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REDUCED-ORDER SYNCHRONIZATION OF UNCERTAIN
CHAOTIC SYSTEMS VIA ADAPTIVE CONTROL

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Abstract

We consider the coupling of two uncertain dynamical systems with different order using an adaptive feedback linearization controller to achieve reduced-order synchronization between the two systems. Reduced-order synchronization is the problem of synchronization of a slave system with projection of a master system. The synchronization scheme is an exponential linearizing-like controller and a state/uncertainty estimator. As an illustrative example, we show that dynamical evolution of second-order driven oscillator can be synchronized with the canonical projection of a fourth-order chaotic system. Simulation results indicated that the proposed scheme can significantly improve the synchronousness performance. These promising results justify the usefulness of the proposed output feedback controller in the application of secure communication.

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1 Introduction

The phenomenon of synchronization of two dynamical systems is fundamental in science and has a wealth of applications in technology. While it is natural to associate synchronization with periodic signals, it has in the last decade been realized that chaotic systems can also be synchronized [1, 2, 3, 4, 5, 6]. Identical dynamical systems have been studied by many authors, for example, following the pioneering work of Pecora and Caroll [1]. This opened the way to envisage engineering applications, among which the most developed on is the transmission of information using chaos.

Recently, various papers in mathematics and physics have been published, which lead to a better understanding of the problem of synchronization [4]. In this sense, some results about the adaptive synchronization between chaotic systems, whose model is strictly different, have been reported [7]-[8]. The problem of synchronizing different chaotic systems is somewhat different from what is usually considered as the synchronization problem: usually synchronization is understood as some adjustment of the system’s dynamics due to their interaction, which is achieved by coupling the system’s variables. However, in the case of the quite different chaotic systems, it seems that coupling the system’s variables is not a good approach, because of the fact that unrelated chaotic systems have quite different behaviour in the time domain, and quite different attractor structures in the phase space (see [9] and references therein).

More recently, the concept of reduced-order synchronization between systems with different order has been analyzed and the more common examples studied in this case concern the third-order systems and second-order driven oscillator [9, 10]. Nevertheless, to the best of authors knowledge, there are only these results in the literature. However, most of the dynamical systems have model (or parametric) uncertainties. These uncertainties may cause chaotic perturbations to originally regular behavior, or induce additional chaos in originally chaotic but known behavior, generating unknown chaotic motion. Therefore, one of actual challenges in problem of synchronization is to achieve and explain the synchronization between uncertain chaotic systems with different order.

Reduced-order synchronization is the problem of synchronizing a slave system with the projection of a master system. The main feature of the reduced-order synchronization is that the order of the slave system is less than the master one. It should be noted that reduced-order synchronization is not partial synchronization. On one hand, partial synchronization is for coupling two chaotic systems whose order is equal. A main feature of the partial synchronization is that, at least, one state of the slave system is, in some sense not synchronous [7]. On the other hand, in reduced-order synchronization, all states of the slave system are synchronous, in some sense. Such a problem is reasonable if, for instance, we think that the order of thalamic neurons can be different from that of hippocampal neurons [11]. One more example is the synchronization between heart and lung. One can observe that both circulatory and respiratory systems behave...
in a synchronous way. However, one can expect that model of the circulatory system is strictly
different than the respiratory system, which can involve different order.

In this paper, we consider the coupling of two uncertain nonidentical dynamical systems via
reduced-order synchronization using an adaptive synchronization technique. Such a problem
is related to the synchronization of strictly different chaotic systems. The control objective
is to achieve the reduced-order synchronization, i.e., all states of the slave system should be
synchronized, in some sense, with any states of the master system. The resulting synchronization
scheme is obtained by posing a control problem in the synchronization error dynamics. The
Synchronization problem consists in the stabilization at the origin of the synchronization error
system. The proposed scheme comprises a linearizing-like control law and a state/uncertainty
estimator. A central feature of our approach is that uncertainties of the underlying vector
field are lumped in an extended state whose dynamics is reconstructed from measurements of
the system output. In this way, the robust exponential control scheme allows the stabilization
of the synchronization error system at the origin at a finite time. Consequently, all states
of the slave system are synchronized with one part of the states of the master system at a
finite time. It should be noted that, similar to the previously reported adaptive schemes, the
proposed adaptive scheme allows the synchronization of uncertain nonidentical chaotic systems
with different order. However, the proposed adaptive strategy is robust, easier to tune, it requires
least prior knowledge, and involves a very simple structure.

The structure of the paper is as follows. Section 2 presents the general class of nonlinear
systems under study, the problem formulation, the proposed feedback and the detailed design.
In Section 3, nonlinear emitter-receiver/\phi^6-Van der Pol synchronization problem is presented.
Numerical simulations are presented to show the effectiveness of the proposed synchronization
scheme. Finally, Section 4 is devoted to the conclusion.

2 Reduced-order synchronization for uncertain nonidentical sys-
tems

The aim of this section is to describe the class of nonlinear systems under study and state
the synchronization problem. Thereafter, we present the proposed feedback and design detail.

