NONCOMMUTATIVE GEOMETRY FRAMEWORK
AND THE FEYNMAN’S PROOF OF MAXWELL EQUATIONS

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Abstract

The main focus of the present work is to study Feynman’s proof of the Maxwell equations using the NC geometry framework. To accomplish this task, we consider two kinds of noncommutativity formulations going along the same lines as Feynman’s approach. This allows us to go beyond the standard case and discover non-trivial results. In fact, while the first formulation gives rise to the static Maxwell equations, the second formulation is based on the following assumption $m[x_j, \dot{x}_k] = i\hbar \delta_{jk} + im \theta_{jk} f$. The results extracted from the second formulation are more significant since they are associated to a non trivial $\theta$-extension of the Bianchi-set of Maxwell equations. We find $\text{div}_\theta B = \eta_\theta$ and $\frac{\partial B_i}{\partial t} + \epsilon_{kji} \frac{\partial E_j}{\partial x_k} = A_1 \frac{df}{dt} + A_2 \frac{d\theta}{dt} + A_3$, where $\eta_\theta$, $A_1$, $A_2$ and $A_3$ are local functions depending on the NC $\theta$-parameter. The novelty of this proof in the NC space is revealed notably at the level of the corrections brought to the previous Maxwell equations. These corrections correspond essentially to the possibility of existence of magnetic charges sources that we can associate to the magnetic monopole since $\text{div}_\theta B = \eta_\theta$ is not vanishing in general.

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1 Introduction

Noncommutative geometry (NCG) stimulated recently by Connes [1] and developed later on by several pioneering authors [2, 3] have played an increasingly important role more notably in the attempts to understand the space-time structure at very small distance. Much attention has been paid also to quantum field theories on NC spaces, in particular NC Yang-Mills gauge theory as well as NC-QED, a subject which has matured as an area of intense research activity in more recent times [4-6]. In fact it has been established by Seiberg and Witten [2] that the existence of noncommutativity in open string boundaries in the presence of the NS-NS B field results in NC D-branes to which the open string endpoints are attached. Related to these stimulating ideas, a wide number of papers were devoted to study several aspects of the NC [7].

One particular property of NCG framework is its richness and also the fact that we can discover the standard results just by requiring the vanishing of the deformed parameter which means also the vanishing of noncommutativity. Note that the passage from commutative to NC space time is simply achieved by replacing the ordinary product, in the space of smooth functions on $\mathbb{R}^2$ with coordinates $(x, t)$, by the NC associative $\star$ product. Works having used this NC formalism are various and the results found are numerous, we will limit ourselves to mention some of them, namely [8-14].

The aim of this paper is to study another aspect of the noncommutativity framework adapted to the Feynman’s proof of Maxwell equations [15-19]. As well known, a century ago, Maxwell brought four basic laws dealing with electromagnetism, these laws describe the evolution in time and space of the electric and magnetic fields $E$ and $B$. Together with the Lorentz force law, the Maxwell equations constitute a complete description of electromagnetism (the Maxwell theory). Furthermore these equations have different forms, vectorial, differential and can be proved in different way. Feynman in 1948 has given a proof of these equations, assuming only Newton’s law of motion and the commutation relation between position and velocity for a single nonrelativistic particle. The importance of this proof emerged notably with the Dyson’s work [15]. As signaled in this work, the motivation of Feynman was to build a new theory outside the framework of conventional physics, but his assumptions using these commutation relations and the Newton’s equation were not lead to new physics [15]. This proof, although based on simple mathematical assumptions, is shown to give rise to nontrivial generalizations [16-19].

Among many possible existing extensions, we are going to adapt thereafter the NC framework to the Feynman’s proof, a fact which leads us to extract important results. We present two kind of NC formulations and show in a first one that the application of the Feynman’s proof in NC space, leads to the static Maxwell equations. Focusing to obtain a new theory, we propose in our second formulation to modify the Moyal bracket between the position $x_i$ and the velocity
\( \dot{x}_j \). This task can be easily accomplished by assuming that the velocity is space dependent and then the star product between \( x_i \) and \( \dot{x}_j \) becomes non trivial. This assumption will modify the Maxwell equations, giving rise to a new theory where extra terms proportional to NC \( \theta \)-parameter appear. The results extracted from the second formulation are more significant as they are associated to a non trivial \( \theta \)-extension of the Bianchi-set of Maxwell equations namely
\[
\text{div} \mathbf{B} = \eta_\theta \text{ and } \frac{\partial B_j}{\partial t} + \epsilon_{kjs} \frac{\partial E_k}{\partial x_s} = A_1 \frac{\partial^2 f}{\partial t^2} + A_2 \frac{\partial f}{\partial t} + A_3, \quad \text{where } \eta_\theta, A_1, A_2 \text{ and } A_3 \text{ are local functions depending on the NC } \theta\text{-parameter.}
\]

