E_0 \oplus E_1 \oplus \cdots \oplus E_n \), then, let \( H = H_0 \oplus H_1 \oplus \cdots \oplus H_n \) be a hermitian metric on \( E \), denote \( h = h_0 \oplus h_1 \oplus \cdots \oplus h_n \), and \( s = s_0 \oplus s_1 \oplus \cdots \oplus s_n \in End (E) \). Let \( l_j = \| s(t_j) \|_1 \) and \( u_j = l_j^{-1} s(t_j) \in End (E) \), from the assumption, we known that \( l_j \to \infty \). Using (5.10), we have

\[
\| u_j \|_1 = 1 \quad \text{and} \quad \| u_j \|_\infty \leq B_{12} \tag{5.16} \]
where $B_{12}$ is a positive constant. By formula (3.7) and the initial assumption (5.15), we have

$$Trs(t) = 0,$$

(5.17)

for every $0 \leq t < \infty$. From

$$l_j \langle \Psi[l_j u_j](\bar{\partial}_E u_j), \bar{\partial}_E u_j \rangle \geq \langle \Psi[u_j](\bar{\partial}_E u_j), \bar{\partial}_E u_j \rangle$$

(5.18)

and (5.11) (5.14), it follows that

$$\int_M \langle \Psi[u_j](\bar{\partial}_E u_j), \bar{\partial}_E u_j \rangle \eta^{[m]} \leq B_{13}.$$ 

Since $u_j$ is bounded uniformly, so $\Psi \geq C > 0$ on the range of the $u_j$'s. Then, we obtain

$$\|\bar{\partial}_E u_j\|_2 \leq B_{14}.$$ 

(5.19)

Then, passing to a subsequence, $u_j$ converges weakly to $u_\infty$ in $L^2_1$; clearly, $u_\infty$ is nontrivial.

We denote $\pi_i : E \to E_i$ the projection to sub-bundle $E_i$, $K = K_0 \oplus K_1 \cdots \oplus K_n$ the initial hermitian metric on bundle $E$, and

$$\tilde{\phi} = \sum_{i=1}^n \phi_i \circ \pi_i.$$ 

(5.20)

The chain $\tau$ vortex equations (2.9) can be re-written as follows:

$$\sqrt{-1} \Lambda F_{H}^{1,1} - \frac{1}{2}(\tilde{\phi}^* \tilde{\phi} - \tilde{\phi} \tilde{\phi}^*) - \sum_{i=0}^n \tau_i \pi_i = 0.$$ 

(5.21)

Moreover,

$$\int_M \langle u_\infty, 2\sqrt{-1} \Lambda F_{K}^{1,1} - 2 \sum_{i=0}^n \tau_i \pi_i \rangle + 2 \int_M \langle \Psi[u_\infty], \partial_E u_\infty \rangle$$

$$= \lim_{j \to \infty} \left( \int_M \langle u_j, 2\sqrt{-1} \Lambda F_{K}^{1,1} - 2 \sum_{i=0}^n \tau_i \phi_i \rangle + 2 \int_M \langle \Psi[u_j], \partial_E u_j \rangle \right)$$

$$\leq \lim_{j \to \infty} l_j^{-1} \left( \int_M \langle s(t_j), 2\sqrt{-1} \Lambda F_{K}^{1,1} - 2 \sum_{i=0}^n \tau_i \phi_i \rangle + 2 \int_M \langle \Psi[s(t_j)], \partial_E s(t_j) \rangle \right)$$

$$+ \int_M \langle \bar{\partial}_H^2 \phi(\tau), \phi^2 \rangle$$

$$\leq \lim_{j \to \infty} l_j^{-1} M_{\phi, \tau}(K, H(t_j)) = 0.$$ 

(5.22)

In the same manner, if $\zeta \in C^\infty(R \times R, R)$ satisfies $\zeta(x, y) \leq (x - y)^{-1}$, whenever $x > y$, then

$$\int_M \langle u_\infty, 2\sqrt{-1} \Lambda F_{K}^{1,1} - 2 \sum_{i=0}^n \tau_i \pi_i \rangle + 2 \int_M \langle \zeta[u_\infty], \partial_E u_\infty \rangle \leq 0.$$ 

(5.23)

For any smooth function $\rho : R \to R$, a straightforward computation gives

$$\bar{\partial} Tr\rho(u_\infty) = Tr(\partial \rho[u_\infty])(\bar{\partial}_E u_\infty)$$

(5.24)

where we set:

$$\delta \rho(\lambda, \mu) = \begin{cases} \frac{\rho(\lambda) - \rho(\mu)}{\lambda - \mu}, & \text{if } \lambda \neq \mu, \\ \rho'(\lambda), & \text{if } \lambda = \mu. \end{cases}$$
For any number \( N \), we can find a smooth function \( \tilde{\rho} : R \times R \to R \) such that: \( \tilde{\rho}(x, x) = \delta\rho(x, x) \); and \( N\tilde{\rho}^2(x, y) \leq (x - y)^{-1} \) whenever \( x > y \). Then

\[
|\partial\text{Tr}(\rho(u_\infty))|^2 = |\text{Tr}(\tilde{\rho}[u_\infty]_E u_\infty)|^2 \leq B_{15}|\tilde{\rho}[u_\infty]_E u_\infty|^2 = \frac{B_{15}}{N}\tilde{\rho}^2[u_\infty]_E(\partial_E u_\infty, \partial_E u_\infty).
\]

By (5.23), we have

\[
\|\partial\text{Tr}(\rho(u_\infty))\|^2 \leq \frac{B_{15}}{N}. \quad (5.25)
\]

Since this holds for all \( N > 0 \), and \( \text{Tr}(\rho(u_\infty)) \) is real valued, we get that \( \text{Tr}(\rho(u_\infty)) \) is constant almost everywhere. This implies that the eigenvalues of \( u_\infty \) are constant almost everywhere, so we have proved the following lemma.