2.1 Problem formulation

We consider two uncertain nonidentical dynamical systems, one with state variable $x_m \in \mathcal{R}^m$, and
the second, with state variable $x_s \in \mathcal{R}^n$, with governing equations of the general form
\[
\begin{align*}
\dot{x}_m &= f_m(x_m, t), \\
\dot{x}_s &= f_s(x_s, t).
\end{align*}
\] (1)

Here $x_m$ is called the master and $x_s$ the slave. In general, we consider that the dynamics of the
two systems are nonlinear, uncertain, smooth and that there is no cross coupling between the
two sets of state variables. Since the vector field is smooth, we assume that the state variable \( x_m \) can be decomposed in two state variables \( x_{ma} \in \mathbb{R}^n \) and \( x_{mb} \in \mathbb{R}^{m-n} \) where \( x_{ma} \) represents the coordinates which require synchronization to achieve coupling between the two systems. So, we will consider the class of systems where Eq. (1) can be expressed as

\[
\begin{align*}
\dot{x}_{ma} &= f_{ma}(x_{ma}, x_{mb}, t), \\
\dot{x}_{mb} &= f_{mb}(x_{ma}, x_{mb}, t), \\
\dot{x}_s &= f_s(x_s, t).
\end{align*}
\] (2)

Then if \( x_{ma} \to x_s \) as \( t \to \infty \) we can say that the two systems are synchronized in reduced order. When such reduced-order synchronization occurs a coupled system is formed which is shown schematically in Fig. 1. The case where \( m = n \) is the standard synchronization problem described previously by Pecora and Caroll and intensively studied in the literature [1].

To achieve reduced-order synchronization, we need to synchronize the dynamics of \( f_{ma} \) and \( f_s \). Thus we add a controller, to a coupled system, such that Eq. (2) can be rewritten as

\[
\begin{align*}
\dot{x}_{ma} &= f_{ma}(x_{ma}, x_{mb}, t), \\
\dot{x}_{mb} &= f_{mb}(x_{ma}, x_{mb}, t), \\
\dot{x}_s &= f_s(x_s, t) + Bu,
\end{align*}
\] (3)

where \( u \) is the control signal, and \( B \) represents a constant matrix which defines the control channel. In this form, the dynamics of \( f_{ma} \) can be thought of as the reference model. We want \( f_s + Bu \) to replicate this dynamic and \( f_s \) to represents the plant. The role of the feedback \( u \) is thus to force convergence of the slave orbit towards a canonical projection of the master orbit. As we shall see below, the synchronization of systems with different order is addressed to as the stabilization of the synchronization error at the origin at a finite time.

Let us assume the following. (a) Only \( y_m = x_{ma1} \) and \( y_s = x_{1s} \) are available for feedback from master and slave systems. \( y_m \) and \( y_s \) are the outputs of both the master and slave systems. (b) Vector fields of master and slave systems are smooth, nonlinear and uncertain. The types of uncertainties comprise the practically important case of the unknown time varying nonlinear parameter and external disturbance.

To carry out such an investigation, let us introduce the variable \( e = x_s - x_{ma} \) which is the measure of the nearness of the state variable \( x_s \) of the slave to the state variable \( x_{ma} \) of the master. Introducing the state variable \( e \) in Eq. (3), we obtain the following error dynamic system

\[
\dot{e} = f_s(e + x_{ma}) - f_{ma}(x_{ma}, x_{mb}, t) + Bu.
\] (4)

Thus, the synchronization problem becomes to find the control signal \( u \) such that Eq. (4) is globally exponentially stable at the origin at a finite time for any initial condition \( e(0) \in \mathbb{R}^n \),

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i.e., $\lim_{t \to T_{syn}} e(t) = 0$ which implies that $x_s(t) \to x_{ma}(t)$ as $t \to T_{syn}$. That is, dynamic evolution of slave system can be manipulated toward the master behavior.

Now, let us define a coordinate transformation $z = T(e)$ such that the error system (4) can be globally transformed into the canonical form [12]:

$$
\begin{align*}
\dot{z}_i &= z_{i+1}, & i = 1, \ldots, n - 1, \\
\dot{z}_n &= \delta(z, x_{ma}, t) + u, \\
y &= z_1
\end{align*}
$$

(5)

where $y = z_1$ is the output of the uncertain system.

Often, the constant matrix $B$ can be selected to make the transformation of system (4) to the assumed format (5) possible. Since $f_s(x_s, t)$ and $f_{ma}(x_{ma}, x_{mb}, t)$ are uncertain, it is clear that $z = T(e)$ is an uncertain nonlinear change of coordinates, hence $\delta(z, x_{ma}, t)$ in the transformed system (5) is also unknown. The idea to deal with the uncertain term $\delta(z, x_{ma}, t)$ is to lump it into a new state variable which can be interpreted as a new observable state. By an observable state we mean that the dynamics of such state can be reconstructed from on-line measurements. Note that system (5) is fully linearizable.

