ON THE ISING MODEL WITH COMPETING INTERACTIONS ON A CAYLEY TREE: GIBBS MEASURES, FREE ENERGY

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Abstract

In the present paper the Ising model with competing binary $J$ and $J_1$ interactions with spin values $\pm 1$, on a Cayley tree is considered. We study translation-invariant Gibbs measures and corresponding free energies ones.

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1. Introduction

Nowadays models of on non-amenable graphs is a modern growing field of statistical mechanics ([L]). One of the non-amenable graphs is a Cayley tree. The Cayley tree is not a realistic lattice, however, its amazing topology makes the exact calculation of various quantities possible [L]. It is believed that several of its interesting thermal properties could persist for regular lattices, for which the exact calculation is far intractable. Here we mention that in the 90’s a lot of research papers were devoted to the study of the classical Ising model, with two spin values $\pm 1$, on such a Cayley tree (see [Pr], [BG], [BRSSZ], [BRZ1], [BRZ2], [GR], [R]). In the present paper we study some generalization of this model, namely, we consider the Ising model with competing interactions, with spin values $\pm 1$, on a Cayley tree. Note that such models were studied extensively (see Refs. [MTA], [SC], [Mo1], [Mo2]) since the appearance of the Vannimenus model (see Ref. [V]), in which the physical motivations for the urgency to study such models was presented. In all of these works no exact solutions of the phase transition problem were found, but some solutions for specific parameter values were presented. On the other hand, while studying such models the appearance of nontrivial magnetic orderings was discovered [MR].

In [MR] we have exactly solved an Ising model on a Cayley tree, the Hamiltonian of which contained ternary interactions. In the present paper we continue investigations of the Ising model, but now we consider a model with the next-nearest-neighbor binary interactions. In the same way as [MR] we exactly solve a phase transition problem for the model, namely, we calculated critical curve such that there is a phase transitions above it, and a single Gibbs state is found elsewhere (cp. [GPW]). Using these facts we will find a form of the free energy of the model under consideration. This leads us to study some asymptotics ones.

2. Definitions and preliminary results

Recall that the Cayley tree $\Gamma^k$ of order $k \geq 1$ is an infinite tree, i.e., a graph without cycles, such that each vertex of which lies on $k + 1$ edges. Let $\Gamma^k = (V, \Lambda)$, where $V$ is the set of vertices of $\Gamma^k$, $\Lambda$ is the set of edges of $\Gamma^k$. The vertices $x$ and $y$ are called nearest neighbors, which are denoted by $l = < x, y >$ if there exists an edge connecting them. A collection of the pairs $< x, x_1>, ..., < x_{d-1}, y >$ is called a path from $x$ to $y$. Then the distance $d(x, y), x, y \in V$, on the Cayley tree, is the length of the shortest path from $x$ to $y$.

For the fixed $x^0 \in V$ we set

$$W_n = \{ x \in V|d(x, x^0) = n \}, \quad V_n = \cup_{m=1}^{n} W_m,$$

$$L_n = \{ l = < x, y > \in L| x, y \in V_n \}.$$  

Denote $|x| = d(x, x^0), x \in V$.

Denote

$$S(x) = \{ y \in W_{n+1} : d(x, y) = 1 \}, \quad x \in W_n.$$
The defined set is called the set of direct successors. Observe that any vertex $x \neq x^0$ has $k$ direct successors and $x^0$ has $k + 1$.

Two vertices $x, y \in V$ are called one level next-nearest-neighboring vertices if there is a vertex $z \in V$ such that $x, y \in S(z)$ and they are denoted by $> x, y <$. In this case the vertices $x, z, y$ are called ternary and denoted by $< x, z, y >$.

**Proposition 2.1.** [G] There exists a one-to-one correspondence between the set $V$ of vertices of the Cayley tree of order $k \geq 1$ and the group $G_k$ of the free products of $k + 1$ cyclic groups of the second order with generators $a_1, a_2, ..., a_{k+1}$.

Consider a left (resp. right) transformation shift on $G_k$ defined as: for $g_0 \in G_k$ we put

$$T_{g_0}h = g_0h \quad (\text{resp. } T_{g_0}h = hg_0) \quad \forall h \in G_k.$$ 

It is easy to see that the set of all left (resp. right) shifts on $G_k$ is isomorphic to the group $G_k$.