Our objectives in reconsidering the Feynman’s proof are, on one hand, to put it in relief and, on the other hand, to show its importance in the NC framework. The novelty of this proof formulated in the NC space is revealed notably at the level of the corrections brought to the standard Maxwell equations. These corrections correspond essentially to the possibility of existence of sources of magnetic charges that we can associate to the magnetic monopole since \( \text{div} \mathbf{B} = \eta_\theta \). Note that these extra terms \( \eta_\theta \) are absent in the ordinary case associated to \( \theta = 0 \). These results may give new insights into the study of the electromagnetic duality and its various physical and mathematical aspects.

This paper is organized as follows. In section 2, we summarize some properties of the Poisson manifold and review the Feynman’s proof of the Maxwell equations. In section 3 we present some useful identities of \(*\) product after that we examine the Feynman proof of Maxwell equations in NC spaces. Section 4 is devoted to our concluding remarks.

## 2 Maxwell Equations: The Feynman’s Proof

Maxwell equations have played a pioneering role in physics and they continue to nourish several axes of research either in physics or in mathematics. Their formulations as well as the survey of their solutions constitute a topic of a big interest [20] and it’s in this context that are located the famous theories of Yang-Mills. Recently, we attended to a new approach leading to the derivation of these equations and based on what is called the Feynman’s proof of Maxwell equations. Details concerning this approach are presented in the Dyson’s work [15]. Later on, several authors took this approach and tried to put in relief the Feynman’s idea and to develop it or sometimes to generalize it to other contexts [16-19]. Before reviewing the Feynman’s proof of the Maxwell equations, lets first start by introducing some basic algebraic properties of the underlying Poisson manifold \( \mathcal{M} \)

### 2.1 Some algebraic properties

In fact the previous approach can be simply stated in the general way as finding all Poisson tensors on a phase space manifold such that they have Hamiltonian vector fields which correspond
to second order differential equations such that \( \{q^i, q^j\} = 0 \), with the symbol \( \{,\} \) standing for the Poisson bracket defined usually as

\[
\{f, g\} = \frac{\partial f}{\partial q^i} \frac{\partial g}{\partial p^i} - \frac{\partial f}{\partial p^i} \frac{\partial g}{\partial q^i},
\]

where \( f \) and \( g \) are two functional of \( q \) and \( p \). Denoting by \( \mathcal{A} \) the algebra of classical observables on the manifold \( \mathcal{M} \), one can define a Poisson structure \( \{,\} : \mathcal{A} \times \mathcal{A} \rightarrow \mathcal{A} \) on this manifold as been a skew-symmetric bilinear map such that:

a) \( (\mathcal{A}, \{,\}) \) satisfies the Jacobi identity

\[
\{F, \{G, H\}\} + \{H, \{F, G\}\} + \{G, \{H, F\}\} = 0
\]

b) The map \( X_F = \{., F\} \) defines a derivation on \( \mathcal{M} \) of the associative algebra \( \mathcal{A}(\mathcal{M}) \), that is, it satisfies the Leibnitz rule

\[
\{F, GH\} = G\{F, H\} + \{F, G\}H
\]

A manifold endowed with such a Poisson bracket on \( \mathcal{A}(\mathcal{M}) \) is called a Poisson manifold. Furthermore, consider a Poisson manifold \( \mathcal{P} \), for any \( H \in \mathcal{A}(\mathcal{P}) \), there is a unique vector field \( X_H \) on \( \mathcal{P} \) such that

\[
X_H G = \{G, H\}
\]

for all \( G \in \mathcal{A}(\mathcal{P}) \). \( X_H \) is nothing but the Hamiltonian vector field of \( H \). Now one can define a dynamical system on the Poisson manifold \( \mathcal{M} \) just by considering for any function \( H \in \mathcal{A} \) the following differential equation

\[
\frac{dF}{dt} = \{F, H\}
\]