Now let $\eta = \delta(z, x_{ma}, t)$ be a state variable such that system (5) can be computed. Then there exists a time-invariant manifold $\psi(z, x_{ma}, \eta, t)$ such that the solution of system (5) is a projection of the solution of the following dynamical system

$$
\begin{align*}
\dot{z}_i &= z_{i+1}, & 1 \leq i \leq n - 1, \\
\dot{z}_n &= \eta + u, \\
\dot{\eta} &= \Xi(z, x_{ma}, \eta, u, t),
\end{align*}
$$

(6)

where $z = (z_1, \ldots, z_n)^T$ is a state variable vector and

$$
\Xi(z, x_{ma}, \eta, u, t) = \sum_{k=1}^{n-1} z_{k+1} \partial_k \delta(z, x_{ma}, t) + (\eta + u) \partial_n \delta(z, x_{ma}, t) + f_{ma}(x_{ma}, x_{mb}, t) \partial_{x_{ma}} \delta(z, x_{ma}, t) + \partial_t \delta(z, x_{ma}, t),
$$

with $\partial_k \delta(z, x_{ma}, t) = \partial \delta(z, x_{ma}, t)/\partial z_k$, $k = 1, 2, \ldots, n$, $\partial \delta(z, x_{ma}, t)/\partial t$ and $\partial_{x_{ma}} \delta(z, x_{ma}, t) = \partial \delta(z, x_{ma}, t)/\partial x_{ma}$, i.e., system (6) is dynamically equivalent to system (5). The manifold $\Psi(z, x_{ma}, \eta, t) = \eta - \delta(z, x_{ma}, t)$ is, by definition, time-invariant, i.e, $d\Psi/dt = 0$.

In fact, it is straightforward to prove that the set

$$
\Psi = \{\Psi(z, x_{ma}, \eta, t) = \eta - \delta(z, x_{ma}, t)\}
$$

satisfies $d\Psi/dt = 0$ for all $t \geq 0$. Now, from the equality $\Psi(z, x_{ma}, \eta, t) = 0$ and condition $d\Psi/dt = 0$, one can take the first integral of system (6) to get $\eta = \delta(z, x_{ma}, t)$. When the
first integral is substituted into system (6), we obtain the solution of system (5). Hence, the solution of system (5) is a projection of system (6) via the module \( \pi(z, \eta) = z \). That is, system (6) is dynamically equivalent to system (5) if initial conditions, \((z(0), \eta(0)) \) are contained in \( \Psi(z, x_{ma}, \eta, t) \).

In the next section, the detailed design procedure of the feedback control law \( u \) is described in detail.

### 2.2 Controller design

Let us consider the following linearizing control law to stabilize the uncertain system (6):

\[
    u = -\eta + \sum_{i=1}^{n} \theta^{(n-i+1)} \kappa_i z_i, \tag{7}
\]

where \( \theta \) is a control gain introduced here to assign the convergence rate of the synchronization process and \( \kappa_i, i = 1, 2, \ldots, n \) are constant parameters which are computed from the following procedure. There has been an increase of interest in developing adaptive synchronization algorithms during the past decade. Important and effectiveness of the following approach reflect in its property that it give us a possibility to assign the convergence rate by tuning a single parameter \( \theta \) of the control law \( u \). To the best of authors’s knowledge, this issue has not been previously studied. Equation (6) under controller (7) action results in the following closed-loop system

\[
    \begin{align*}
    \dot{z} &= \theta \Phi^{-1}_\theta A(\kappa) \Phi_\theta z, \\
    \dot{\eta} &= \Xi(z, x_{ma}, \eta, u, t),
    \end{align*}
\]

where the matrices \( A(\kappa) \) and \( \Phi_\theta \) are given by

\[
    A(\kappa) = \begin{bmatrix}
    0 & 1 & \cdots & 0 \\
    \vdots & \vdots & \ddots & \vdots \\
    0 & 0 & \cdots & 1 \\
    \kappa_1 & \kappa_2 & \cdots & \kappa_n
    \end{bmatrix}
\]

and

\[
    \Phi_\theta = \begin{bmatrix}
    \theta^{-1} & 0 & \cdots & 0 \\
    \vdots & \ddots & \ddots & \vdots \\
    0 & 0 & \cdots & 0 \\
    0 & 0 & \cdots & \theta^{-n}
    \end{bmatrix}
\]

with \( \Phi^{-1}_\theta \) the inverse matrix of \( \Phi_\theta \). Then \( \kappa_i, i = 1, 2, \ldots, n \) are chosen such that the matrix \( A(\kappa) \) has all its eigenvalues at the open left-hand complex plane (i.e., all the roots of polynomial \( s^n + \kappa_n s^{n-1} + \cdots + \kappa_2 s + \kappa_1 = 0 \) have negative real parts).

If the initial conditions satisfy \( \psi(z(0), x_{ma}(0), \eta(0), 0) = 0 \), then under the feedback control law (7), the state \( z(t) \) of the closed-loop system (8) converges exponentially at zero at a finite time, i.e., \( \lim_{t \to T_{syn}} z(t) = 0 \) where \( T_{syn} \) is defined as:

\[
    T_{syn} = -\frac{2}{\theta} \lambda_{max}(S) \ln \frac{c_0}{\theta}, \tag{9}
\]

where

\[
    c_0 = \frac{h}{\|z(0)\|} \sqrt{\frac{\lambda_{min}(S)}{\lambda_{max}(S)}}
\]
with \( h \) the synchronization precision or tolerance, \( \lambda_{\text{min}}(S) \) and \( \lambda_{\text{max}}(S) \) are the minimum and maximum eigenvalues of a positive definite matrix \( S \) to be determined later; and \( z(0) \) the initial state of \( z(t) \). Convergence of \( \eta(t) \) to zero follows from the fact that the closed-loop system is in a cascade form [13]. From Eq. (9), one can see that the synchronization time decreases logarithmically when \( \theta \) increases. In fact, the \( \theta \)-parametrization of the feedback (7) provides a simple tuning procedure. Indeed the larger the value, the faster the convergence of the synchronization error \( z(t) \). This can be qualitatively analyzed with the Lyapunov stability theory as follows.