We consider the models where the spin takes values in the set $\sigma = \{-1, 1\}$. A configuration $\sigma$ on $V$ is then defined as a function $x \in V \rightarrow \sigma(x) \in \sigma$; the set of all configurations coincides with $\Omega = \Phi^V$. The Hamiltonian of the Ising model with competing interactions has the form

$$H(\sigma) = -J \sum_{x,y <} \sigma(x)\sigma(y) - J_1 \sum_{< x,y >} \sigma(x)\sigma(y)$$

(2.1)

where $J, J_1 \in \mathbb{R}$ are coupling constants and $\sigma \in \Omega$.

We consider a standard $\sigma$-algebra $\mathcal{F}$ of subsets of $\Omega$ generated by cylinder subsets, all probability measures are considered on $(\Omega, \mathcal{F})$. A probability measure $\mu$ is called a Gibbs measure (with Hamiltonian $H$) if it satisfies the DLR equation: $\forall n = 1, 2, ..., \text{and } \sigma_n \in \Phi^{V_n}$:

$$\mu\left(\{\sigma \in \Omega : \sigma|_{V_n} = \sigma_n\}\right) = \int_{\Omega} \mu(d\omega)\nu^{V_n}_{\omega|_{W_{n+1}}} (\sigma_n)$$

where $\nu^{V_n}_{\omega|_{W_{n+1}}}$ is the conditional probability

$$\nu^{V_n}_{\omega|_{W_{n+1}}} (\sigma_n) = Z^{-1}(\omega|_{W_{n+1}}) \exp(-\beta H(\sigma_n||\omega|_{W_{n+1}})).$$

where $\beta > 0$. Here $\sigma_n|_{V_n}$ and $\omega|_{W_{n+1}}$ denote the restriction of $\sigma, \omega \in \Omega$ to $V_n$ and $W_{n+1}$ respectively. Next, $\sigma_n : x \in V_n \rightarrow \sigma_n(x)$ is a configuration in $V_n$ and $H(\sigma_n||\omega|_{W_{n+1}})$ is defined as the sum $H(\sigma_n) + U(\sigma_n, \omega|_{W_{n+1}})$ where

$$H(\sigma_n) = -J \sum_{x,y <} \sigma_n(x)\sigma_n(y) - J_1 \sum_{< x,y >} \sigma_n(x)\sigma_n(y)$$

(2.2)

$$U(\sigma_n, \omega|_{W_{n+1}}) = -J \sum_{x,z <} \omega(z) - \sum_{< x,y >} \sigma_n(x)\omega(y)$$

Finally, $Z(\omega|_{W_{n+1}})$ stands for the partition function in $V_n$ with the boundary condition $\omega|_{W_{n+1}}$:

$$Z(\omega|_{W_{n+1}}) \equiv Z_\beta(\omega|_{W_{n+1}}) = \sum_{\sigma_n \in \Phi^{V_n}} \exp(-\beta H(\sigma_n||\omega|_{W_{n+1}})).$$

(2.3)
It is known (see Ref. [S]) that for any sequence \( \omega(n) \in \Omega \), any limiting point of the measures \( \nu_{\omega(n)}^{V_{n+1}} \) is a Gibbs measure. Here \( \nu_{\omega(n)}^{V_{n+1}} \) is a measure on \( \Omega \) such that \( \forall n' > n \):

\[
\nu_{\omega(n)}^{V_{n+1}} \left( \{ \sigma \in \Omega : \sigma_{|V_{n'}} = \sigma_{n'} \} \right) = \left\{ \begin{array}{ll}
\nu_{\omega(n)}^{V_{n+1}} (\sigma_{n'}|V_n), & \text{if } \sigma_{n'}|V_n = \omega(n)|V_n \setminus V_n \\
0, & \text{otherwise.}
\end{array} \right.
\]

The free energy is defined as

\[
F(\beta) = - \lim_{n \to \infty} \frac{1}{\beta|V_n|} \ln Z_\beta(\omega|W_{n+1}).
\]

Observe that \(|W_n| = 3 \cdot 2^{n-1}\) and \(|V_n| = 3 \cdot 2^n - 2\).