Moreover, one can also express the Poisson bracket \( \{F, G\} \) in any set of local coordinates \( (x^a) \) in the following way

\[
\{F, G\} = X_G F = x^a, G \frac{\partial F}{\partial x^a}
\]

### 2.2 The Feynman’s Proof

This section is devoted to an explicit remind of the main steps involved into the Feynman’s proof of the Maxwell equations in their classical form [15, 16]. Our objective is to present these calculations in order to make a comparison with the NC case to be discussed later. This proof is essentially based on the Newton’s laws of the non relativistic classical mechanics and on the relations of commutation joining the coordinates of position and velocity of a single non relativistic particle. An extension to the relativistic case is naturally possible [17, 19] and may leads to important results more notably in connection with quantum field theory approaches.

The manifold we consider is parameterized by local coordinate variables \( (w^a) = (x^i, \dot{x}^i) \) of a non relativistic particle whose position \( x_j (j = 1, 2, 3) \) and velocity \( \dot{x}_j \) satisfy the Newton’s equation

\[
m\ddot{x}_j = F_j (x, \dot{x}, t),
\]
with commutation relations
\[ \{x_j, x_k\} = 0, \quad \{x_j, \dot{x}_k\} = i\hbar \delta_{j,k}. \]

(8) \hspace{1cm} (9)

Then, there exist a couple of fields \( E(x, t) \) and \( B(x, t) \) that we can identify with the electric and the magnetic fields respectively such that we get the Lorentz force law
\[ F_j = E_j + \epsilon_{jkl} \dot{x}_k B_l, \]

(10)

and the first couple of the Maxwell equations
\[ \text{div} B = 0, \]

(11)
\[ \frac{\partial B}{\partial t} + \nabla \times E = 0. \]

(12)

The second couple of Maxwell equations
\[ \text{div} E = 4\pi \rho, \]

(13)
\[ \frac{\partial E}{\partial t} - \nabla \times B = 4\pi j, \]

(14)

merely defines the external charge and the current densities \( \rho \) and \( j \) respectively.

The Feynman’s proof starts by differentiating the bracket (9) with respect to time and using (7), we have
\[ \{x_j, F_k\} + m\{\dot{x}_j, \dot{x}_k\} = 0. \]

(15)

Then using the Jacobi identity
\[ \{x_l, \{x_j, \dot{x}_k\}\} + \{\dot{x}_j, \{x_k, x_l\}\} + \{\dot{x}_k, \{x_l, \dot{x}_j\}\} = 0 \]

(16)
as well as the bilinearity of the Poisson bracket we find the following constraint equation
\[ \{x_l, \{x_j, F_k\}\} = 0. \]

(17)

Furthermore, since the bracket is antisymmetric the tensor \( \{x_j, F_k\} \) satisfy
\[ \{x_j, F_k\} = -\{x_k, F_j\}, \]

(18)
and therefore we may write
\[ \{x_j, F_k\} = -\frac{i\hbar}{m} \epsilon_{jkl} B_l. \]

(19)

This equation gives a definition of the field \( B \) whose components are \( B_l \). This shows that \( B \) would in general depend on coordinates \( x, \dot{x} \) of the Poisson manifold \( \mathcal{M} \) and possibly time \( t \).

Combining (17) with equation for \( B_l \) (19) lead to
\[ \{x_l, B_m\} = 0. \]

(20)
On account of the basic equations (8-9), this means that $B$ is a function of the coordinates $x$ and $t$ of the particle. Therefore, its shown that the vectors $E$ and $B$ are not independent as we have

$$\{x_m, E_j\} = 0,$$

which says that $E$ is also a function of $x$ and $t$ only.

Now we have two equations (15) and (19) that we naturally need to compare. The way to do this consist simply in writing $B$ as

$$B_t = -i \frac{m^2}{2\hbar} \epsilon_{jkl}\{x_j, \dot{x}_k\}. \quad (22)$$

Another application of the Jacobi identity gives

$$\epsilon_{jkl}\{\dot{x}_l, \{x_j, \dot{x}_k\}\} = 0, \quad (23)$$

leading naturally to the first Maxwell equation $\text{div}B = 0$ (11) namely

$$\{\dot{x}_l, B_l\} = 0. \quad (24)$$

Indeed,

$$\{B_l, \dot{x}_l\} = \{x_a, \dot{x}_l\} \frac{\partial B_l}{\partial x_a} = \frac{i\hbar}{m} \frac{\partial B_l}{\partial x_a} \delta_{al} = \frac{i\hbar}{m} \text{div}B = 0. \quad (25)$$