Define the Lyapunov candidate function as

\[
V(z) = z^T \Phi_\theta S \Phi_\theta z, \tag{10}
\]

where \( S \) is a symmetric positive definite matrix solution of the Lyapunov matrix equation

\[
A(\kappa)^T S + SA(\kappa) = -I_{\mathbb{R}^n} \text{ with } I_{\mathbb{R}^n} \text{ the identity matrix of dimension } n. \]

The derivative of this function along the trajectories of the closed-loop system (8) satisfies

\[
\dot{V}(z) = -\theta \| \Phi_\theta z \|^2. \tag{11}
\]

Remark that

\[
\lambda_{\text{min}}(S) \| \Phi_\theta z \|^2 \leq V(z) \leq \lambda_{\text{max}}(S) \| \Phi_\theta z \|^2
\]

where \( \lambda_{\text{min}}(S) \) and \( \lambda_{\text{max}}(S) \) stand for the minimum and the maximum eigenvalues of \( S \). Then Eq. (11) becomes

\[
\dot{V}(z) \leq -\frac{\theta}{\lambda_{\text{max}}(S)} V(z). \tag{12}
\]

Note that the derivative of the Lyapunov function (10) can be rewritten as

\[
\dot{V}(z) = 2 \| \Phi_\theta z \|_S \frac{d}{dt} \| \Phi_\theta z \|_S
\]

where \( \| \Phi_\theta z \|_S = [z^T \Phi_\theta S \Phi_\theta z]^{\frac{1}{2}} \). Therefore, it follows that

\[
\frac{d}{dt} \| \Phi_\theta z \|_S \leq -\frac{\theta}{2\lambda_{\text{max}}(S)} \| \Phi_\theta z \|_S
\]

Then, it comes that

\[
\| z(t) \|_S \leq \| z(0) \|_S e^{\left\{ \frac{-\theta}{2\lambda_{\text{max}}(S)} t \right\}} \rightarrow 0, \tag{13}
\]

which implies that for some \( T_{\text{syn}} > 0 \)

\[
\lim_{t \rightarrow T_{\text{syn}}} z(t) = 0. \tag{14}
\]

Convergence of \( \eta(t) \) to zero follows from the fact that the closed-loop system is in a cascade form [13], that is, since \( \lim_{t \rightarrow T_{\text{syn}}} z(t) = 0 \), the part \( \lim_{t \rightarrow T_{\text{syn}}} \Xi(z, x_{ma}, \eta, u, t) = \lim_{t \rightarrow T_{\text{syn}}} \Xi(0, x_{ma}, \eta, u, t) = 0. \)

We shall now prove that the convergence to zero is attained at a finite time. One way to compute the synchronization time is to follow the time trajectory of the state \( x_s \) of the slave.
system relative to the state $x_{ma}$ of the master system. In this case, synchronization is achieved when the synchronization error $z(t)$ obeys the following condition:

$$|z(t)| \leq h, \quad \forall t > T_{syn},$$

(15)

where $h$ is the synchronization precision or tolerance. Using Eqs. (13) and (15), a simple algebraic manipulation can prove that the reduced-order synchronization occurs at a finite time defined as in Eq. (9).

Note that the controller (7) requires full information about the states of system (6). In this sense the following comments are in order: (i) the augmented state $\eta$ is not available for feedback. This fact is obvious because $\eta$ represents, by definition, the mismatches between master and slave systems; (ii) it is desired that only one state is measurable in chaos synchronization schemes, thus only one state $y = z_1$ is available for feedback from one-line measurements. Consequently, estimated values of the states $(z, \eta)$ are required for practical implementation. As it has been established in [15], the problem of estimating $(z, \eta)$ can be addressed by using a high-gain observer. Thus, the dynamics of the states $(z, \eta)$ can be reconstructed from measurements of the output $y = z_1$ in the following way [14, 16]:

$$\left\{ \begin{array}{l}
\hat{z}_i = \hat{z}_{i+1} + \rho^i C_{n+1}^i (z_1 - \hat{z}_1), \quad 1 \leq i \leq n - 1, \\
\hat{z}_n = \hat{\eta} + u + \rho^n C_{n+1}^n (z_1 - \hat{z}_1), \\
\hat{\eta} = \rho^{n+1} (z_1 - \hat{z}_1),
\end{array} \right.$$  

(16)

where $(\hat{z}, \hat{\eta})$ are estimated values of $(z, \eta)$, $C_{n+1}^i = \frac{(n+1)!}{i!(n+1-i)!}$, $i = 1, \ldots, n + 1$ and $\rho > 0$ a high-gain estimation parameter.