3. On Gibbs measures

In this section we give the construction of a special class of limiting Gibbs measures for the Ising model on a Cayley tree with competing interactions.

Let \( h : x \to \mathbb{R} \) be a real valued function of \( x \in V \). Given \( n = 1, 2, ... \) consider the probability measure \( \mu^{(n)} \) on \( \Phi^{V_n} \) defined by

\[
\mu^{(n)}(\sigma) = Z_n^{-1} \exp\{-\beta H(\sigma) + \sum_{x \in W_n} h_x \sigma(x)\},
\]

Here, as before, \( \beta = \frac{1}{T} \) and \( \sigma_n : x \in V_n \to \sigma_n(x) \) and \( Z_n \) is the corresponding partition function:

\[
Z_n = Z_n(\beta, h) = \sum_{\tilde{\sigma}_n} \exp\{-\beta \tilde{H}(\tilde{\sigma}) + \sum_{x \in W_n} h_x \tilde{\sigma}(x)\}.
\]

The consistency condition for \( \mu^{(n)}(\sigma_n), n \geq 1 \) is

\[
\sum_{\sigma_{n-1}} \mu^{(n)}(\sigma_{n-1}, \sigma_n) = \mu^{(n-1)}(\sigma_{n-1}),
\]

where \( \sigma_{n} = \{\sigma(x), x \in W_n\} \).

Let \( V_1 \subset V_2 \subset ... \cup_{n=1}^\infty V_n = V \) and \( \mu_1, \mu_2, ... \) be a sequence of probability measures on \( \Phi^{V_1}, \Phi^{V_2}, ... \) satisfying the consistency condition (3.2). Then, according to the Kolmogorov theorem, (see, e.g. Ref. [Sh]) there is a unique limit Gibbs measure \( \mu_h \) on \( \Omega \) such that for every \( n = 1, 2, ... \) and \( \sigma_n \in \Phi^{V_n} \) the following equality holds

\[
\mu \left( \{ \sigma|_{V_n} = \sigma_n \} \right) = \mu^{(n)}(\sigma_n).
\]

The following statement describes conditions on \( h_x \) guaranteeing the consistency condition of measures \( \mu^{(n)}(\sigma_n) \). For simplicity In the sequel we will consider the case \( k = 2 \).

**Theorem 3.1**. The measures \( \mu^{(n)}(\sigma_n), n = 1, 2, ... \) satisfy the consistency condition (3.2) if and only if for any \( x \in V \) the following equation holds:

\[
h_x = \frac{1}{2} \log \left( \frac{\theta^2 \theta_2 e^{2(h_y + h_z)} + \theta_1 (e^{2h_y} + e^{2h_z}) + \theta}{\theta e^{2(h_y + h_z)} + \theta_1 (e^{2h_y} + e^{2h_z}) + \theta_1^2 \theta} \right)
\]

where \( \theta = e^{2\beta J}, \ \theta_1 = e^{2\beta J_1} \) and \( < y, x, z > \) are ternary neighbors.

**Proof.** Necessity. According to the consistency condition (3.2) we have
\[
\frac{Z_{n-1}}{Z_n} \sum_{\sigma^{(n)}(x) \in W_{n-1}} \exp\{-\beta H_{n-1}(\sigma_{n-1}) + \beta J_1 \sum_{x \in W_{n-1}, y, z \in S(x)} \sigma(x)(\sigma(y) + \sigma(z)) + \beta J \sum_{x \in W_{n-1}, y, z \in S(x)} \sigma(y)\sigma(z) + \sum_{x \in W_{n-1}, y \in S(x)} h_y \sigma(y)} = \exp\{-\beta H_{n-1}(\sigma_{n-1}) + \sum_{x \in W_{n-1}} h_x \sigma(x)\}. \tag{3.5}
\]

Whence we get
\[
\frac{Z_{n-1}}{Z_n} \sum_{\sigma^{(n)}(x) \in W_{n-1}} \prod_{x \in W_{n-1}} \exp\{\beta J_1 \sigma(x)(\sigma(y) + \sigma(z)) + \beta J \sigma(y)\sigma(z) + h_y \sigma(y) + h_x \sigma(z)\} = \prod_{x \in W_{n-1}} \exp\{h_x \sigma(x)\}. \tag{3.6}
\]