The proof of the second Maxwell equation (12) starts from deriving both sides of (22) with respect to time. This gives

$$\frac{\partial B_l}{\partial t} + \dot{x}_m \frac{\partial B_l}{\partial x_m} = -i \frac{m^2}{\hbar} \epsilon_{jkl}\{\dot{x}_j, \dot{x}_k\}. \quad (26)$$

Now by virtue of (7) and (10), the right side of (26) becomes

$$-i \frac{m}{\hbar} \epsilon_{jkl}\{E_j + \epsilon_{jab} \dot{x}_a B_b, \dot{x}_k\} = -i \frac{m}{\hbar} (\epsilon_{jkl}\{E_j, \dot{x}_k\} + \{\dot{x}_k B_l, \dot{x}_l\} - \{\dot{x}_l B_k, \dot{x}_k\})$$

$$= \epsilon_{jkl} \frac{\partial E_j}{\partial x_k} + \dot{x}_k \frac{\partial B_l}{\partial x_k} - \dot{x}_l \frac{\partial B_k}{\partial x_k} + i \frac{m}{\hbar} B_k \{\dot{x}_l, \dot{x}_k\}. \quad (27)$$

On the right side of (27), the last term is zero by virtue of (22), the third term vanishes also as it describes exactly the first Maxwell equation. Now identifying l.h.s and r.h.s of (26), we get

$$\frac{\partial B_l}{\partial t} = \epsilon_{jkl} \frac{\partial E_j}{\partial x_k}, \quad (28)$$

which is nothing but the second Maxwell equation (12).

This is the way followed by Feynman to prove the Maxwell equations in their classical form. His
motivation was to “discover a new theory not to reinvent the old one”, but the proof showed him that his assumptions (7-9) were not leading to new physics. As was the case for several authors who find interesting the Feynman’s approach, we project in the forthcoming section to go beyond this approach and setup the Feynman’s proof in a non-commutative space. The way to apply the noncommutativity is by replacing ordinary product by star product and Poisson bracket or ordinary commutators by the Moyal bracket.

\section{The Feynman’s proof in the NC geometry framework}

The passage to NC geometry, based essentially on the noncommutativity of space-time coordinates, is justified among others by its importance in different currents of research more particularly in high energy physics. The deep idea behind the noncommutativity of coordinates is that in a certain microscopic regime our standard conception of the space-time is not more applicable. Such a regime is characterized by domains of area $\theta$ where the space-time loses its condition of continuum and becomes subject to the following new structure

$$[x_\mu, x_\nu]_\star = x_\mu \star x_\nu - x_\nu \star x_\mu = i\theta^{\mu\nu},$$

where $\theta^{\mu\nu}$ is a real antisymmetric constant matrix and $[\cdot, \cdot]_\star$ is the Moyal bracket. One way of incorporating noncommutativity of coordinates in the context of field theory is through the Moyal product based on the $\star$-product that will be introduced latter on. To avoid hard notations, we will later on simply denote the Moyal bracket by $[\cdot, \cdot]$. Before going on, let us first recall briefly some useful identities of the $\star$-product.

\subsection{Some properties of $\star$-product}

Recently the star product marks a remarkable success due to its intervention in different aspects of string theory greatly related to NC geometry. In this section, we give some useful properties of this star product as well as of the Moyal bracket \cite{9,10}. To define this object, lets start by considering two functions $f(x)$ and $g(x)$ such that

$$f(x) \star g(x) = e^{\frac{i}{2} \theta^{ab} \frac{\partial}{\partial x^a} \frac{\partial}{\partial x^b}} f(x + \xi)g(x + \eta)/\xi = \eta = 0,$$

where $\theta^{ab}$ is a constant, of dimension $[L]^2$, known as the NC parameter\footnote{In all the parts of this paper the parameter $\theta$ is considered as a constant matrix.}. This formula leads naturally to what is often called the Moyal bracket of functions.

$$[f(x), g(x)] = f(x) \star g(x) - g(x) \star f(x).$$

According to this definition, the commutation relation for the space coordinates becomes

$$[x_i, x_j] = i\theta_{ij}.$$
depend on space-time coordinates.
We collect here below some useful properties.