Note that since $\delta(z, x_{ma}, t)$ is uncertain, the function $\Xi(z, \eta, x_{ma}, u, t)$ is correspondingly unknown. Thus, such a term was not used in the construction of the observer (16). This feature yields a low-order parametrization (only a tuning parameter is required) of the dynamic compensator of the adaptive strategy.

Thus, the robust exponential feedback (7) and the synchronization time (9) become

$$u = -\hat{\eta} + \sum_{i=1}^{n} \theta^{(n-i+1)} \kappa_i \hat{z}_i,$$

(17)

and

$$T_{syn} = -\frac{2}{\theta} \lambda_{max}(S) \ln \frac{c_0}{\theta},$$

(18)

where

$$c_0 = \frac{h}{\|\hat{z}(0)\|} \sqrt{\frac{\lambda_{min}(S)}{\lambda_{max}(S)}}$$

with $\hat{z}(0)$ the initial state of $\hat{z}(t)$. 

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Under actions of the dynamic output feedback (17), the solutions \( z(t) \) of the uncertain nonlinear system (6) converges exponentially to the origin at a finite time defined as in Eq. (18). This can be qualitatively analyzed as follows.

Combining systems (6) and (16), the dynamics of the estimation error \( \tilde{e}_j = \rho^{n-j}(z_j - \hat{z}_j), \quad j = 1, \ldots, n \) and \( \tilde{e}_{n+1} = \eta - \hat{\eta} \) can be written as follows

\[
\dot{\tilde{e}} = \rho D \tilde{e} + \Gamma(z, x_{ma}, \eta, u, t), \quad (19)
\]

where \( \Gamma(z, x_{ma}, \eta, u, t) = [0, \ldots, 0, \Xi(z, x_{ma}, \eta, u, t)^T] \) and \( D \in \mathbb{R}^{(n+1)\times(n+1)} \) is the companion matrix

\[
D = \begin{bmatrix}
-C_{n+1}^1 & 1 & 0 & \cdots & 0 \\
-C_{n+1}^2 & 0 & 1 & \cdots & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
-C_{n+1}^m & 0 & 0 & \cdots & 1 \\
-1 & 0 & 0 & \cdots & 0
\end{bmatrix}
\]

One may easily check that the matrix \( D \) has all its eigenvalues located at \(-1\), i.e., \( D \) is a Hurwitz matrix. In addition, since the trajectories \( x_m(t) \) and \( x_s(t) \) are contained in a chaotic attractor, \( \Xi(z, x_{ma}, \eta, u, t) \) is bounded. Consequently, for any sufficiently large value of the high-gain estimation parameter \( \rho \), \( \tilde{e} \to 0 \) as \( t \to \infty \), which implies that \( (\dot{z}, \dot{\eta}) \to (z, \eta) \). In consequence, the output state feedback controller (17) tends to the state feedback given by Eq. (7). Then, control actions counteract the nonlinear uncertainties and induce a stable behavior.

Note that the proposed feedback controller comprises two parts: the output feedback controller (17) and the uncertainties estimator given by the observer (16). Feedback based on Eq. (17) has three advantages: (i) it does not require a priori knowledge about the unknown function nor parameters structure; (ii) only one controller parameter is required to tune and (iii) the expression of the synchronization time has been explicitly computed.

In addition, since the dynamic estimator is a high-gain observer, peaking phenomenon can be induced [17]. To diminish overshoot, one can use the saturated version of the dynamic output feedback [18]:

\[
u = Sat \left\{ -\hat{\eta} + \sum_{i=1}^{n} \theta^{(n-i+1)} \kappa_i \hat{z}_i \right\}, \quad (20)
\]

where \( Sat: \mathbb{R} \to S \) and \( S \subset \mathbb{R}^n \) is a bounded set.

In the next section, we will show that the use of the proposed algorithm yields continuous feedback such that the reduced-order synchronization can be achieved in fourth and second order chaotic systems.

3 Illustrative example and simulation results

Let us now move our focus from the general discussion of detecting synchronization by means of different order auxiliary system to a specific example. We consider reduced-order
synchronization of chaotic oscillator in a two-dimensional $\phi^6$-Van der Pol system when it is driven by a chaotic signal from a fourth-dimensional nonlinear emitter-receiver system.

The drive system is a nonlinear emitter-receiver. A nonlinear emitter-receiver is a nonlinear electrostatic transducer model [19]. The equations of motion can be written as a fourth-order system that is given in the following dimensionless form:

$$
\begin{align*}
\dot{x}_{1m} &= x_{2m}, \\
\dot{x}_{2m} &= -\lambda_1 x_{2m} - \omega_1^2 x_{1m} - \gamma_1 x_{1m}^3 + \alpha_1 x_{3m} + \beta_1 x_{1m} x_{3m} + s(t), \\
\dot{x}_{3m} &= x_{4m}, \\
\dot{x}_{4m} &= -\lambda_2 x_{4m} - \omega_2^2 x_{3m} - \gamma_2 x_{3m}^3 + \alpha_2 x_{1m} + \beta_2 x_{1m}^2 + F_m \cos \omega_m t,
\end{align*}
$$

