ON SOLUTION OF LAMÉ EQUATIONS IN AXISYMMETRIC DOMAINS WITH CONICAL POINTS

Boniface Nkemzi\textsuperscript{1}

\textit{Department of Mathematics, Faculty of Science, University of Buea, Cameroon}

\textit{and}

\textit{The Abdus Salam International Centre for Theoretical Physics, Trieste, Italy.}

\textbf{Abstract}

Partial Fourier series expansion is applied to the Dirichlet problem for the Lamé equations in axisymmetric domains $\Omega \subseteq \mathbb{R}^3$ with conical points on the rotation axis. This leads to dimension reduction of the three-dimensional boundary value problem resulting to an infinite sequence of two-dimensional boundary value problems on the plane meridian domain $\Omega_a \subseteq \mathbb{R}_+^2$ of $\Omega$ with solutions $u_n (n = 0, 1, 2, \cdots)$ being the Fourier coefficients of the solution $\hat{u}$ of the 3D BVP. The asymptotic behavior of the Fourier coefficients $u_n (n = 0, 1, 2, \cdots)$ near the angular points of the meridian domain $\Omega_a$ is fully described by singular vector-functions which are related to the zeros $\alpha_n$ of some transcendental equations involving Legendre functions of the first kind. Equations which determine the values of $\alpha_n$ are given and a numerical algorithm for the computation of $\alpha_n$ is proposed with some plots of values obtained. The singular vector functions for the solution of the 3D BVP is obtained by Fourier synthesis.

MIRAMARE – TRIESTE

October 2003

\textsuperscript{1}Junior Associate of the Abdus Salam ICTP. nkemzi@yahoo.com
1 Introduction

Many problems in physics and engineering belong to the class of boundary value problems for the elliptic partial differential equations with solutions having singularities due to boundary irregularities. The construction of suitable finite element (boundary element) methods for treating BVPs with singularities (e.g. adaptive mesh refinement, singular function augmentation, etc.) requires a priori knowledge of the form of the singular functions. Moreover, some physical useful parameters such as stress intensity factors can be computed if the singular forms are known. Thus, the need to analyze and characterize singular forms and regularity results for BVPs is crucial and much work has been done in this area (cf. [16]-[21], [4, 5, 8, 11, 12, 15, 22], [23]-[26]).

In this paper we consider the homogeneous Dirichlet problem for the Lamé system in axisymmetric domains $\hat{\Omega} \subset \mathbb{R}^3$ with conical points on the rotation axis, i.e.

$$
-\mu \Delta \hat{u}(x) - (\lambda + \mu) \text{grad} \, \text{div} \, \hat{u}(x) = \hat{f}(x) \quad \text{for} \quad x \in \hat{\Omega},
$$
(1.1)

$$
\hat{u}(x) = 0 \quad \text{for} \quad x \in \hat{\Gamma} := \partial \hat{\Omega},
$$
(1.2)

where $\hat{u}$ denotes the displacement vector field, $\hat{f} \in (L_2(\hat{\Omega}))^3$ is the vector of the volume forces and $\mu > 0, \lambda > 0$ are the Lamé coefficients. Analysis of solutions of BVPs for the Lamé equations in domains with conical points and re-entrant vertices has been quite intensive (cf. [4, 11, 15, 16, 22, 23, 24, 25, 26]). More precisely, suppose for convenience that the domain $\hat{\Omega} \subset \mathbb{R}^3$ has only one conical point which coincides with the origin $O$, then according to the general theory (cf. [4, 15, 16, 17, 23, 25, 26]), in the vicinity of the conical point $O$ the solution $\hat{u}$ of (1.1), (1.2) behaves like a linear combination of terms of the form

$$
|x|^\alpha \sum_{s=0}^k (\ln |x|)^s u^{k-s}(x/|x|).
$$
(1.3)

The exponents $\alpha$ (in general complex) are eigenvalues of a certain operator pencil which arises from the Mellin transformation of the principal parts of the differential and boundary operators in (1.1), (1.2) on the tangent cone, and the functions $u^s$ ($0 \leq s \leq k$) are the generalized eigenvectors corresponding to $\alpha$.

In view of the fact that boundary value problems in axisymmetric domains $\hat{\Omega} \subset \mathbb{R}^3$ with nonaxisymmetric data, are treated numerically, preferably by the Fourier-finite-element method (FFEM) (cf. [2, 6, 13, 28, 30, 33]), the characterization of singular forms and regularity results for the Fourier coefficients of the solution of the 3D BVP is crucial for an efficient application of this method. In this paper we give a precise description of the singular vector-functions that describe the asymptotic behavior of the Fourier coefficients of the solution of the 3D BVP near the angular points of the plane meridian domain $\Omega_a$. The singular vector-functions depend on a parameter $\alpha$ which is related to the roots of some transcendental equations involving Legendre functions. We present equations which determine the values of $\alpha$ and give a numerical procedure for computing $\alpha$. Some graphs of the computed values of $\alpha$ are presented. The analysis shows
that Fourier coefficients $u_n$ for $n \geq 2$ of the solution $\hat{u}$ of (1.1), (1.2) have the required regularity for any conical points, whenever the right-hand side $\hat{f}$ belong to $(L_2(\hat{\Omega}))^3$. The singular forms for the solution of the 3D BVP is obtained by means of Fourier synthesis.

2 Analytical preliminaries

Let $\hat{\Omega} \subset \mathbb{R}^3$ be an open bounded and simply connected set with Lipschitz boundary $\hat{\Gamma}$ and let $(x_1, x_2, x_3)$ denote the Cartesian co-ordinates of the point $x \in \mathbb{R}^3$. Suppose that $\hat{\Omega}$ is axisymmetric with respect to the $x_3$-axis and that the set $\hat{\Omega} \setminus \Gamma_0$ ($\Gamma_0$ is part of the $x_3$-axis contained in $\hat{\Omega}$) is obtain by rotation of a bounded plane meridian domain $\Omega_a$ about the $x_3$-axis. Let $\Gamma_a := \partial \Omega_a \setminus \Gamma_0$ and suppose that $\Gamma_a \in C^2$ and that $\Gamma_a$ is straight line in some neighbourhood of the points of intersection of $\tilde{\Gamma}_a$ and $\tilde{\Gamma}_0$ (see Figure 2.1). With this assumption we exclude axisymmetric edges on the boundary of $\hat{\Omega}$, this case has been handled in [32]. We assume for convenience that the axisymmetric domain $\hat{\Omega}$ has only one conical point on its boundary which coincides with the origin $O$ and let the interior opening angle be equal $\theta_c$.

Subsequently we employ cylindrical co-ordinates $r, \varphi, z$ ($\varphi \in (-\pi, \pi]$) which are related to the Cartesian co-ordinates by $x_1 = r \cos \varphi$, $x_2 = r \sin \varphi$, $x_3 = z$. Thus the sets $\hat{\Omega} \setminus \Gamma_0$ and $\hat{\Gamma} \setminus \Gamma_0$ are mapped into the sets $\Omega := \Omega_a \times (-\pi, \pi]$ and $\Gamma := \Gamma_a \times (-\pi, \pi]$, respectively, in cylindrical co-ordinates. For any vector function $\hat{u}(x) = (\hat{u}_1(x), \hat{u}_2(x), \hat{u}_3(x))^T$, $x \in \hat{\Omega} \setminus \Gamma_0$, some vector function $u = (u_r(r, \varphi, z), u_\varphi(r, \varphi, z), u_z(r, \varphi, z))^T$, $(r, \varphi, z) \in \Omega$ is uniquely defined by

$$u_r := \hat{u}_1 \cos \varphi + \hat{u}_2 \sin \varphi, \quad u_\varphi := -\hat{u}_1 \sin \varphi + \hat{u}_2 \cos \varphi, \quad u_z := \hat{u}_3.$$

(2.1)

Figure 2.1

Let us introduce with respect to the origin $O$ local polar co-ordinates $R, \theta$ viz. $r = R \sin \theta$ $z = -R \cos \theta$ and define a circular sector $G_a$ in $\Omega_a$ with radius $R_c$ and angle $\theta_c$ by (see Figure 2.1)

$$G_a := \{(r, z) \in \Omega_a : 0 \leq R \leq R_c, 0 \leq \theta \leq \theta_c\}, \quad G_a := G_a \setminus \partial G_a,$$

(2.2)
where $\partial G_0$ denotes the boundary of $G_0$. Let $\hat{G}$ denote the domain generated by rotation of $G_0$ about the $x_3$-axis and $\partial \hat{G}$ its boundary. Then the images of the sets $\hat{G} \setminus \Gamma_0$ and $\partial \hat{G} \setminus \Gamma_0$ in cylindrical co-ordinates are $G := G_0 \times (-\pi, \pi)$ and $\partial G := \partial_0 G_0 \times (-\pi, \pi)$, respectively, where $\partial_0 G_0 := \partial G_0 \setminus \Gamma_0$.

For the analysis of the behaviour of the solution $\hat{u}$ of (1.1), (1.2) and its Fourier coefficients $u_n, n \in \mathbb{N}_0 := \{0, 1, 2, \cdots\}$, near the conical point $O$, we consider in cylindrical co-ordinates the homogeneous Lamé system in the neighbourhood $G$ of $O$ with homogeneous boundary conditions.

$$
\begin{align*}
\Delta_{r\varphi z} u_r - \frac{1}{r^2} u_r - \frac{2}{r^2} \frac{\partial u_{\varphi r}}{\partial r} + \frac{1}{1 - 2\nu} \frac{\partial e}{\partial r} &= 0 \quad \text{in} \ G, \\
\Delta_{r\varphi z} u_{\varphi r} - \frac{1}{r^2} u_{\varphi r} + \frac{2}{r^2} \frac{\partial u_r}{\partial \varphi} + \frac{1}{1 - 2\nu} \frac{\partial e}{\partial \varphi} &= 0 \quad \text{in} \ G, \\
\Delta_{r\varphi z} u_z + \frac{1}{1 - 2\nu} \frac{\partial e}{\partial z} &= 0 \quad \text{in} \ G, \\
\partial G
\end{align*}
$$

Here

$$
\Delta_{r\varphi z} := \frac{\partial^2}{\partial r^2} + \frac{1}{r} \frac{\partial}{\partial r} + \frac{1}{r^2} \frac{\partial^2}{\partial \varphi^2} + \frac{\partial^2}{\partial z^2}, \quad e := \frac{\partial u_r}{\partial r} + \frac{1}{r} u_r + \frac{1}{r} \frac{\partial u_{\varphi r}}{\partial \varphi} + \frac{\partial u_z}{\partial z}
$$

and the Poisson’s ratio $0 < \nu < 0.5$ is related to the Lamé coefficients by $\nu = \frac{\lambda}{2(\lambda + \mu)}$.

Using the orthogonal and complete system $\{1, \sin \varphi, \cos \varphi, \cdots, \sin n\varphi, \cos n\varphi, \cdots\}$ in $L_2(-\pi, \pi)$ we can represent the solution $u$ of (2.3) in partial Fourier series in the form (cf. [2, 31]):

$$
\begin{align*}
u_r(r, \varphi, z) &= \sum_{n=0}^{\infty} (u_{rn}^s(r, z) \cos n\varphi + u_{rn}^a(r, z) \sin n\varphi), \\
u_{\varphi}(r, \varphi, z) &= \sum_{n=0}^{\infty} (u_{\varphi n}^s(r, z)(- \sin n\varphi) + u_{\varphi n}^a(r, z) \cos n\varphi), \\
u_z(r, \varphi, z) &= \sum_{n=0}^{\infty} (u_{zn}^s(r, z) \cos n\varphi + u_{zn}^a(r, z) \sin n\varphi),
\end{align*}
$$

where the Fourier coefficients are defined the usual way (cf. [31]). Substituting (2.4) in (2.3) we obtain in $G_0$ the following sequence of decoupled two-dimensional boundary value problems (we omit the superscript $s$ and $a$).

$$
\begin{align*}
\Delta_{rz} u_{rn} - \frac{n^2 + 1}{r^2} u_{rn} + \frac{2n}{r^2} u_{\varphi n} + \frac{1}{1 - 2\nu} \frac{\partial e_{rn}}{\partial r} &= 0 \quad \text{in} \ G_0, \\
\Delta_{rz} u_{\varphi n} - \frac{n^2 + 1}{r^2} u_{\varphi n} + \frac{2n}{r^2} u_{rn} + \frac{n}{1 - 2\nu} e_{rn} &= 0 \quad \text{in} \ G_0, \\
\Delta_{rz} u_{zn} - \frac{n^2}{r^2} u_{zn} + \frac{1}{1 - 2\nu} \frac{\partial e_{rn}}{\partial z} &= 0 \quad \text{in} \ G_0, \\
u_{rn} = u_{\varphi n} = u_{zn} &= 0 \quad \text{on} \ \partial_0 G_0,
\end{align*}
$$

where

$$
\begin{align*}
\Delta_{rz} := \frac{\partial^2}{\partial r^2} + \frac{1}{r} \frac{\partial}{\partial r} + \frac{\partial^2}{\partial z^2}, \quad e_{rn} := \frac{\partial u_{rn}}{\partial r} + \frac{1}{r} u_{rn} + \frac{\partial u_{zn}}{\partial z} - \frac{n}{r} u_{\varphi n}.
\end{align*}
$$
For the analysis of the generalized solutions \( u_n, n \in \mathbb{N}_0, \) of (2.5) we introduce the following spaces (see also [28, 31]):

\[
\begin{align*}
L_2(G_a) & := \{ w = w(r, z) : \int_{G_a} |w|^2 r dr dz < \infty \}, \\
L_{2,1/2}(G_a) & := \{ w = w(r, z) : r^{1/2} w \in L_2(G_a) \}, \\
W^{1,2}_{1/2}(G_a) & := \{ w \in L_{2,1/2}(G_a) : \frac{\partial w}{\partial r}, \frac{\partial w}{\partial z} \in L_{2,1/2}(G_a) \}, \\
W^{2,2}_{1/2}(G_a) & := \{ w \in W^{1,2}_{1/2}(G_a) : \frac{\partial^2 w}{\partial r^2}, \frac{\partial^2 w}{\partial z^2}, \frac{\partial^2 w}{\partial r \partial z} \in X(G_a) \}.
\end{align*}
\]

These spaces are endowed with the norms:

\[
\begin{align*}
\|w\|_{L_{2,1/2}(G_a)} & := \left( \int_{G_a} |w|^2 r dr dz \right)^{1/2}, \\
\|w\|_{W^{1,2}_{1/2}(G_a)} & := \left\{ \|w\|_{L_{2,1/2}(G_a)}^2 + \left\| \frac{\partial w}{\partial r} \right\|_{L_{2,1/2}(G_a)}^2 + \left\| \frac{\partial w}{\partial z} \right\|_{L_{2,1/2}(G_a)}^2 \right\}^{1/2}, \\
\|w\|_{W^{2,2}_{1/2}(G_a)} & := \left\{ \left( \frac{\partial^2 w}{\partial r^2} \right)_{L_{2,1/2}(G_a)}^2 + \left( \frac{\partial^2 w}{\partial z^2} \right)_{L_{2,1/2}(G_a)}^2 + \left( \frac{\partial^2 w}{\partial r \partial z} \right)_{L_{2,1/2}(G_a)}^2 \right\}^{1/2}, \\
\|w\|_{W^{2,2}_{1/2}(G_a)} & := \left\{ \|w\|_{W^{1,2}_{1/2}(G_a)}^2 + \|w\|_{W^{1,2}_{1/2}(G_a)}^2 \right\}^{1/2}.
\end{align*}
\]

**Remark 2.1:** We note here that if the function \( \hat{u} \) belong to the Sobolev space \( (W^2_2(\hat{\Omega}))^3 \), then its Fourier coefficients \( u_n, n \in \mathbb{N}_0 \) belong naturally in the space \( (W^{2,2}_{1/2}(\Omega_a))^3 \) (cf. [30, 32]).

Our main concern in the next sections is to determine the maximum value of the angle \( \theta_c \) at the conical point which is permissible if the Fourier coefficients \( u_n, n \in \mathbb{N}_0 \) of the solution \( \hat{u} \) of the BVP (1.1), (1.2) to belong to \( (W^{2,2}_{1/2}(\Omega_a))^3 \). We also determine the singular forms for the Fourier coefficients.

### 3 The singularity functions

We are interested on the solutions of (2.5) which have the form

\[
u_n = R^{\alpha_n} U_n(\alpha_n, \theta),
\]

where \( R, \theta \) are the local polar co-ordinates (cf. (2.2)). To determine expressions involving the parameter \( \alpha_n \), we use the Boussinesq-Papkovich-Neuber representation of the general solution of homogeneous Lamé equations in three-dimensional domains in terms of three harmonic functions \( \Phi, \Psi, \Lambda \) (cf. [3, 4, 16]). Thus in \( \hat{G} \) the solution \( \hat{u} \) of the homogeneous Lamé system in Cartesian co-ordinates has the form

\[
\hat{u} = a \nabla \Psi + 2b \nabla \times (\Phi e_3) + c(\nabla (x_3 \Lambda) - 4(1 - \nu)\Lambda e_3),
\]

where \( \nabla \) denotes the gradient operator in Cartesian co-ordinates, \( e_3 \) is the unit vector in \( x_3 \)-direction and \( a, b, c \) are arbitrary constants. We now seek to find the specific harmonic functions
which have the required property. In cylindrical co-ordinates (3.2) has the form

\[
\begin{pmatrix}
u_r \\
u_\varphi \\
u_z
\end{pmatrix}
= a
\begin{pmatrix}
\frac{\partial \psi_n}{\partial r} \\
\frac{1}{r} \frac{\partial \psi_n}{\partial \varphi} \\
\frac{\partial \psi_n}{\partial z}
\end{pmatrix}
+ 2b
\begin{pmatrix}
-\frac{1}{r} \frac{\partial \phi_n}{\partial r} \\
-\frac{\partial \phi_n}{\partial \varphi} \\
0
\end{pmatrix}
+ c
\begin{pmatrix}
z \frac{\partial \Lambda_n}{\partial r} \\
z \frac{1}{r} \frac{\partial \Lambda_n}{\partial \varphi} \\
z \frac{\partial \Lambda_n}{\partial z} - (3 - 4\nu)\Lambda_n
\end{pmatrix}.
\]

We expand the functions \(\Psi, \Phi, \Lambda\) in partial Fourier series with respect to the co-ordinate variable \(\varphi\) in the form

\[
\Psi = \sum_{n=0}^{\infty} (\Psi_n^r(r, z) \cos n\varphi + \Psi_n^z(r, z) \sin n\varphi).
\]

By comparing coefficients with the representation of \(u\) according to (2.4) we obtain the following relation for the solutions \(u_n\) \((n \in \mathbb{N}_0)\) of (2.5)

\[
\begin{pmatrix}
u_{rn} \\
u_{zn}
\end{pmatrix}
= a
\begin{pmatrix}
\frac{n}{r} \Psi_n \\
\frac{\partial \psi_n}{\partial z}
\end{pmatrix}
+ 2b
\begin{pmatrix}
-\frac{n}{r} \phi_n \\
0
\end{pmatrix}
+ c
\begin{pmatrix}
z \frac{\partial \Lambda_n}{\partial r} \\
z \frac{n}{r} \Lambda_n
\end{pmatrix}.
\]

We introduce for simplicity the \(3 \times 3\) matrix \(B(R, \theta)\) with columns

\[
\begin{pmatrix}
\sin \theta \frac{\partial \psi_n}{\partial R} + \frac{\cos \theta \partial \psi_n}{R \partial \theta} \\
\frac{n}{R \sin \theta} \Psi_n \\
-\cos \theta \frac{\partial \psi_n}{\partial R} + \frac{\sin \theta \partial \psi_n}{R \partial \theta}
\end{pmatrix};
\begin{pmatrix}
-\frac{2n}{R \sin \theta} \phi_n \\
-2 \sin \theta \frac{\partial \phi_n}{\partial R} - \frac{2 \cos \theta \partial \phi_n}{R \partial \theta} \\
0
\end{pmatrix};
\begin{pmatrix}
-R \cos \theta (\sin \theta \frac{\partial \Lambda_n}{\partial R} + \frac{\cos \theta \partial \Lambda_n}{R \partial \theta}) \\
-n \cot \theta \Lambda_n \\
-R \cos \theta (-\cos \theta \frac{\partial \Lambda_n}{\partial R} + \frac{\sin \theta \partial \Lambda_n}{R \partial \theta}) - (3 - 4\nu)\Lambda_n
\end{pmatrix}
\]

Relation (3.4) expressed in terms of local polar co-ordinates \(R, \theta\) has the form

\[
\begin{pmatrix}
u_{rn} \\
u_{zn}
\end{pmatrix}
= B(R, \theta)
\begin{pmatrix}
a \\
b \\
c
\end{pmatrix}.
\]

Now, let \(T(r, \varphi, z), (r, \varphi, z) \in G\) be a harmonic function in \(G\), i.e. \(\triangle_{r, \varphi, z} T = 0\) in \(G\). Suppose the Fourier coefficients \(T_n(r, z), (r, z) \in G, \ n \in \mathbb{N}_0\), of \(T\) have the form \(R^n \chi_n^0(\theta)\) with respect to the local co-ordinates \(R, \theta\), then the functions \(\chi_n^0(\theta)\) must satisfy the homogeneous differential equation

\[
\frac{d^2 \chi_n^0}{d\theta^2} + \cot \theta \frac{d \chi_n^0}{d\theta} + (\alpha_n(\alpha_n + 1) - \frac{n^2}{\sin^2 \theta}) \chi_n^0 = 0.
\]

Equation (3.6) is the associated Legendre equation and the general solution is given by (cf. [1, 9])

\[
\chi_n^0 = c_1 P_{\alpha_n}^{-n}(\cos \theta) + c_2 Q_{\alpha_n}^{n}(\cos \theta),
\]

\[\text{(3.7)}\]
where $P_{\alpha n}^{-n}(\cos \theta)$ and $Q_{\alpha n}^{-n}(\cos \theta)$ are the associated Legendre functions of the first and second kind, respectively. But, since the functions $T_n$ are expected to be bounded, we must choose $c_2 = 0$ in (3.7) as $Q_{\alpha n}^{-n}(\cos \theta)$ is unbounded at $\cos \theta = \pm 1$. Thus

$$T_n(r, z) = R^{\alpha n} P_{\alpha n}^{-n}(\cos \theta).$$

(3.8)

This suggests that we must choose the functions $\Psi, \Phi, \Lambda$ in (3.2) such that their Fourier coefficients on $G_n$ have the form (3.8) and such that the solutions $u_n (n \in \mathbb{N}_0)$ of (2.5) having the form (3.1) satisfy (3.5). Thus

$$\Psi_n(r, z) = R^{\alpha n+1} P_{\alpha n+1}^{-n}(\cos \theta),$$

$$\Phi_n(r, z) = R^{\alpha n+1} P_{\alpha n+1}^{-n}(\cos \theta),$$

$$\Lambda_n(r, z) = R^{\alpha n} P_{\alpha n}^{-n}(\cos \theta).$$

(3.9)

We introduce for simplicity the $3 \times 3$ matrix $M_n(\alpha_n, \theta)$ with columns

$$\begin{pmatrix}
(\alpha_n + 1) \sin^{-1} \theta P_{\alpha n+1}^{-n} - (\alpha_n + 1 - n) \cot \theta P_{\alpha n}^{-n} \\
2 n \sin^{-1} \theta P_{\alpha n+1}^{-n} \\
-(\alpha_n + 1 - n) P_{\alpha n}^{-n}
\end{pmatrix};
$$

$$\begin{pmatrix}
-2 n \sin^{-1} \theta P_{\alpha n+1}^{-n} \\
-3(\alpha_n + 1) \sin^{-1} \theta P_{\alpha n+1}^{-n} + 2(\alpha_n + 1 - n) \cot \theta P_{\alpha n}^{-n} \\
0
\end{pmatrix};
$$

$$\begin{pmatrix}
-2(\alpha_n + 1) \cos^2 \theta - (3 - 4\nu) P_{\alpha n}^{-n} - (\alpha_n + n + 1) \cos \theta \cot \theta P_{\alpha n+1}^{-n} \\
-n \cot \theta P_{\alpha n}^{-n} \\
((2\alpha_n + 1) \cos^2 \theta - (3 - 4\nu)) P_{\alpha n}^{-n} - (\alpha_n + n + 1) \cos \theta P_{\alpha n+1}^{-n}
\end{pmatrix}.$$

Substitution of (3.9) in (3.5) gives the proof of the following lemma.

**Lemma 3.1:** Let $\{\alpha_n\}_{n \in \mathbb{N}_0}$ be a sequence of real numbers. Then the Fourier coefficients given by (3.9) define uniquely harmonic functions $\Psi, \Phi$ and $\Lambda$ in $G$ and the functions

$$u_n = R^{\alpha n} U_n(\alpha_n, \theta) \quad \text{with} \quad U_n(\alpha_n, \theta) = M_n(\alpha_n, \theta) \begin{pmatrix} a \\ b \\ c \end{pmatrix}. \quad (3.10)$$

are solutions of the differential equations (2.5) for any constant vector $(a, b, c)^T$.

Now we determine a non-trivial vector $(a, b, c)^T$ in (3.10) such that the homogeneous boundary condition at $\theta = \theta_c$ is satisfied, that is

$$M_n(\alpha_n, \theta_c) \begin{pmatrix} a \\ b \\ c \end{pmatrix} = 0. \quad (3.11)$$

Equation (3.11) has a non-trivial solution only if

$$\det M_n(\alpha_n, \theta_c) = 0. \quad (3.12)$$
The solutions $\alpha_n$ of the transcendental equation (3.12) are the eigenvalues of the operator pencil in $G_a$ that determines the asymptotic behaviour of the coefficients $u_a^{s/a}$ ($n \in \mathbb{N}_0$) near the vertex $O$. The vector functions $U_n(\alpha_n, \theta)$ from (3.10), where the vector $(a, b, c)^T$ is the nontrivial solution of the algebraic equation (3.11), are the corresponding eigenvector functions to $\alpha_n$. For $\theta_c \in (0, \pi)$ given, we have
\[
\det M_n = 2(n^2(P_{\alpha_n+1}^{-n})^2 - ((\alpha_n + 1)P_{\alpha_n+1}^{-n} - (\alpha_n + n + 1)P_{\alpha_n}^{-n} \cos \theta_c)^2)
\]
\[
- (\alpha_n + n + 1)P_{\alpha_n+1}^{-n} \cos \theta_c + P_{\alpha_n}^{-n}(-3 + 4\nu + (2\alpha_n + 1) \cos^2 \theta_c) \sin^{-2} \theta_c
\]
\[
+ (\alpha_n + n - 1)P_{\alpha_n}^{-n} \cot \theta_c(2n^2P_{\alpha_n}^{-n}P_{\alpha_n+1}^{-n} \sin^{-1} \theta_c - (\alpha_n + n + 1)P_{\alpha_n+1}^{-n} \cos \theta_c)
\]
\[
+ \frac{1}{2}P_{\alpha_n}^{-n}(1 + (2\alpha_n + 1) \cos 2\theta_c))(2(\alpha_n - n + 1)P_{\alpha_n}^{-n} \cot \theta_c - 2(\alpha_n + 1)P_{\alpha_n+1}^{-n} \sin^{-1} \theta_c)) = 0.
\]
For each $n \in \mathbb{N}_0$ and $\theta_c \in (0, \pi)$, the roots of equation (3.13) consist of an infinite set $\{\alpha_{n_i}\}_{i \in \mathbb{Z}}$ of isolated real numbers with no accumulation point. At each point $(\alpha_{n_i}, \theta_c)$ where equation (3.13) is satisfied, we seek for a nontrivial vector $(a, b, c)^T$ that satisfies (3.11), and verify via (3.10) if $U_n(\alpha_{n_i}, \theta)$ is a nontrivial eigenvector function. We notice, for example, that for any $\theta_c \in (0, \pi)$ and $n \in \mathbb{N}_0$, the number $\alpha_n = n - 1$ satisfies equation (3.13), but for this eigenvalue there is no corresponding nontrivial eigenvector function. Also, for $\alpha_n = 0$, equation (3.13) is satisfied for any $\theta_c \in (0, \pi)$, but for this value there is no nontrivial eigenvector function. We also note that the geometric multiplicity of each eigenvalue $\alpha_{n_i}$ is one, that is to each eigenvalue there is only one linearly independent eigenvector function. This is easy to justify, since for there to be two linearly independent eigenvector functions corresponding to the same eigenvalue $\alpha_{n_i}$, the determinant of the matrix $M_n$ must vanish at all points $(\alpha_{n_i}, \theta)$ for $\theta \in (0, \pi)$. This is only possible if $M_n$ is a null matrix.

We are interested in positive solutions of equation (3.13). We assume that for each $n \in \mathbb{N}_0$ the roots $\{\alpha_{n_i}\}_{i \in \mathbb{N}}$ are arranged in ascending order. Firstly, we are interested in the range of values of $\alpha_n$ for which the Fourier coefficients $u_a^{s/a}$ would exhibit singularities near the vertex $O$, and secondly to determine which solutions of equation (3.13) lie in that range. Of course, equation (3.13) cannot be solved by analytical means and we must use numerical means to approximate the solutions. This will be done in the next section. Now by Remark 2.1, we expect regular solutions $u_n^{s/a}$ of the 2D problem (2.5) to belong to the space $(W^{2,2}_{1/2}(G_a))^3$. The following lemma gives us a useful criterion for deciding if a function of the form $v(r, z) = R^aT(\theta)$ belongs to the space $W^{2,2}_{1/2}(G_a)$ or not.

**Lemma 3.2:** Let $v \in W^{1,2}_{1/2}(G_a)$ be a function defined on the circular sector $G_a$ with vertex $O$, and whose expression in terms of the local polar co-ordinates $R, \theta$ is of the form $v(r, z) = R^aT(\theta)$ for some real number $\alpha > 0$ and with $T(\theta) \in C^\infty([0, \pi])$. Then $v \in W^{2,2}_{1/2}(G_a)$ if and only if $\alpha > 1/2$.

**Proof:** The norm $\|v\|_{W^{2,2}_{1/2}(G_a)}$ of functions $v \in W^{2,2}_{1/2}(G_a)$ in local polar co-ordinates $R, \theta$ is given.
by

\[ \|v\|_{W^{2,2}_{1/2}(G_a)} = \int_{G_a} \left\{ |v|^2 + \left| \frac{\partial v}{\partial R} \right|^2 + \frac{1}{R^2} \left| \frac{\partial^2 v}{\partial \theta^2} \right| + \left| \frac{\partial^2 v}{\partial R \partial \theta} \right|^2 \right\} R^2 \sin \theta dRd\theta. \]

We need to show that each term on the right-hand side of (3.14) is bounded for \( v = R^a \hat{T}(\theta) \) if and only if \( \alpha > 1/2 \). Let us consider the term

\[ \int_{G_a} \left| \frac{1}{R} \frac{\partial^2 v}{\partial R \partial \theta} - \frac{1}{R^2} \frac{\partial v}{\partial \theta} \right|^2 R^2 \sin \theta dRd\theta. \] (3.15)

Substituting \( v = R^a \hat{T}(\theta) \) in (3.15) and taking into account that \( \int_0^\pi |T'(\theta)|^2 \sin \theta d\theta < C < \infty \), we get the identity

\[ \int_{G_a} \left| \frac{1}{R} \frac{\partial^2 v}{\partial R \partial \theta} - \frac{1}{R^2} \frac{\partial v}{\partial \theta} \right|^2 R^2 \sin \theta dRd\theta = (\alpha - 1)^2 \int_0^\pi |T'(\theta)|^2 \sin \theta d\theta \int_0^R R^{2\alpha-2} dR. \] (3.16)

We see that the integral on the right-hand side of (3.16) is bounded only if \( 2\alpha - 2 > -1 \), i.e. \( \alpha > 1/2 \). The same argument holds for the other terms.

**Lemma 3.3:** Let \( \hat{u} \in (W^1_2(\tilde{\Omega}))^3 \) be the generalized solution of the boundary value problem (1.1), (1.2) for the right-hand side \( \tilde{f} \in (L_2(\tilde{\Omega}))^3 \). Let \( u_n^s, u_n^a \) and \( f_n^s, f_n^a \ (n \in N_0) \) denote the Fourier coefficients of \( \hat{u} \) and \( \tilde{f} \), respectively, defined almost everywhere on \( \Omega_a \). For each \( n \in N_0 \), let \( \alpha_{n_l} \ (l = 1, 2, \cdots) \) denote the positive roots of equation (3.13) associated with non-trivial eigenvector functions \( U_n(\alpha_{n_l}, \theta) \). Then

(i) if \( \alpha_{n_l} > 1/2, \ l = 1, 2, \cdots \), for all \( n \in N_0 \) and \( l \in N \), then the coefficients \( u_n^s \) and \( u_n^a \) satisfy the relations

\[ u_n^{s/a} \in (W^{2,2}_{1/2}(\Omega)) \cap (W^{2,2}_{1/2}(\Omega_a)) \cap (W^{2,2}_{1/2}(\Omega_a)), \]

\[ \|u_n^{s/a}\|_{(W^{2,2}_{1/2}(\Omega))} \leq M_0 \|f_n^{s/a}\|_{(L^2_{2,1}(\Omega_a))}. \]

(ii) if \( \alpha_{n_l} \leq 1/2, \ l = 1, \cdots, L_n \) holds for some coefficients \( u_n^s \) and \( u_n^a \), then there exist real constants \( \gamma_n^s \) and \( \gamma_n^a \) so that these coefficients can be represented in the form

\[ u_n^{s/a} = s_n^{s/a} + w_n^{s/a}, \quad s_n^{s/a} = \eta(R) \sum_{l=1}^{L_n} \gamma_n^{s/a} R^{\alpha_{n_l}} U_n(\alpha_{n_l}, \theta), \quad w_n^{s/a} \in (W^{2,2}_{1/2}(\Omega_a))^3, \]

\[ \sum_{l=1}^{L_n} |\gamma_n^{s/a}| + \|w_n^{s/a}\|_{(W^{2,2}_{1/2}(\Omega_a))} \leq M_1 \|f_n^{s/a}\|_{(L^2_{2,1}(\Omega_a))}. \] (3.19)

In (3.18) \( \eta \) denotes a smooth cut-off function.

Proof: Lemma 3.3 follows from the preceding analysis and Lemma 3.2 together with classical results regarding the regularity of solutions of elliptic boundary value problems (cf. [11, 15]). Formulas for computing the coefficients \( \gamma_n^{s/a} \) are given, for example, in [25, 26].

**Remark 3.1:** A regularity result for the solution \( \hat{u} \) of the three-dimensional boundary value
problem (1.1), (1.2) can be obtained from the representation (3.18) of the Fourier coefficients by means of Fourier synthesis. However \( \hat{\mathbf{u}} \in (W_2^2(\Omega))^3 \) cannot be proved by assuming that the Fourier coefficients satisfy the relation \( u_{n/l}^{s/a} \in (W_2^{1,2}(\Omega_{a}))^3 \), \( n \in \mathbb{N}_0 \). That is if \( \alpha_{n/l} \leq 1/2 \), \( n = 0, \cdots, K \), \( l = 1, \cdots, L_n \), then the solution \( \hat{\mathbf{u}} \) can be represented in the form

\[
\hat{\mathbf{u}} = \hat{\mathbf{u}}_s + \hat{\mathbf{w}}, \quad \hat{\mathbf{u}}_s := \eta(R) \sum_{n=0}^{L_n} \sum_{l=1}^{K} \left( \gamma_{n/l}^s \mathbf{R}_{n}^s(\varphi) + \gamma_{n/l}^a \mathbf{R}_{n}^a(\varphi) \right) R^{n_{m/l}} \mathbf{U}_{n}(\alpha_{n/l}, \theta),
\]

where the diagonal matrices \( \mathbf{R}_{n}^s(\varphi) \) and \( \mathbf{R}_{n}^a(\varphi) \) \((n \in \mathbb{N}_0, \varphi \in (-\pi, \pi))\) are defined by

\[
\mathbf{R}_{n}^s(\varphi) := \text{diag} \{ \cos n\varphi, -\sin n\varphi, \cos n\varphi \}, \quad \mathbf{R}_{n}^a(\varphi) := \text{diag} \{ \sin n\varphi, \cos n\varphi, \sin n\varphi \}.
\]

### 4 Numerical experiment

Now we introduce a numerical algorithm with which one can compute the roots of equation (3.13). From the computation we are going to answer principally the following three questions:

a) What is the minimum value of the opening angle \( \theta_c \) at the conical point of \( \hat{\Omega} \) which can affect the regularity of the Fourier coefficients?

b) How many Fourier coefficients can possibly be affected by the presence of a conical point on \( \hat{\Omega} \)?

c) In each case, what is the value of \( L_n \) for which \( \alpha_{n/l} \leq 1/2, l = 1, \cdots, L_n \)?

We use the Mehler-Dirichlet formula (cf. [9])

\[
P_{\alpha}^{-n}(\cos \theta) = \sqrt{\frac{2}{\pi}} \frac{\sin^{-\theta} \theta}{\Gamma(n + \frac{1}{2})} \int_0^\theta \frac{\cos(\alpha + \frac{1}{2})t}{\cos t - \cos \theta} \frac{1}{2^n} dt, \quad n \geq 0. \quad (4.1)
\]

We notice that the integrand in (4.1) is singular at \( t = \theta \). By trigonometric identity, we can write the integrand in the form

\[
\frac{\cos(\alpha + \frac{1}{2})t}{\left[ -\sin(\frac{t+\theta}{2}) \right]^{\frac{1}{2}-n} \left[ 2 \sin(\frac{t-\theta}{2}) \right]^{\frac{1}{2}-n}}. \quad (4.2)
\]

and use

\[
\frac{\cos(\alpha + \frac{1}{2})t}{\left[ -\sin(\frac{t+\theta}{2}) \right]^{\frac{1}{2}-n} (t-\theta)^{n-\frac{1}{2}}} \quad (4.3)
\]
to approximate (4.2) for \( t \) close to \( \theta \). Using Gauss-Legendre quadrature formula with forty five partition points the integral (4.1) is approximated by a sum containing the parameter \( \alpha \). The expression is then used in (3.13) and with a root finding procedure values for \( \alpha \) are computed. The softwares MATLAB 6 and MATHEMATICA were used for the computation. From the computation it was established that the Fourier coefficients \( u_n \) for \( n \geq 2 \) of the solution \( \mathbf{u} \in (W^1_2(\Omega))^3 \) of (1.1), (1.2) belong to \( (W^{2,2}_{1/2}(\Omega_q))^3 \), for any conical point, whenever the right-hand side \( \mathbf{f} \) belong to \( (L^2_2(\Omega))^3 \). Moreover, the effect of the conical points on \( u_1^{s/a} \) is more than on \( u_0^{s/a} \). In both cases the sum in (3.18) has only one term, i.e. \( L_n = 1 \) for \( n = 0, 1 \). The critical angle depends on the Poisson’s ratio. Figure 4.1 shows the dependence of the eigenvalues \( 0 < \alpha_0 \leq 1 \) for \( n = 0 \) on \( \theta \in (\pi, \pi) \) for different Poisson’ratios. For fixed \( \theta \) the eigenvalues increase monotonically with respect to the Poisson’s ratio \( \nu \). In Table 4.1 the roots \( \alpha \) of equation (3.13) for a few selected values of \( \theta \) and for \( n = 1, \nu = 0.2, 0.3, 0.4 \) are presented. We note that the eigenvalues in this case are lower than the corresponding eigenvalues for \( n = 0 \) and are monotonically decreasing for increasing \( \nu \) and fixed \( \theta \).
<table>
<thead>
<tr>
<th>$\theta$</th>
<th>$\nu = 0.2$</th>
<th>$\nu = 0.3$</th>
<th>$\nu = 0.4$</th>
</tr>
</thead>
<tbody>
<tr>
<td>60</td>
<td>0.994145</td>
<td>0.993887</td>
<td>0.993500</td>
</tr>
<tr>
<td>100</td>
<td>0.812927</td>
<td>0.806797</td>
<td>0.797492</td>
</tr>
<tr>
<td>120</td>
<td>0.553730</td>
<td>0.544059</td>
<td>0.529804</td>
</tr>
<tr>
<td>150</td>
<td>0.3234873</td>
<td>0.315972</td>
<td>0.304933</td>
</tr>
<tr>
<td>160</td>
<td>0.277894</td>
<td>0.271532</td>
<td>0.262139</td>
</tr>
</tbody>
</table>

Table 4.1: Solutions of (3.13) for $n = 1$ and selected values for $\nu$ and $\theta$

Acknowledgments

This work was done while the author was visiting the Abdus Salam International Centre for Theoretical Physics (ICTP), Trieste, Italy, under the Junior Associateship Scheme. The visit was sponsored by the Swedish International Development Cooperation Agency (SIDA). The author gratefully acknowledges this support.

References


[34] Software Packages: *MATLAB 6* and *MATHEMATICA*. 13