1) **Associativity**

\[(f * g) * h = f * (g * h). \]  \hspace{1cm} (32)

2) **Jacobi identity**

\[[f, [g, h]] + [g, [h, f]] + [h, [f, g]] = 0. \]  \hspace{1cm} (33)

3) **Leibnitz rule**

\[[f, g * h] = g * [f, h] + [f, g] * h. \]  \hspace{1cm} (34)

4) **Linearity**

\[f * (g + h) = (f * g) + (f * h). \]  \hspace{1cm} (35)

The star product is also compatible with integration

\[\int Tr(f * g) = \int Tr(g * f), \]  \hspace{1cm} (36)

where \(Tr\) is the ordinary trace of the \(N \times N\) matrices, and \(f\) is the ordinary integration of functions.

Another useful identity is given in term of local coordinates \((x^a)\). For two functions \(F(x)\) and \(G(x)\), the coordinate expression for the Moyal bracket \([F, G]\) is

\[[F, G] = [x^a, G] \frac{\partial F}{\partial x^a}. \]  \hspace{1cm} (37)

More details about the origin of the \(*\) product and other important properties are available in literature [9-13]. Next, we will study the Feynman’s proof of Maxwell equations using the NC framework, a fact which consist also in using the Moyal bracket instead of the Poisson bracket. In what follows we will present two kind of NC framework associated to two distinguished scenarios to conceive the proof of the Maxwell equations in a NC space. These two scenarios offering two possibilities to make the space NC permit among others to debate the novelty extracted from this extension relatively to each case.

### 3.2 Noncommutativity: First kind

One way to make the space NC is to consider the following commutation relations

\[[x_j, x_k] = i \theta_{jk}, \]  \hspace{1cm} (38)
\[ m[x_j, \dot{x}_k] = i\hbar \delta_{jk}, \]  

where \([,]\) stands for the Moyal bracket and where (38) is simply a NC extension of (8). We assume in this first kind of noncommutativity that the r.h.s of (39) is not affected by the deformation parameter. Differentiating this equation with respect to time and using (7) we find the same equation as in the ordinary product (9), since the NC parameter \(\theta_{jk}\) is a constant

\[ [x_j, F_k] + m[\dot{x}_j, \dot{x}_k] = 0. \]  

(40)

On the other hand, the bilinearity of the Moyal bracket implies

\[ [[x_i, F_j], x_k] + m[[\dot{x}_i, \dot{x}_j], x_k] = 0. \]  

(41)

Computing the second term of this equation, using the Jacobi identity of \(\_x_i, \_x_j, \_x_k\) as well as the Moyal bracket of \(x_j\) and \(\dot{x}_k\) (39), we find the following constraint

\[ [[\dot{x}_i, \dot{x}_j], x_k] = 0, \]  

(42)

or by virtue of (40)

\[ [[x_i, F_j], x_k] = 0. \]  

(43)

Compared to the standard computations, the present case shows a new property namely the quantity \([x_i, F_j]\) is coordinate space independent, and hence the field \(B\) defined by

\[ B_l = -i \frac{m^2}{2\hbar} \epsilon_{jkl} [\dot{x}_j, \dot{x}_k] \]  

(44)

is also independent of \(x_i\). Consequently, the corresponding equations for \(H\) read as

\[ \text{div} B = \frac{\partial B_l}{\partial x_l} = 0, \]  

(45)

and

\[ \text{rot} B = \nabla \times B = 0. \]  

(46)

Moreover, using (39) and (44), the field \(E\) defined by the Lorentz force equation (10), satisfies then

\[ [x_m, E_j] = 0. \]  

(47)

The above equation shows that the field \(E\) is also space independent which, in turn, gives the following equations

\[ \text{div} E = 0, \]  

(48)

and

\[ \text{rot} E = 0. \]  

(49)

Few remarks are in order:
1. The fact to introduce a parameter of noncommutativity to the manner of (38), induces
necessarily the static Maxwell equations which means also the absence of the charge and current
densities $\rho$ and $j$. We can advance at this level that the noncommutativity of the first kind is
equivalent to cancel the charge and current densities for the Maxwell equations.

2. It’s important to look for the meaning of the commutative limit $\theta = 0$. In fact, once the
previous limit is performed, the behavior of the Lorentz force $F$ as well as of the field $B$ change
completely as they depend on the behavior of the space coordinates $x_m$. Setting $\theta = 0$ one
discover the Poison bracket $\{x_j, x_k\} = 0$ which, by virtue of the standard computations, means
the restoration of the densities $\rho$ and $j$.