(21)

where $x_m = (x_{1m}, x_{2m}, x_{3m}, x_{4m})^T$ is a state vector, $\lambda_1$ and $\lambda_2$ are the viscous damping coefficients, $\omega_1$ and $\omega_2$ are the natural frequencies, $\gamma_1$ and $\gamma_2$ are the nonlinear coefficients, $\alpha_1$, $\alpha_2$, $\beta_1$ and $\beta_2$ characterize the coupling between the oscillators. $F_m$ and $\omega_m$ are the amplitude and frequency of the external excitation respectively. The message $s(t)$ will take either the form of a single tone message

$$
s(t) = S \cos \omega t,
$$

(22)
or an amplitude-modulated message of the form

$$
s(t) = S(1 + g \sin \omega_p t) \cos \omega t,
$$

(23)

where $S$ and $\omega$ are respectively the amplitude and frequency of the message, while $g$ is a parameter modulating at frequency $\omega_p$, the amplitude of the message. This second message $s(t)$ will be sent to a second system enslaved in a chaotic state by a first system (master).

The model (21) can be found in an electrostatic microphone and loudspeaker which consists of a mechanical part (described here by $x_{1m}$ and $x_{2m}$) connected capacitively to an R-L-C circuit whose capacitance can be controlled by acoustic waves (for instance, harmonic acoustic waves). The interest for such devices is justified by the fact that with the revival of the electret old idea, electrostatic microphones are widely used for various types of technological applications such as monoscope tubes for TV set signals, cassette recorders devices and evidently telephone devices.

In the present study the non-linear equation of motion (21) is integrated numerically in order to obtain the chaotic behavior of the emitter receiver. The Poincaré map is used to examine the behavior of the system. It is the "slice" through the attractor. Thus the system is "checked" for every full orbit around the attractor. If we are dealing with periodic evolution of period $n$ ($n$ an integer), then this sequence consists of $n$ dots repeating indefinitely in the same order. If the evolution is quasi-periodic, then the Poincaré section is a sequence of points defining a closed limit cycle. If the evolution is chaotic, then the Poincaré section is a collection of points that show interesting patterns with no obvious repetition. We are interest here with the chaotic motion. Figure 2(a) and (b) show the Poincaré maps for $F_m = 0.7$, $\lambda_1 = 0.1$, $\gamma_1 = 0.1$, $\alpha_1 = 0.4$, $\omega_1 = 0.1$, $\omega_2 = 0.2$. The system is in a chaotic state. The Poincaré maps show a typical chaotic attractor with a complex structure.
\( \beta_1 = 0.4, \omega_1 = 0.4, \lambda_2 = 0.3, \gamma_2 = 0.6, \omega_2 = 0.1, \alpha_2 = 0.1, \beta_2 = 0.05, S = 0 \) and \( \omega_m = 1 \), with the initial condition arbitrarily located at origin. Interesting attractors, which configuration looks like a "flower" is presented. Figure 2(b) and 2(c) present the corresponding phase trajectories.

Now, let us consider the \( \phi^6 \) Van der Pol oscillator as the slave system. The equations of motion of the oscillator are given

\[
\begin{align*}
\dot{x}_{1s} &= x_{2s}, \\
\dot{x}_{2s} &= \mu(1-x_{1s}^2)x_{2s} - \omega_0^2 x_{1s} - \alpha x_{1s}^3 - \lambda x_{1s}^5 + F_s \cos \omega_s t + u,
\end{align*}
\]

where \( \mu, \omega_0, \alpha, \lambda, F_s \) and \( \omega_s \) are the system’s parameters and \( u \) is the coupling force (controller) to be chosen. Following the process of the precedent paragraph, system (24) is integrated numerically. From Figure 3(a) and (b) when \( u = 0 \) the attractors of the chaotic state of the oscillator for \( F_s = 4.5, \mu = 0.4, \omega_0 = 0.46, \alpha = 1, \lambda = 0.1, \) and \( \omega_s = 0.86 \) is presented. As seen, the Poincaré map (see Fig. 3(a)) shows an interesting appearance, in which the configuration looks like a "shell-fish". Initial conditions were arbitrarily located at the point \((x_{1s}(0),x_{2s}(0)) = (0.1,0)\).

Obviously, there is a difference between attractors of systems (21) and (24) [see attractor in Fig. 3(b) and projection in Fig. 2(c) and 2(d)], i.e., systems (21) and (24) are not synchronized neither for phase nor frequency. Moreover, there is no synchronization in any sense (see [7] for details concerning the definition of synchronization kinds). In order to study reduced-order synchronization of these systems it is important to note that system (21) can be decomposed into two second order sub-systems coupled by the terms with the coefficients \( \alpha_i \) and \( \beta_i \) (i=1,2). We are interested in the synchronization of the \( \phi^6 \)-Van der Pol attractor with the mechanical part \((x_{ma} = (x_{1m},x_{2m}))\) of the nonlinear emitter-receiver system. Thus, the chaos control objective is to design a feedback \( u \) such that the discrepancy error \( e = x_s - x_{ma} \) tends to zero as \( t \to T_{syn} \).