**Lemma 5.7** The eigenvalues of \( u_\infty \) are constant almost everywhere.

Let \( \lambda_1, \ldots, \lambda_l \) denote the distinct eigenvalues of the \( u_\infty \), listed in ascending order. On the other hand, by (5.17), we have

\[
\text{Tr} u_\infty = 0
\]

almost everywhere. So \( l \geq 2 \).

For \( \alpha < l \) define \( p_\alpha : R \to R \) to be a smooth positive function such that

\[
p_\alpha(x) = \begin{cases} 1, & \text{if } x \leq \lambda_\alpha, \\ 0, & \text{if } x \geq \lambda_{\alpha+1} \end{cases}
\]

(5.26)

Define

\[
\pi_\alpha = p_\alpha(u_\infty).
\]

(5.27)

**Proposition 5.8** Let \( \pi_\alpha \) be as above for \( \alpha < l \). Then

1. \( \pi_\alpha \in L^1(S_K(E)) \);
2. \( \pi_\alpha^2 = \pi_\alpha = \pi_\alpha^* K \);
3. \((\text{Id} - \pi_\alpha)\partial_E^* \otimes E(\pi_j) = 0 \) almost everywhere;
4. \( \|(\text{Id} - \pi_\alpha)\hat{\phi}\pi_\alpha^* \hat{\phi}\|_2^2 = 0 \).

**Proof.** (1), (2), (3) can be deduced directly by the same argument as in [5; position 3.10.2] or [7; proposition 4.6]. So we only need to prove (4). It is not hard to check that, for large enough \( j \),

\[
\int_M |\hat{\phi}|_{H(t_j)}^2 = \int_M \langle h(t_j)\hat{\phi}h^{-1}(t_j), \hat{\phi} \rangle_K = \int_M \langle e^{t_j u_j}\hat{\phi} e^{-t_j u_j}, \hat{\phi} \rangle_K \geq \int_M \langle \Omega[u_j]\tilde{\phi}, \tilde{\phi} \rangle_K,
\]

where \( \Omega(\lambda, \mu) = \omega(\mu - \lambda), \omega : R \to R \) is a smooth positive function such that \( \omega \) be compactly supported and \( \omega(\lambda) = 0 \) whenever \( \lambda \leq \epsilon \) for some \( \epsilon > 0 \). In (5.28), we have used the fact ([5,
So, we have obtained a sequence of proper weakly $J$-holomorphic sub-chains $C_\alpha$, of $C = (E, J, \phi)$; 

\[ C_\alpha : \pi'_{\alpha n} \rightarrow \pi'_{\alpha n-1} \rightarrow \cdots \rightarrow \pi'_{\alpha 0}, \] 

where $\phi'_{\alpha i} = \phi_i|_{\pi'_{\alpha i}}$. We define 

\[ Q(\tau) := \lambda_1 \text{deg}_\tau (C) - \sum_{\alpha=1}^{l-1} (\lambda_{\alpha+1} - \lambda_\alpha) \text{deg}_\tau (C_\alpha). \] 

Then 

\[
Q(\tau) = \frac{1}{l^2} \sum_{\alpha=1}^{l-1} (\lambda_{\alpha+1} - \lambda_\alpha) \text{deg}_\tau (C_\alpha) \] 

\[
= \frac{1}{l^2} \sum_{\alpha=1}^{l-1} \left( \lambda_{\alpha+1} - \lambda_\alpha \right) \left( \partial_{E^* \otimes E} \pi'_\alpha \right)^2 
- \frac{1}{l^2} \sum_{\alpha=1}^{l-1} \left( \lambda_{\alpha+1} - \lambda_\alpha \right) \text{deg}_\tau (C_\alpha).
\] 

Using the result and notation of [5, lemma 3.12.1],

\[
\sum_{\alpha=1}^{l-1} (\lambda_{\alpha+1} - \lambda_\alpha) \left( \partial_{E^* \otimes E} \pi'_\alpha \right)^2 
= \sum_{\alpha=1}^{l-1} (\lambda_{\alpha+1} - \lambda_\alpha) \left( \partial_{E^* \otimes E} \pi'_\alpha \right)^2 
= \sum_{\alpha=1}^{l-1} (\lambda_{\alpha+1} - \lambda_\alpha) \left( \partial_{E^* \otimes E} \pi'_\alpha \right)^2 
= \sum_{\alpha=1}^{l-1} (\lambda_{\alpha+1} - \lambda_\alpha) \left( \partial_{E^* \otimes E} \pi'_\alpha \right)^2 
= \left( \partial_{E^* \otimes E} \pi'_\alpha \right)^2.
\]
Here $\zeta : R \times R \to R$ is defined by $\zeta = \sum_{\alpha=0}^{l-1}(\lambda_{\alpha+1} - \lambda_{\alpha})(\delta p_{\alpha})^2$, hence it satisfies the conditions that $\zeta(\lambda, \mu) \leq (\lambda - \mu)^{-1}$ for $\lambda > \mu$. Then, we make use of (5.23), (5.35), (5.36) to deduce that

$$Q(\tau) \leq 0. \quad (5.37)$$

On the other hand, from the definition of the $\tau$-stability of the chain $C$ we deduce that $Q(\tau) > 0$, thus we get a contradiction. So, we have proved the main theorem.

Acknowledgments

This paper was written while the author was visiting ICTP. He wishes to acknowledge the generous support of the Centre. He would also like to thank Prof. Yibing Shen and Prof. Jiayu Li for their useful discussions.

References