Let \(x \in W_{n-1}\) and \(S(x) = \{y, z\}\), \(\sigma^{(n)}_x = \{\sigma(y), \sigma(z)\}\). Then it is easy to see that \(\sigma^{(n)} = \cup_{x \in W_{n-1}} \sigma^{(n)}_x\). Hence
\[
\frac{Z_{n-1}}{Z_n} \prod_{x \in W_{n-1}} \sum_{\sigma^{(n)}(x) \in W_{n-1}} \exp\{\beta J_1 \sigma(x)(\sigma(y) + \sigma(z)) + \beta J \sigma(y)\sigma(z) + h_y \sigma(y) + h_x \sigma(z)\} = \prod_{x \in W_{n-1}} \exp\{h_x \sigma(x)\}. \tag{3.7}
\]

Now fix \(x \in W_{n-1}\) and rewrite (3.7) for the cases \(\sigma(x) = 1\) and \(\sigma(x) = -1\) then we can find
\[
\frac{\sum_{\sigma^{(n)}_x = \{\sigma(y), \sigma(z)\}} \exp\{\beta J_1 (\sigma(y) + \sigma(z)) + \beta J \sigma(y)\sigma(z) + h_y \sigma(y) + h_x \sigma(z)\}}{\sum_{\sigma^{(n)}_x = \{\sigma(y), \sigma(z)\}} \exp\{-\beta J_1 (\sigma(y) + \sigma(z)) + \beta J \sigma(y)\sigma(z) + h_y \sigma(y) + h_x \sigma(z)\}} = \exp\{2h_x\}. \tag{3.8}
\]

Denote
\[
W_1 = \exp(2J_1 \beta + J \beta + h_y + h_z) + \exp(-J \beta - h_y + h_z) + \exp(-J \beta + h_y - h_z) + \exp(-2J_1 \beta + J \beta - h_y + h_z)
\]
\[
W_{-1} = \exp(-2J_1 \beta + J \beta + h_y + h_z) + \exp(-J \beta - h_y + h_z) + \exp(-J \beta + h_y - h_z) + \exp(2J_1 \beta + J \beta - h_y - h_z).
\]

It then follows from (3.8) that
\[
\exp\{2h_x\} = \frac{W_1}{W_{-1}}. \tag{3.9}
\]

The equality (3.9) implies (3.4).

**Sufficiency.** Now assume that (3.4) is valid, then it implies (3.9), and hence (3.8). From (3.8) we obtain
\[
\sum_{\sigma^{(n)}_x = \{\sigma(y), \sigma(z)\}} \exp\{\beta J_1 (\sigma(y) + \sigma(z))\sigma(x) + \beta J \sigma(y)\sigma(z) + h_y \sigma(y) + h_x \sigma(z)\} = a(x) \exp\{\sigma(x)h_x\}, \tag{3.10}
\]

where \(\sigma(x) = \pm 1\). This equality implies
\[
\prod_{x \in W_{n-1}} \sum_{\sigma^{(n)} = \{\sigma(y), \sigma(z)\}} \exp\{\beta J_1(\sigma(y) + \sigma(z))\sigma(x) + \beta J_2(\sigma(y) + \sigma(z)) + h_y \sigma(y) + h_x \sigma(z)\} = \prod_{x \in W_{n-1}} a(x) \exp\{\sigma(x)h_x\}. \tag{3.11}
\]

Writing \(A_n = \prod_{x \in W_n} a(x)\) we find from (3.11)
\[
Z_{n-1}A_{n-1} \mu^{(n-1)}(\sigma_{n-1}) = Z_n \sum_{\sigma^{(n)}} \mu_n(\sigma_{n-1}, \sigma^{(n)}).
\]
Since each \(\mu^{(n)}\), \(n \geq 1\) is a probability measure, we have
\[
\sum_{\sigma_{n-1}} \sum_{\sigma^{(n)}} \mu^{(n)}(\sigma_{n-1}, \sigma^{(n)}) = 1, \quad \sum_{\sigma_{n-1}} \mu^{(n-1)}(\sigma_{n-1}) = 1.
\]
Therefore from these equalities we find
\[
Z_{n-1}A_{n-1} = Z_n, \tag{3.12}
\]
which means that (3.2) holds. This completes the proof.

According to Theorem 3.1 the problem of describing of Gibbs measures is reduced to the description of solutions of the functional equation (3.4).

By Proposition 2.1 any transformation \(S\) of the group \(G_k\) induces a shift automorphism \(\tilde{S}\) by
\[
(\tilde{S} \sigma)(h) = \sigma(Sh), \quad h \in G_k, \quad \sigma \in \Omega.
\]

By \(G_k\) we denote the set of all shifts of \(\Omega\).

We say that a Gibbs measure \(\mu\) on \(\Omega\) is translation - invariant if for any \(T \in G_k\) the equality \(\mu(T(A)) = \mu(A)\) is valid for all \(A \in \mathcal{F}\).

The analysis of the solutions of (3.4) is rather tricky. It is natural to begin with the translation - invariant solutions where \(h_x = h\) is constant for all \(x \in V\). It is clear that a Gibbs measure corresponding to this solution is translation-invariant. This case has been investigated in Ref.[GPW].

In this case from (3.4) we infer
\[
u = \frac{\theta_1^2 \theta u^2 + 2 \theta_1 u + \theta}{\theta u^2 + 2 \theta_1 u + \theta_1^2 \theta} \tag{3.13}
\]
where \(u = e^{2h}\).

**Proposition 3.2 [GPW].** If \(\theta_1 > \sqrt{3}\) and \(\theta > \frac{2 \theta_1}{\theta_1^2 - 3}\) then for all pairs \((\theta, \theta_1)\) equation (3.13) has three positive solutions \(u_1^* < u_2^* < u_3^*, \) here \(u_2^* = 1\). Otherwise the eq. (3.13) has a unique solution \(u_4 = 1\).

**Remark.** The numbers \(u_1^*\) and \(u_3^*\) are the solutions of the following equation
\[
u^2 + (1 + \alpha)u + 1 = 0, \tag{3.14}
\]
here \(\alpha = \frac{2 \theta_1}{\theta} - \theta_1^2\). Hence \(u_1^* u_3^* = 1\) and if \(\beta \to \infty\) then \(u_3^* \to \infty\) and \(u_1^* \to 0\).
By $\mu_1, \mu_2, \mu_3$ we denote Gibbs measures corresponding to these solutions. Denote $u_x = \exp(2h_x), x \in V$. Then the functional equation (3.4) is rewritten as follows

$$u_x = \frac{\theta_1^2 \theta_y u_y + \theta_1 (u_y + u_z) + \theta}{\theta_y u_y + \theta_1 (u_y + u_z) + \theta_1^2 \theta} \tag{3.15}$$

here as before $<y,x,z>$ are ternary neighboring vertices.

**Proposition 3.3.** Let $\theta_1 > \sqrt{3}$, $\theta > \frac{2\theta_1}{\theta_1^2 - 3}$ and $u_x$ be a solution of equation (3.15). Then

$$u_1^* \leq u_x \leq u_3^* \text{ for any } x \in V.$$

The proof is similar to the proof of Proposition 4.3. of [MR].

From this proposition we infer the following (See [Ge])

**Theorem 3.4.** For the model (2.1) with parameters $J_1 > 0$ and $J \in \mathbb{R}$ on the Cayley tree $\Gamma^2$ the following assertions hold

(i) if $\theta_1 > \sqrt{3}, \theta > \frac{2\theta_1}{\theta_1^2 - 3}$ then the measures $\mu_1$ and $\mu_3$ are extreme;

(ii) in the opposite case there is a Gibbs measure $\mu_\star (= \mu_2)$ and it is extreme.

**Remark.** This theorem specifies the result obtained in [GPW] where they proved that a phase transition occurs if and only if the above indicated conditions were satisfied, and the extremity were open. The formulated theorem shows that the found Gibbs measures are extreme.