3.3 Noncommutativity: Second Kind

As its shown through the previous calculations, the relation (9) constitutes a crucial step in
the Feynman’s proof. Any changes at the level of this relation will necessarily lead important
modifications and all the standards results are then suspected to change. Here we propose to
consider the NC space (38) and modify the equation (39) while supposing that velocity is a
quantity that depends on spatial coordinates. We suppose the following NC expressions

$$[x_j, x_k] = i\theta_{jk},$$

$$m[x_j, \dot{x}_k] = i\hbar \delta_{jk} + im\theta_{jk}f,$$

where $f$ is a function which can depend on $x$ and $t$ and it’s given by

$$f = \left(\frac{\partial \hat{x}_l}{\partial \eta_l}(x + \eta)\right)_{\eta=0}. \quad (52)$$

Note that the equation (51) is established by using the definition of the star product and assum-
ing that

$$\frac{\partial \dot{x}_k}{\partial \eta_b} = \delta_{kb} \frac{\partial \dot{x}_j}{\partial \eta_i}. \quad (53)$$

Note also that the velocity should not be proportional to the position due to the presence of the
term $i\hbar \delta_{jk}$ in (51).

However, following step by step the Feynman’s analysis [15], one shows that the derivation of
(51) with respect to time $t$ drives naturally to the following expression

$$m[\dot{x}_j, \dot{x}_k] + m[x_j, \frac{d\dot{x}_k}{dt}] = im\theta_{jk} \frac{df}{dt},$$

or equivalently

$$m[\dot{x}_j, \dot{x}_k] + [x_j, F_k] = im\theta_{jk} \frac{df}{dt}. \quad (55)$$

Since the Moyal bracket is also bilinear we can write

$$[x_l, [x_j, F_k]] = -m[x_l, [x_j, \dot{x}_k]] + im\theta_{jk}[x_l, \frac{d\dot{x}_k}{dt}], \quad (56)$$
Furthermore, using the Jacobi identity of $x_l$, $\dot{x}_j$ and $\dot{x}_k$, the first term in the right side of equation (56) gives
\[
[x_l, [\dot{x}_j, \dot{x}_k]] = i[(\theta_{lk}\dot{x}_j - \theta_{lj}\dot{x}_k), f]
\] (57)
and we can write
\[
[x_l, [x_j, F_k]] = -i m[(\theta_{lk}\dot{x}_j - \theta_{lj}\dot{x}_k), f] + i m \theta_{jk}[x_l, \frac{df}{dt}].
\] (58)
Note that, in spite of the fact that (55) extends the standard relation (15) it preserves the antisymmetry property of $x_j$ and $F_k$, because of the antisymmetry of the NC parameter $\theta$, namely
\[
[x_j, F_k] = -[x_k, F_j],
\] (59)
and therefore the field $B$ can also be defined as
\[
[x_j, F_k] = -(i\hbar/m)\epsilon_{jkl} B_l.
\] (60)
Equations (58) and (60) give the following Moyal bracket
\[
[x_l, B_s] = \frac{m^2}{2\hbar} \epsilon_{jks} \left( [(\theta_{lk}\dot{x}_j - \theta_{lj}\dot{x}_k), f] - \theta_{jk}[x_l, \frac{df}{dt}] \right),
\] (61)
which vanishes for $\theta = 0$, giving rise then to the standard Poisson bracket (20).
The field $B$ can be written using (55) and (60) as follows
\[
B_s = -\frac{i m^2}{2\hbar} \epsilon_{jks} [\dot{x}_j, \dot{x}_k] - \frac{m^2}{2\hbar} \epsilon_{jks} \theta_{jk} \frac{df}{dt}.
\] (62)
Note that the second term in the right hand side of (62) didn’t appear in standard calculations (22). On the other hand, using (60) as well as the expression of the Lorentz force (10) we can write for the electric field $E$
\[
[x_j, E_k] = -\epsilon_{knm} x_m [x_j, B_n] - i \epsilon_{knm} \theta_{jm} f B_n,
\] (63)
To explicit much more this expression one need only to substitute the bracket $[x_j, B_n]$ and $B_n$ by their explicit formulas (61-62). Now, in order to obtain the NC analogous of the first Maxwell equation $\text{div} B = 0$, one should compute, as previously, the Moyal bracket between the velocity and the field $B$
\[
[x_s, B_s] = -\frac{i m^2}{2\hbar} \epsilon_{jks} [\dot{x}_s, [\dot{x}_j, \dot{x}_k]] - \frac{m^2}{2\hbar} \epsilon_{jks} \theta_{jk} [\dot{x}_s, \frac{df}{dt}],
\] (64)
or simply
\[
[B_s, \dot{x}_s] = \frac{m^2}{2\hbar} \epsilon_{jks} \theta_{jk} [\dot{x}_s, \frac{df}{dt}],
\] (65)
since the first term of (64) vanishes using the analogous of the Jacobi identity(23).
Afterwards, using (37), this equation becomes
\[
(i\hbar \delta_{as} + i m \theta_{as} f) \frac{\partial B_s}{\partial x_a} = \frac{m^3}{2\hbar} \epsilon_{jks} \theta_{jk} [\dot{x}_s, \frac{df}{dt}],
\] (66)
or equivalently
\[
\frac{\partial B_s}{\partial x_s} = -\frac{i}{2} \frac{m^3}{\hbar^2} \epsilon_{jk} \frac{\partial}{\partial x'} [\dot{x}'_s + 2x_s] \left\{ \frac{df}{dt} - \frac{m}{\hbar} \theta_{as} f \frac{\partial B_s}{\partial x_a} \right\}.
\] (67)

Using once again (61) and the following identity
\[
[B_s, x_s] = [x_a, x_s] \frac{\partial B_s}{\partial x_a} = i \theta_{as} \frac{\partial B_s}{\partial x_a},
\] (68)

the first NC Maxwell equation corresponding to (51) reads finally as
\[
\text{div}_B B = \frac{\partial B_s}{\partial x_s} = -\frac{i}{2} \frac{m^3}{\hbar^2} \epsilon_{jk} \left\{ \frac{d^2 f}{dt^2} - \frac{m^2}{\hbar^2} \epsilon_{jk} \frac{\partial^2 f}{dt^2} \right\}.
\] (69)

This equation can be simply rewritten as
\[
\text{div}_B B = \eta_\theta,
\] (70)

where we have introduced the notation \( \text{div}_B B \equiv \frac{\partial B_s}{\partial x_s} \) for the first NC Maxwell equation to distinguish it from the standard case. A remarkable fact is that the r.h.s. of (69) namely \( \eta_\theta \), is completely dependent of the NC parameter \( \theta \), setting \( \theta = 0 \) we obtain exactly the ordinary Maxwell equation (24). Here, one could anticipate and give a significance to this new immersing term \( \eta_\theta \) as being a density of magnetic charges in analogy with the density of electric charge.

Next, to obtain the second NC Maxwell equation, we derive with respect to time the field \( B_s \)
\[
\frac{\partial B_s}{\partial t} + \epsilon_{jk} \frac{\partial B_s}{\partial x'_m} = -\frac{m^2}{\hbar^2} \epsilon_{jk} \left\{ \frac{d^2 f}{dt^2} - \frac{m^2}{\hbar^2} \epsilon_{jk} \frac{\partial^2 f}{dt^2} \right\},
\] (71)

this is because the magnetic field \( B \) is \((x, t)\)-coordinates dependent, since the velocity is also considered as depending on the space coordinate. Furthermore, using the Lorentz force (10), one have
\[
\frac{\partial B_s}{\partial t} + \epsilon_{jk} \frac{\partial B_s}{\partial x'_m} = -\frac{m^2}{\hbar^2} \epsilon_{jk} \left\{ \frac{d^2 f}{dt^2} - \frac{m^2}{\hbar^2} \epsilon_{jk} \frac{\partial^2 f}{dt^2} \right\} = -\frac{m^2}{\hbar^2} \left\{ \epsilon_{jk} \left[ \epsilon_{jk} + [x_k B_s, x_k] - [x'_s B_k, x_k] \right] \right\}
\] (72)

Explicitly we find the following expression for the second NC Maxwell equation
\[
\frac{\partial B_s}{\partial t} + \epsilon_{jk} \frac{\partial E_j}{\partial x_k} = -\frac{m^2}{\hbar^2} \epsilon_{jk} \left[ \epsilon_{jk} + \frac{df}{dt} - \frac{m}{\hbar} \theta_{mn} \epsilon_{mnk} \frac{df}{dt} \right] - \frac{m^2}{\hbar^2} \epsilon_{jk} \frac{\partial E_j}{\partial x_k}
\] (73)
leading then, after some algebraic manipulations, to the following compact formula

$$\frac{\partial B_s}{\partial t} + \epsilon_{kjs} \frac{\partial E_i}{\partial x_k} = A_1 \frac{d^2 f}{dt^2} + A_2 \frac{df}{dt} + A_3$$

(74)

where the r.h.s. term of (74) is a non linear second order differential equation in the arbitrary function $f$ whose coefficients are explicitly given by

$$A_1 = -\frac{m^2}{2\hbar} \epsilon_{jks} \theta_{jk}$$

$$A_2 = \frac{m^3}{2\hbar^2} \theta_{jt} \left\{ \theta_{ks} \epsilon_{jls} f^2 - i \epsilon_{jlk} [\dot{x}_s, \dot{x}_k] \right\}$$

(75)

$$A_3 = \frac{m^3}{2\hbar^2} \left\{ \epsilon_{jlk} [\dot{x}_s, \dot{x}_k] + i \theta_{ks} \epsilon_{jls} f^2 \right\} [\dot{x}_j, \dot{x}_i] - \dot{x}_s \eta_\theta.$$

where $\eta_\theta = div B$ (69-70). As we can easily check, all the local coefficients functions $A_1$, $A_2$ and $A_3$ are $\theta$-depending. Thus, the standard limit $\theta = 0$ is natural as it leads to the standard Feynman’s proof computations. Our last forthcoming section is devoted to a conclusion with a discussion about the derived results.

4 Concluding Remarks

Let us summarize what has been the scope of this work. The importance of the so called Feynman’s proof of the Maxwell equations was essentially revealed by the Dyson’s work [15]. This paper resuscitated a former idea of Feynman who made a proof of the Maxwell equations assuming only the Newton’s law of motion and the commutation relations between position and velocity. This proof that Feynman refused to publish, believing that it was a simple joke [21], was appreciated and taken with a great seriousness by several scientists [16-19].

However, one of the things that caused some discussions around the Feynman’s proof is the fact that the derivation mixes classical and quantum concepts and the small confusion that seems to appear when we see the relativistic Maxwell equations derived from the classical Newton’s law. The point is that the consideration of non relativistic equations and the relations of commutation between position and velocity are only a process well arranged to get the Maxwell equations. As it is signaled in [18], one may wonder then how truly relativistic Maxwell equations are derived from Newton’s classical assumptions?. The confusion could be shaped if we remark that the Feynman’s proof concerns only half of Maxwell equations, namely $div B = 0$ and $\frac{\partial B}{\partial t} + \nabla \times E = 0$, describing the Bianchi set of equations. It doesn’t anymore pose problem since this couple of equations is compatible with the nonrelativistic Galilean invariance.

Following the Feynman’s proof of the Maxwell equations, assuming only the Newton’s law of motion and the commutation relation between position and velocity, we try in this paper to study this proof using the NC geometry framework. To accomplish this task, we consider two
kinds of NC formulations going along the same way as Feynman’s approach. This allows us to
discover, in a first formulation, the static Maxwell equations. Afterwards, motivated by the hope
to find a new theory using NC framework, we assume that the velocity is also space dependent
and write the modified NC relation (51). The results extracted from the second formulation are
more significant as they are associated to a non trivial $\theta$-extension of the Bianchi-set of Maxwell
equations namely $\text{div}_0 B = \eta_0$ and $\frac{\partial B_s}{\partial t} + \epsilon_{kjs} \frac{\partial E_i}{\partial x_k} = A_1 \frac{d^2 f}{dt^2} + A_2 \frac{df}{dt} + A_3$, where $A_1$, $A_2$ and $A_3$ are
local coefficient functions depending on the NC parameter $\theta$. Our objectives in reconsidering
the Feynman’s proof are, on one hand, to put it in relief and, on the other hand, to show its
importance in the NC framework.

The novelty of this proof in the NC space is revealed notably at the level of the corrections
brought to the previous Maxwell equations. These corrections correspond essentially to the
possibility of existence of sources of magnetic charges that we can associate to the magnetic
monopole since $\text{div}_0 B = \eta_0$. Note that these extra terms $\eta_0$ are absent in the ordinary case
associated to $\theta = 0$. These results may give new insights into the study of the electromagnetic
duality and its various physical and mathematical aspects.

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