Now let us define \( e = (e_1, e_2)^T \in \mathcal{R}^2 \) by \( e_1 = x_{1m} - x_{1s} \) and \( e_2 = x_{2m} - x_{2s} \) (\( e \) is a vector whose component represents the synchronization error). Then the resulting error dynamical system can be expressed as

\[
\begin{align*}
\dot{e}_1 &= e_2, \\
\dot{e}_2 &= \Theta(e, x_m, t) + u,
\end{align*}
\]

where

\[
\Theta(e, x_m, t) = \mu [1 - (e_1 + x_{1m})^2](e_2 + x_{2m}) - \omega_0^2(e_1 + x_{1m}) - \alpha(e_1 + x_{1m})^3 - \lambda(e_1 + x_{1m})^5 \\
+ F_s \cos \omega_s t + \lambda_1 x_{2m} + \omega_0^2 x_{1m} + \gamma_1 x_{1m}^3 - \alpha_1 x_{3m} - \beta_1 x_{1m} x_{3m} - \sigma(t)
\]

contains the two system’s model difference, which is unknown to us. Thus, the coordinate transformation is given by \( z_1 = e_1 \) and \( z_2 = e_2 \). In such a way, system (25) is transformed into

\[
\begin{align*}
\dot{z}_1 &= z_2, \\
\dot{z}_2 &= \Theta(z, x_m, t) + u, \\
y &= z_1
\end{align*}
\]
where $\Theta(z, x_m, t) = \Theta(e, x_m, t)$ and $y = z_1$ is the output of the uncertain system (26). Defining $\eta = \Theta(z, x_m, t)$, system (26) can be transformed into its equivalent form (6). In this way, we can get the extended state observer (16) in the following form

\[
\begin{aligned}
\dot{\hat{z}}_1 &= \hat{z}_2 + 3 \rho (z_1 - \hat{z}_1), \\
\dot{\hat{z}}_2 &= \hat{\eta} + u + 3 \rho^2 (z_1 - \hat{z}_1), \\
\dot{\hat{\eta}} &= \rho^3 (z_1 - \hat{z}_1).
\end{aligned}
\]

So, the robust output feedback (17) can be described by

\[
u = -\hat{\eta} + \theta^2 \kappa_1 \hat{z}_1 + \theta \kappa_2 \hat{z}_2.
\]

In the simulations, the values of the control parameters $\kappa_1 = -4$, $\kappa_2 = -4$ were chosen for the output feedback controller (28). This implies that all eigenvalues of the matrix $A(\kappa)$ defined in Eq. (8) are located at $-2$. In this case

\[
S = \begin{bmatrix}
\frac{9}{8} & \frac{1}{8} \\
\frac{1}{8} & \frac{5}{32}
\end{bmatrix}.
\]

Hence, $\lambda_{\text{max}}(S) = 1.1409$ and $\lambda_{\text{min}}(S) = 0.1404$. Without loss of generality, one can consider that parameters and initial conditions have the same value as Figs. 2(c)-(d) and 3(b) which present the phase trajectories of the two oscillators. The control gain and the high-gain estimation parameter were chosen to be $\theta = 20$ and $\rho = 100$, respectively. The initial conditions of the observer (27) have been chosen as $(\hat{z}_1(0), \hat{z}_2(0), \hat{\eta}(0)) = (0.1, 0, 0)$.

Figure 4 shows the resulting control performance of the output feedback controller (28). The output controller was activated at time $t = 50s$. Figures 4(a) and 4(b) show that states $x_{1m}$ and $x_{1s}$; $x_{2m}$ and $x_{2s}$ evolve in a complete synchronous behavior. Note that synchronization has been attained. States $x_{1m}$ and $x_{2m}$ of the master system (solid line) are tracked by states $x_{1s}$ and $x_{2s}$ of the slave system (dashed line). However, as we state below, trajectories of $\phi^6$-Van der Pol oscillator tracks the nonlinear emitter-receiver system projection, i.e., reduced-order synchronization is performed by output feedback controller (28). It should be pointed out that the synchronization achieved by controller (28) is complete in the sense that the trajectories of the synchronization error system (25) converges to the origin. This means that error $e(t)$ converges exactly to zero.

To illustrate the fact that an arbitrary convergence rate of the synchronization error can be prescribed, Fig. 4(c) presents the synchronization error norm $\| \vec{E}(t) \| = \sqrt{e_1^2(t) + e_2^2(t)}$, for three different values of the control gain $\theta$. As expected, the synchronization error norm $\| \vec{E}(t) \|$ converges exponentially to zero and the larger the value of $\theta$, the faster the convergence.

We also have checked for the validity of the analytical results by comparing the values given by Eq. (9) to those obtained from the numerical simulation. We use Eq. (15) to compute the

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synchronization time as a function of \( \theta \) with the precision \( h = 10^{-4} \). The results are reported in Fig. 5. The agreement between the analytical (lines) and numerical (lines with stars) results is good. One can see that when the control gain \( \theta \) increases, the analytical and numerical results are very closed.

After the reduced-order synchronization of the nonlinear emitter-receiver system and the \( \phi^6 \)-Van der Pol oscillator, we can use many ways to recover the transmitted information \( s(t) \). Here we use chaotic modulation \([20]\) to discuss it simply. The transmitter \((27)\) contains a chaotic oscillator with an input that is modulated by the information signal. The receiver is composed by a copy of the transmitter driven by a synchronization signal \( x_{1s} \). Figure 6 illustrates the typical frame of communication transmission.