It is clear from the construction of the Gibbs measures that the measures $\mu_1$ and $\mu_3$ depend on parameter $\beta$. Now we are interested on the behavior of these measures when $\beta$ goes to $\infty$. Put

$$\sigma_+ = \{\sigma(x) : \sigma(x) = 1, \forall x \in \Gamma^2\},$$

$$\sigma_- = \{\sigma(x) : \sigma(x) = -1, \forall x \in \Gamma^2\}.$$

**Theorem 3.5.** Let $\theta_1 > \sqrt{3}$ and $\theta > \frac{2\theta_1}{\theta_1^2 - 3}$, then

$$\mu_1 \to \delta_{\sigma_-}, \quad \mu_3 \to \delta_{\sigma_+} \text{ as } \beta \to \infty,$$

here $\delta_\sigma$ is a delta-measure concentrated on $\sigma$.

**Proof.** The theorem follows from the fact that $u_1^* \to 0$ and $u_3^* \to \infty$ as $\beta \to \infty$.

From Theorem 3.5 it follows that ground states $\mu_\star, i = 1, 3$ are concentrated on $\sigma_-$ and $\sigma_+$ -spin configurations, respectively.

**4. A FORMULA OF THE FREE ENERGY**

Consider the partition function $Z_n(\beta, h)$ of the state $\mu_\beta^h$ (which corresponds to solution $h = \{h_x, x \in V\}$ of equation (3.4))

$$Z_n(\beta, h) = \sum_{\delta_n \in \Omega_n} \exp\{-\beta H(\delta_n) + \sum_{x \in W_n} h_x \delta(x)\}.$$
The free energy is defined as

$$F(\beta, h) = -\lim_{n \to \infty} \frac{1}{3 \beta \cdot 2^n} \ln Z_n(\beta, h).$$  \hfill (4.1)

The goal of this section is to prove the following

**Theorem 4.1.**

(i) The free energy exists for all $h$, and is given by the formula

$$F(\beta, h) = -\lim_{\beta \to \infty} \lim_{n \to \infty} \frac{1}{3 \cdot 2^n} \sum_{k=0}^{n} \sum_{x \in W_{n-k}} D(\beta, J_1, J h_y, h_z),$$  \hfill (4.2)

where $y = y(x), z = z(x)$ are direct successors of $x$;

$$D(\beta, J_1, h_y, h_z) = d(\beta, J_1 + h_z) + d(\beta, -J_1 + h_z) +$$

$$\Delta(\beta, h_y + f(-J_1 + h_z; \theta); h_y + f(J_1 + h_z; \theta));$$

$$d(\beta, x) = \frac{1}{4} \ln [4 \cosh(x - \beta) \cosh(x + \beta)];$$

$$\Delta(\beta, x, y) = \frac{1}{2} \ln [4 \cosh(x - \beta) \cosh(y + \beta)];$$

$$f(x, \theta) = \tanh^{-1}(\theta \tanh x), \theta = \tanh(J \beta).$$  \hfill (4.3)

(ii) For any solution $h = \{h_x, x \in V\}$ of (3.4)

$$F(\beta, h) = F(\beta, -h),$$

where $-h = \{-h_x, x \in V\}$.

**Proof.** (i) We shall use the recursive equation (3.12):

$$Z_n = A_n Z_{n-1},$$

where $A_n = \prod_{x \in W_n} a(x)$ and $a(x) = a(x, J_1, J, \beta), x \in V$ is some function, which we will define below. Using (3.10) we have

$$a(x) = 4 \sqrt{\cosh(J_1 \beta - J \beta + h_z) \cosh(J_1 \beta + J \beta + h_z)} \times$$

$$\sqrt{\cosh(-J_1 \beta - J \beta + h_z) \cosh(-J_1 \beta + J \beta + h_z)} \times$$

$$\sqrt{\cosh(-J_1 \beta + h_y + f(-J_1 + h_z; \theta)) \cosh(J_1 \beta + h_y + f(J_1 + h_z; \theta)).}$$

Thus, the recursive equation (3.12) has the following form

$$Z_n(\beta, h) = \exp \left( \sum_{x \in W_{n-1}} D(\beta, J_1, J h_y, h_z) \right) Z_{n-1}(\beta, h).$$  \hfill (4.4)

This gives (4.2). Now we prove the existence of the RHS limit of (4.2). By proposition 3.4 we have $h_x \in [\frac{1}{2} \ln u; -\frac{1}{2} \ln u]$ consequently functions from (4.3) and so $D$ is bounded i.e. $|D(\beta, J h_y, h_z)| \leq C_\beta$ for all $h_y, h_z$. Hence we get

$$\frac{1}{3 \cdot 2^n} \sum_{k=l+1}^{n} \sum_{x \in W_{n-k}} D(\beta, J_1, J h_y, h_z) \leq$$

$$\frac{C_\beta}{2^n} \sum_{k=l+1}^{n} 2^{n-k-1} \leq C_\beta \cdot 2^{-l}.$$  \hfill (4.5)
Therefore, from (4.6) we get the existence of the limit at RHS of (4.2).

(ii) Now, we shall prove that

\[ F(\beta, -h) = F(\beta, h). \]  \tag{4.6}

It is easy to see that if \( h = \{h_x, x \in V\} \) is a solution of equation (3.4) then \( -h = \{-h_x, x \in V\} \) is so. The equality (4.6) follows from the following equality

\[ D(J_1, J, -h_y, -h_z) = D(J_1, J, h_y, h_z), \]

which is a consequence of the following properties

1) \( d_\beta(-x) = d_\beta(x); \)

2) \( f(-x; \theta) = -f(x; \theta); \)

3) \( \Delta_\beta(-x, -y) = \Delta_\beta(y, x). \)

The theorem is proved.

The rest of the section is devoted to the study of asymptotical properties of the free energy \( F(\beta, h) \) as \( \beta \to \infty \) for any constant \( h \), i.e. \( h_x = \text{const}, x \in V \). In this setting \( F(\beta, h) \) has the form:

\[ F(\beta, h) = D(J_1, J, h, h). \]  \tag{4.7}

Denote

\[ F(\infty) = \lim_{\beta \to \infty} F(\beta, h). \]  \tag{4.8}

By Proposition 3.2, we know that equation (3.4) has exactly three constant (translation - invariant) solutions: \( h = h(\beta) = \{h_x = \frac{1}{12} \ln u_1^*, x \in V\}, \{h_x = 0, x \in V\} \) and \( \{h_x = \frac{1}{2} \ln u_3^*, x \in V\} \). Using (3.14) one can find that \( h \) has the following asymptotic

\[ h(\beta) = M \beta + o(\beta^{-N}), \quad N \geq 2, \quad \text{as} \quad \beta \to \infty \]  \tag{4.9}

where \( M = \frac{1}{2} \max \{2(J_1 - J); 3J_1 - J; 4J_1; J_1 - J, 0\} \). Using (4.9) and

\[ \ln \left( 2 \cosh(\alpha \beta + o(\beta^{-N})) \right) = \beta(|\alpha| + o(\beta^{-N})), \]

where \( N \geq 2 \) and \( \alpha \in \mathbb{R} \), it is easy to see that

\[ d_{J_1}(J_1 \beta \pm h) = \frac{\beta}{4} \sum_{\varepsilon = \pm 1} |J_1 + \varepsilon J \pm M| + o(\beta^{-N}), \]

\[ f(\pm J_1 \beta + M \beta + o(\beta^{-N}); \theta) = \frac{\beta}{2} (| \pm J_1 - J + M| - | \pm J_1 + J + M|) + o(\beta^{-N}). \]

Consequently,

\[ F(\beta, h) = F(\infty) + o(\beta^{-N}), \]  \tag{4.10}

where

\[ F(\infty) = \sum_{\delta = \pm 1} \left( \frac{1}{2} |\delta J_1 + M - \frac{1}{2} \sum_{\varepsilon = \pm 1} \varepsilon [\delta J_1 + \beta J + M| \varepsilon | + \frac{1}{4} \sum_{\varepsilon = \pm 1} |J_1 + \varepsilon J + M| \right). \]
5. Discussion of results

It is known that much attention has been paid to exact calculations in statistical mechanics many of researchers, because such investigations are important not only for their own interest but also to deeper understand of the critical properties of spin systems which are not obtained from approximations. Therefore, those are very useful for testing the credibility and efficiency of any new method or approximation before it is applied to more complicated spin systems. In this paper we have exactly solved an Ising model on a Cayley tree, the Hamiltonian of which contains the nearest-neighbor and competing interactions, namely, we calculated critical curve such that there is a phase transition above it, and a single Gibbs state is found elsewhere. The obtained facts give us to explicitly express the free energy of the model under consideration, associated with the translation invariant Gibbs measures.

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