Thus, the recovered information signal \( s_r(t) \) is given by

\[
s_r(t) = y_T - x_{1s}
\]

where \( y_T = x_{1m} + s(t) \) is the transmitted signal and \( x_{1s} \) the receiver synchronization signal.

As derived earlier, since the error dynamics globally exponentially converges to zero at a finite time under the feedback control law \((28)\), we have

\[
\lim_{t \to T_{syn}} s_r(t) = \lim_{t \to T_{syn}} (y_T - x_{1s}) = \lim_{t \to T_{syn}} (e_1 + s(t)) = s(t)
\]

This implies that the information signal \( s(t) \) can be recovered exponentially from the receiver by means of the output-feedback controller \((27)-(28)\).

Figure 7 presents the behavior of the communication system described in Fig. 6 for \( \theta = 20 \), \( \rho = 100 \) and a single tone message \( s(t) = S \cos \omega t \) with \( S = 1 \) and \( \omega = 8 \). The frequency was chosen such that the dynamic behavior of the master system remain chaotic. Figure 7(a) presents the \((x_{1m}, x_{2m})\)-phase plane which shows the chaotic nature of transmitter dynamics. The waveform of the transmitted signal \( y_T = x_{1m} + s(t) \) is depicted in Fig. 7(b). Figure 7(c) presents the recovered signal \( s_r(t) = e_1 + s(t) \). One can see that the message signal is well recovered in about 0.6s which corresponds approximately to the analytical value of the synchronization time (see Fig. 5).

Figure 8 presents similar results for the case of an amplitude modulated message \( s(t) = S(1 + g \sin \omega_p t) \cos \omega t \) with \( S = 1 \), \( \omega = 8 \), \( g = 1 \) and \( \omega_p = 0.3 \).

4 Conclusion

In this work, an adaptive strategy for reduced-order chaos synchronization has been developed. The proposed scheme allows the synchronization of two uncertain chaotic systems with
different order. This means that two uncertain chaotic systems can be synchronized in spite of the fact that the order of the response system is less than the order of the drive system. Of course, since the order of the response oscillator is smaller than the master system, the synchronization is only attained in reduced order. Results are focussed on geometrical features of synchronization phenomenon. The output feedback controller is designed from a simple algorithm based on chaos suppression. The proposed strategy is an input-output control scheme which comprises an uncertainty estimator and an exponential controller. The use of this adaptive feedback controller is significant, in that it can be used to synchronize two uncertain chaotic systems with different order at a finite time in spite of the system’s uncertainties. As an illustrative example, the synchronization of a fourth order chaotic nonlinear emitter-receiver system and a second order chaotic $\phi^6$-Van der Pol oscillator is achieved in reduced-order, i.e., a $\phi^6$-Van der Pol equation can be synchronized under feedback actions with a canonical projection of the nonlinear emitter-receiver system. Numerical simulations are presented to show the effectiveness of the proposed synchronization scheme. Finally, we have discussed how the proposed synchronization technique can be implemented to secure communication. Nevertheless, other practical applications can be physically carried out.

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References


Figure captions

**Figure 1**: Schematic representation of a system formed by reduced-order synchronization.

**Figure 2**: (a) Poincaré map of the first oscillator; (b) Poincaré map of the second oscillator; (d) Phase trajectory of the first oscillator; (e) Phase trajectory of the second oscillator;

**Figure 3**: (a) Poincaré map of the $\phi^6$-Van der Pol oscillator without control actions. (b) Phase trajectory of the $\phi^6$-Van der Pol oscillator without control actions.

**Figure 4**: Complete exact synchronization of the master and slave systems via the dynamic output (28). (a) $x_{1m}(t)$ (---) and $x_{1s}(t)$ (- - -); (b) $x_{2m}(t)$ (---) and $x_{2s}(t)$ (- - -) (c) Synchronization error norm for three different values of the control gain $\theta$.

**Figure 5**: Synchronization Time $T_{syn}$ as a function of $\theta$ when $h = 10^{-4}$: numerical results (lines with stars) and analytical results (lines).

**Figure 6**: Schematic diagram of the secure communication system.

**Figure 7**: Behavior of the communication system for the single tone message $s(t) = \cos 8t$ performed with $\theta = 20$ and $\rho = 100$. (a) Phase portrait $(x_{1m}, x_{2m})$ of the transmitter dynamics; (b) Transmitted signal $y_T(t) = x_{1m} + s(t)$; (c) Information signal $s(t)$ (---) and recovered signal $s_r(t)$ (- - -).

**Figure 8**: Behavior of the communication system for the incoming amplitude modulated message $s(t) = (1 + \sin 0.3t) \cos 8t$ performed with $\theta = 20$ and $\rho = 100$. (a) Phase portrait $(x_{1m}, x_{2m})$ of the transmitter dynamics; (b) Transmitted signal $y_T(t) = x_{1m} + s(t)$; (c) Information signal $s(t)$ (---) and recovered signal $s_r(t)$ (- - -).
Figure 1: Schematic representation of a system formed by reduced-order synchronization.
Figure 2:
Figure 4:
Figure 5:

Figure 6:
Figure 7:
Figure 8: