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TORUS THEORY

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Abstract

Geometrical structure and physical characteristics of a torus are investigated in detail. Newtonian and electromagnetic potentials of the torus are defined at short and long distances. It is shown that torus potential at small distances has attractive oscillator behaviour. Motion of a particle in the torus potential is studied. The inertia tensor of the torus and its dynamics are obtained. Rotating torus whose tip is held fixed by two massless rigid threads and moves in a gravitational field is considered.

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I. INTRODUCTION

Starting from Newton the successful attack on the problem of string as distinct from particles has been done by a majority of physicists at the end of the second half of the 20^{th} century and for which far reaching detailed studies exist (Green, Schwarz and Witten, 1986; Polchinski, 1998). [1] In the geometrical structure aspects string at least closed one is a particular case of the torus. Recently, mathematical problems of the torus theory have received much attention, for example, in the higher dimensional theory, where the extra dimensions are the L two-dimensional noncommutative tori with noncommutativity θ (Huang, 2001) [2]. Aharonov-Bohm and Casimir effects (Chaichian et al.,2001) [3] and quantum mechanics (Morariu and Polychronakos, 2001) [4] on the noncommutative tori are also studied.

In the previous papers (Namsrai and von Geramb, 2001) [6] and (Namsrai, 2001) [5] we considered nonlocal potentials and photon propagators arising from charged torus, and carried out quantization of elementary particles masses and charges within the framework of square-root operator formalism and the concept of extended particles associated with the structure of the torus. An assumption that masses and electric charges of fermions are distributed on the surface of the torus allowed us to link between masses and charges of elementary particles by means of the definite number of steps with universal constant defining the structure of the tori. Moreover, it is shown that the torus electric charges are radiated or absorbed complicated fields consisting of the nonolocal photon and nonlocal massless spinor (supersymmetric partner of photon) fields.

The purpose of this paper is to study geometrical and physical problems of the torus in the classical physical level and to prepare its relativistic and quantum field descriptions further. In Sect. 2 we obtain gravitational and electromagnetic potentials of the torus in the general form, where mass and electric charge are distributed on the surface of the torus. Sect. 3 is devoted to another representation of the torus potentials, where mass and charge are distributed in a volume of the torus. Study of obtained potentials at short and long distances is the topic of Sect. 4. Motion of a particle near torus and on its surface is investigated in Sect. 5. Sect. 6 deals with the study of dynamics of a rigid torus, where we will calculate inertia tensor of the torus and derivation of the Euler equations for the motion of the rigid torus. Force-free motion of the torus is given in Sec. 7. In Sect. 8 we consider a rotating torus whose tip is held fixed by two massless rigid threads and which moves in a gravitational field.

II. EXTENDED MASSES AND CHARGES ON THE SURFACE OF THE TORUS

Let us consider a 3-dimensional object where its mass and electric charge are distributed uniformly on the surface of the torus. In general, the torus has two different main structures: ellipsoid and circular. Without loss of generality, we consider here the simplest geometrical shape of the torus, i.e., its cross section on the $z = 0$ plane yields two circles with radiuses R and r, and through the $\varphi = constant$ plane is also circle with radius $(R-r)/2$ in the polar coordinate system. For the ellipsoid torus these two radiuses depend on the polar angle φ:

$$r \to r(\varphi) = \frac{a}{\sqrt{\cos^2\varphi + \frac{a^2}{b^2}\sin^2\varphi}}, \quad R \to R(\varphi) = \frac{A}{\sqrt{\cos^2\varphi + \frac{A^2}{B^2}\sin^2\varphi}}$$

where constants a, b, A and B are parameters of two ellipses. We see that this case leads to the complication of calculations of torus geometrical characteristics.

At first sight one can think that the circular torus has also a complicated geometrical structure to set up its Newtonian and Coulomb potentials. However, if we look at it carefully we can observe its nice geometrical construction having common characteristics with respect to the ring except an extra degree freedom winding around the torus.

Let ds be the surface differential element (protuberant trapezium) on the torus, center of which belongs to the point N (Figure 1). Further, we trace two lines from the point N: $AN = h$ is perpendicular to the plane OXY and ND crosses with the central line of the torus, i.e., $ND = (R-r)/2$, where R and r are the big (outer) and small (inner) radiuses of the torus. An angle $< ADN$ is denoted by α, $(0 \le \alpha \le 2\pi)$ that is the winding angle around the torus. Let $M = M(z, \rho, \varphi)$ be an observable point at which we would like to find the Newtonian and Coulomb potentials of the torus. Here we use the cylinderical coordinate system, which is associated to a rectangular one by the formula

$$x = \rho\cos\varphi, \quad y = \rho\sin\varphi, \quad z = z.$$

Thus, by construction, the distance between the surface differential element ds and the point M is defined as

$$NM^2 = (z-h)^2 + AB^2 \tag{1}$$

According to the law of cosines

$$AB^2 = \rho^2 + L^2 - 2\rho L\cos\varphi, \tag{2}$$

where

$$L = r + \frac{R-r}{2}(1-\cos\alpha), \quad h = \frac{R-r}{2}\sin\alpha \tag{3}$$

Since the medium line of the trapezium ds is $ON \cdot d\varphi = \sqrt{h^2+L^2}d\varphi$ then its area is

$$ds = \frac{R-r}{2}d\alpha\sqrt{(r + \frac{R-r}{2}(1-\cos\alpha))^2 + (\frac{R-r}{2})^2\sin^2\alpha}d\varphi \tag{4}$$

Thus, the element of charge corresponding to the surface differential form ds of the torus reads

$$de = \lambda ds = \lambda\frac{R-r}{2}\sqrt{L^2+h^2}d\alpha d\varphi = \lambda r\frac{R-r}{2}\sqrt{1+q^2\sin^2\frac{\alpha}{2}}d\alpha d\varphi \tag{5}$$

where $q^2 = R^2/r^2 - 1$, λ is the surface charge density, φ is the polar angle $(0 \le \varphi \le 2\pi)$. The potential element $dU_C(M)$ generated by the charge de of the torus at the point M is given by

$$dU_C(M) = \frac{de}{4\pi\sqrt{NM^2}} \tag{6}$$

where de and NM^2 are defined by expressions (5) and (1).

Integration over the polar angle $d\varphi$ becomes

$$U_C(z,\rho) = \pi\lambda(R-r)\int_0^{2\pi} d\alpha\sqrt{L^2+h^2}\frac{1}{4\pi\sqrt{(z-h)^2+(\rho+L)^2}}\times$$

$$\frac{2}{\pi}F(\frac{\pi}{2},\frac{2\sqrt{L\rho}}{\sqrt{(z-h)^2+(\rho+L)^2}}) \tag{7}$$

where parameters L and h are given by (3). This is the Coulomb potential of the torus, where the electric charge is distributed on its surface. Let us calculate the surface of the torus. From (5) it follows

$$S_t = \oint_\Sigma ds = 4\pi R(R-r)E(\frac{\pi}{2},\sqrt{1-\frac{r^2}{R^2}}) \tag{8}$$

The torus potential (7) is also finite at the origin

$$U_C(0) = \frac{e}{8RE(\frac{\pi}{2},\sqrt{1-\frac{r^2}{R^2}})} \tag{9}$$

Functions $F(\pi/2,x)$ and $E(\pi/2,x)$ in expressions (7) and (8) are complete elliptic integrals of first and second kinds, respectively.

It is natural that the Newtonian potential of the torus is given by the same formula (7) where the quantity λ should be changed by the surface mass density σ: $\lambda \to \sigma$. Then, the mass of the torus becomes

$$m_{tori} = m_{sphere}\cdot(1-\frac{r}{R})E(\frac{\pi}{2},\sqrt{1-\frac{r^2}{R^2}}) \tag{10}$$

where $m_{sphere} = 4\pi\sigma R^2$ is a mass of a spherical object.

III. EXTENDED MASSES AND CHARGES IN THE VOLUME OF THE TORUS

Let us consider extended objects masses and charges which are distributed uniformly in the volume of the torus. An elementary volume, i.e., the parallelepiped with two curvilinear trapezium bases, center of which belongs to the point A (Figure 1) is given by

$$dv = 2h\cdot ds = 2\cdot\frac{R-r}{2}\sin\alpha\cdot ds \tag{11}$$

where the differential element ds is expressed by formula (4). Then, the potential element $dU_C'(M)$ generated by the volume charge de' of the torus at the point M becomes

$$dU_C'(M) = \frac{de'}{4\pi\sqrt{AM^2}} \tag{12}$$

where

$$AM^2 = NM^2(z=0) = z^2 + AB^2 \tag{13}$$

Here AB^2 is defined by the same expression (2). In this case, the Coulomb potential (7) acquires the form

$$U'_C(z,\rho) = \pi\lambda'(R-r)^2 \int_0^\pi d\alpha \cdot \sin\alpha \sqrt{L^2+h^2} \frac{1}{4\pi\sqrt{z^2+(\rho+L)^2}} \times$$

$$\frac{2}{\pi} F(\frac{\pi}{2}, \frac{2\sqrt{L\rho}}{\sqrt{z^2+(\rho+L)^2}}) \tag{14}$$

where λ' is the volume charge density.

Let us calculate the volume of the torus. From (11) it follows

$$V_{tori} = \frac{(R-r)^2}{2} \int_0^{2\pi} d\varphi \int_0^\pi d\alpha \cdot \sin\alpha \cdot r\sqrt{1+q^2\sin^2\frac{\alpha}{2}} \tag{15}$$

Since

$$1+q^2\sin^2\frac{\alpha}{2} = 1 + (\frac{R^2}{r^2}-1)\sin^2\frac{\alpha}{2} =$$

$$\frac{R^2}{r^2}[1-(1-\frac{r^2}{R^2})\cos^2\frac{\alpha}{2}]$$

Further, changing the integration variable $\alpha \to 2\psi + \pi$, and using the identities

$$\cos(\psi+\frac{\pi}{2}) = -\sin\psi, \quad \sin(2\psi+\pi) = -\sin 2\psi$$

one gets

$$V_{tori} = 2\pi R(R-r)^2 \int_0^{\pi/2} d\psi \sin 2\psi \cdot \sqrt{1-\gamma^2\sin^2\psi} \tag{16}$$

where $\gamma^2 = 1 - r^2/R^2$. The last integral is calculated by means of elementary functions :

$$V_{tori} = \frac{4\pi}{3} R^3 \frac{(y-1)^2}{y^3} \frac{y^2+y+1}{y+1} \tag{17}$$

Here $y = R/r$. For the mass distributed uniformly in the volume of the torus the Newtonian potential is also defined by the same formula (14) where one can perform the change $\lambda' \to \sigma'$, σ' is the volume mass density.

IV. LARGE AND SHORT DISTANCES BEHAVIORS OF THE TORUS POTENTIALS

A. Distribution of Mass and Charge in the Volume of the Torus

1. The Long Distance Behavior of the Volume Potentials

In this case, the parameter $Q = z^2 + (\rho + L)^2$ in the integrand (14) takes the form

$$Q = r_0^2(1 + \frac{2\rho}{r_0^2}\Omega + \frac{\Omega^2}{r_0^2}) \tag{18}$$

where $r_0^2 = z^2 + \rho^2$ and

$$\Omega = R[1 - (1 - \frac{r}{R})x^2], \quad x = \sin^2\psi$$

Further, using the series representations of the elliptic integral $F(\pi/2, x)$:

$$F(\frac{\pi}{2}, x) = \frac{\pi}{2}F(\frac{1}{2}, \frac{1}{2}; 1; x^2) =$$

$$\frac{\pi}{2}\{1 + (\frac{1}{2})^2 x^2 + (\frac{1\cdot 3}{2\cdot 4})^2 x^4 + \cdots + [\frac{(2n-1)!!}{2^n n!}]^2 x^{2n} + \cdots\} \tag{19}$$

and decomposing expressions in the integrand (14) over the small parameters $\epsilon = 1/r_0^2$, one gets ($r_0^2 \to \infty$):

$$U'_N(M) = \frac{m'_{tori}}{4\pi r_0} - \frac{4\pi\sigma' R(R-r)^2}{2r_0^3}(1 - \frac{3}{2}\sin^2\theta)\times$$

$$\int_0^{\pi/2} dx \cdot x\sqrt{1-\gamma^2 x^2}\frac{1}{4\pi}R^2[1 - (1 - \frac{r}{R})x^2]^2 \tag{20}$$

where

$$\gamma^2 = 1 - \frac{r^2}{R^2}, \quad r_0 = \sqrt{z^2 + \rho^2}, \quad x = \sin\psi$$

$\rho = r\sin\theta$ and $z = r\cos\theta$ in the spherical coordinate system. The first term in (20) coincides with the Newtonian potential for the torus, as it should. After elementary integration, the potential (20) acquires the form

$$U'_N(M) = \frac{m'_{tori}}{4\pi r_0} - \frac{1}{2r_0^3}(1 - \frac{3}{2}\sin^2\theta)\frac{R^2}{4\pi}m'_{tori}\times$$

$$[1 - \frac{6}{\gamma^2 y^3}\frac{1}{y^2+y+1}(\frac{1}{5} - \frac{1}{3}y^2 + \frac{2}{15}) -$$

$$\frac{3}{\gamma^4 y^6}\frac{y-1}{y^2+y+1}(\frac{1}{7} - \frac{2}{3}y^2 + \frac{1}{3}y^4 - \frac{8}{105}y^7)] \tag{21}$$

in the limit $r_0 \gg R$. For the Coulomb potential case, the mass of the torus m'_{tori} should be changed by the electric charge e'_{tori} of the torus in expressions (20) and (21).

2. The Short Distance Behavior of the Volume Potentials

In this case, decomposition of the expression (18) is carried out in powers of the variables r_0 and ρ. The series

$$\frac{1}{\Omega}[1-\frac{\rho}{\Omega}-\frac{1}{2}\frac{r_0^2}{\Omega^2}+\frac{3}{2}\frac{\rho^2}{\Omega^2}+$$

$$\frac{3}{2}\frac{\rho r_0^2}{\Omega^3}+\frac{3}{8}\frac{r_0^4}{\Omega^4}-\frac{5}{2}\frac{\rho^3}{\Omega^3}-\frac{15}{4}\frac{\rho^2 r_0^2}{\Omega^4}] \tag{22}$$

appears due to the multiplier $(z^2+(\rho+L)^2)^{-1/2}$ in (14), while the elliptic integral $F(\frac{\pi}{2},k)$ in (14) gives the series:

$$1+\frac{\rho}{\Omega}+\frac{1}{4}\frac{\rho^2}{\Omega^2}-\frac{\rho r_0^2}{\Omega^3}-\frac{27}{4}\frac{\rho^3}{\Omega^3}-\frac{17}{2}\frac{\rho^2 r_0^2}{\Omega^4}-\frac{2263}{64}\frac{\rho^4}{\Omega^4} \tag{23}$$

Multipling expressions (22) and (23) together and collecting terms of the same powers of the variables r_0 and ρ one gets

$$U_N'(z,\rho)=R^3\sigma'(1-\frac{r}{R})^2\int_0^{\pi/2}dx\cdot x\sqrt{1-\gamma^2x^2}\frac{1}{R[1-(1-\frac{r}{R})x^2]}\times$$

$$[1-\frac{1}{2}\frac{r_0^2}{\Omega^2}+\frac{3}{4}\frac{\rho^2}{\Omega^2}-\frac{79}{8}\frac{\rho^2 r_0^2}{\Omega^4}-8\frac{\rho^3}{\Omega^3}+\frac{3}{8}\frac{r_0^4}{\Omega^4}-\frac{1967}{64}\frac{\rho^4}{\Omega^4}] \tag{24}$$

where Ω is defined in equation (18), and $x=\sin\psi$. Here we restrict ourselves terms in square powers in r_0^2 and ρ^2, and calculate integrals:

$$i_1=\int_0^{\pi/2}dx\cdot x\sqrt{1-\gamma^2x^2}\frac{1}{1-(1-\frac{r}{R})x^2} \tag{25}$$

and

$$i_2=\int_0^{\pi/2}dx\cdot x\sqrt{1-\gamma^2x^2}\frac{1}{[1-(1-\frac{r}{R})x^2]^3} \tag{26}$$

After some elementary calculations, we find

$$i_1=-\frac{1}{2}\frac{1}{\tau}\{2\sqrt{1-\gamma^2u}+\sqrt{\frac{r}{R}}\arcsin[\frac{\gamma^2+2\frac{r}{R}\frac{1}{u-\frac{1}{\tau}}}{\gamma^2}]\} \tag{27}$$

where $\tau=1-r/R$, $u=x^2=\sin^2\psi$ and ψ runs from zero to $\pi/2$, and therefore

$$i_1=1+\sqrt{\frac{r}{R}}\frac{1}{\tau}\arcsin(\frac{\tau}{1+y^{-1}}) \tag{28}$$

Here as before $y=R/r$ and $\tau=1-y^{-1}$. After some transformations the second integral (26) becomes

$$i_2=-\frac{1}{4\tau^3}\cdot\frac{d^2}{dC^2}[\int_0^{-\sin^2\psi}du\sqrt{1+\gamma^2u}\frac{1}{u+C}] \tag{29}$$

where $C=1/\tau$.

Further, use the explicit form (27) of the integral (25) and differentiate it twice with respect to the variable C. Then, we have

$$i_2=-\frac{1}{4}\frac{1}{\tau^3}\frac{d^2}{dC^2}\{2\sqrt{1+\gamma^2u}-\frac{1-\gamma^2C}{\sqrt{-(1-\gamma^2C)}}\arcsin[\frac{2(1-\gamma^2C)}{\gamma^2(-u-C)}-1]\}$$

where $u = -\sin^2\psi$, ψ runs from zero to $\pi/2$, and therefore

$$i_2 = 1 + \frac{y}{4} - \frac{1}{8y} + \frac{(1+y^{-1})^2}{8\tau} y^{\frac{3}{2}} \arcsin(\frac{y-1}{y+1}) \tag{30}$$

Thus, the volume potential of the torus at short distances acquires the form

$$U'_N(M) = \frac{3}{4\pi}\frac{m'_{tori}}{R}\frac{1-y^{-2}}{1+y^{-1}+y^{-2}}[1 + \frac{\arcsin(\frac{y-1}{y+1})}{\tau\sqrt{y}}$$

$$-\frac{1}{2}\frac{r_0^2}{R^2}(1-\frac{3}{2}\sin^2\theta)(1+\frac{y}{4}-\frac{1}{8y}+$$

$$\frac{1}{8\tau}y^{\frac{3}{2}}(1+y^{-1})^2\arcsin(\frac{y-1}{y+1}))] \tag{31}$$

We see that the torus potential is finite at the origin and has attractive oscillator behavior.

B. Distribution of Mass and Charge on the Surface of the Torus

1. The Large Distance Behavior of the Surface Potentials

Let us consider another case when mass and electric charge of a particle are distributed on the surface of the torus. We would now like to evaluate torus surface potentials at long distances. In this case, the parameter (18) has the form

$$Q' = r_0^2(1 + \frac{2\rho\Omega_0}{r_0^2} + \frac{\Omega'^2}{r_0^2}) \tag{32}$$

where

$$\Omega_0 = \Omega - \frac{z}{\rho}h, \quad \Omega'^2 = \Omega^2 + h^2 \tag{33}$$

and h is given by the expression (3). Series in the variable r_0^{-2} yield

$$U_N = 4\pi\sigma R^2\tau\int_0^{\pi/2} d\psi\sqrt{1-\gamma^2\sin^2\psi}\frac{1}{4\pi r_0}\times$$

$$\{1 - \frac{R^2}{2r_0^2}(1-\frac{3}{2}\sin^2\theta)(1-\tau\sin^2\psi)^2 - \frac{1}{2}\frac{\tau^2R^2}{r_0^2}\sin^2\psi\cos^2\psi\}$$

Integration over the variable ψ is reduced to the elliptic integrals. Thus, asymptotic behavior of the torus surface potentials at large distances takes the form

$$U_N(r_0 \gg R) = \frac{m_{tori}}{4\pi r_0}\{1 - \frac{R^2}{2r_0^2}(1-\frac{3}{2}\sin^2\theta)[1 - \frac{2}{3}\frac{\tau}{\gamma^2}\times$$

$$((1-\gamma^2)O + 2\gamma^2 - 1) + \frac{\tau^2}{15\gamma^4}(-(2\gamma^4-\gamma^2-1)O + 8\gamma^4 - 3\gamma^2 - 2)] -$$

$$\frac{R^2\tau^2}{30r_0^2\gamma^4}[-(1-\gamma^2)(2-\gamma^2)O + 2(\gamma^4-\gamma^2+1)]\} \tag{34}$$

where we have denoted $O = F(\pi/2), k)/E(\pi/2, k)$. We obtain again the Newtonian potential. The second term in expression (34) gives a small contribution to the Newtonian law.

2. The Short Distance Behavior of the Surface Potentials

In this case, we decompose expression (32) over the small variables r_0^2 and ρ^2. That is

$$Q' = \Omega'^2[1 + \frac{2\rho\Omega_0}{\Omega'^2} + \frac{r_0^2}{\Omega'^2}] \tag{35}$$

The term $[z^2 + (\rho + L)^2]^{-1/2}$ in (7) gives the series

$$W_1 = \frac{1}{\Omega'}[1 - \frac{\rho\Omega_0}{\Omega'^2} - \frac{1}{2}\frac{r_0^2}{\Omega'^2} + \frac{3}{2}\frac{\rho^2\Omega_0^2}{\Omega'^4} +$$

$$\frac{3}{2}\frac{\rho\Omega_0 r_0^2}{\Omega'^4} + \frac{3}{8}\frac{r_0^4}{\Omega'^4} - \frac{5}{2}\frac{\rho^3\Omega_0^3}{\Omega'^6} - \frac{15}{4}\frac{\rho^2 r_0^2}{\Omega'^6}\Omega_0^2] \tag{36}$$

While the elliptic integral $F(\frac{\pi}{2}, k)$ in (7) is decomposed in series:

$$W_2 = 1 + \frac{\rho\Omega}{\Omega'^2} + \frac{\rho^2\Omega}{\Omega'^4}(-2\Omega_0 + \frac{9}{4}\Omega) - \frac{\rho r_0^2}{\Omega'^4}\Omega +$$

$$\frac{\rho^3\Omega}{\Omega'^6}(-4\Omega_0^2 - 9\Omega\Omega_0 + \frac{25}{4}\Omega^2) + \frac{\rho^2 r_0^2}{\Omega'^6}\Omega(-4\Omega_0 - \frac{9}{2}\Omega) +$$

$$\frac{\rho^4\Omega}{\Omega'^8}[-8\Omega_0^3 - 9\Omega\Omega_0^2 - \frac{75}{2}\Omega^2\Omega_0 + \frac{1225}{64}\Omega^3] \tag{37}$$

Multiply two expressions (36) and (37) with each other and classify the terms of the same orders. Then substituting $\Omega_0 = \Omega - zh/\rho$ and $\Omega'^2 = \Omega^2 + h^2$ in the obtained series one gets

$$U_N = \frac{\sigma}{2}R(R - r)\int_{-\pi/2}^{\pi/2} d\psi\{1 + \frac{zh}{\Omega'^2} - \frac{1}{2}\frac{r_0^2}{\Omega'^2} +$$

$$\frac{3}{2}\frac{\rho^2}{\Omega'^4}(\frac{1}{2}\Omega^2 + \frac{z^2h^2}{\rho^2}) + \frac{\rho^3}{\Omega'^6}(-8\Omega^3 + \frac{39}{2}\Omega^2\Lambda - 8\Omega\Lambda^2 +$$

$$\frac{5}{2}\Lambda^3) - \frac{3}{2}zh\frac{r_0^2}{\Omega'^4} + \frac{\rho^2 r_0^2}{\Omega'^6}(-\frac{109}{8}\Omega^2 + 8\Omega\Lambda - \frac{15}{4}\Lambda^2) + \frac{3}{8}\frac{r_0^4}{\Omega'^4} +$$

$$\frac{\rho^4\Omega}{\Omega'^8}(-\frac{381}{8}\Omega^3 + \frac{131}{2}\Omega^2\Lambda - \frac{201}{8}\Omega\Lambda^2 + \frac{19}{2}\Lambda^3)\} \tag{38}$$

where

$$h = \frac{R - r}{2}(-\sin 2\psi), \quad \Lambda = \frac{z}{\rho}h$$

$$\Omega' = R\sqrt{1 - \gamma^2\sin^2\psi}, \quad \Omega = R[1 - \tau\sin^2\psi]$$

We see that odd terms in the integration variable ψ turn to zero, i.e., terms are proportional to Λ, Λ^3 quantities. The remaining integrals of the type

$$i=\int_0^{\pi/2}d\psi\frac{\sin^\nu\psi\cos^\mu\psi}{[1-\gamma^2\sin^2\psi]^n}$$

are calculated by means of the changing variable $\sin^2\psi=y$ and $dy=2\sin\psi\cos\psi d\psi$. They are reduced to the hypergeometric functions. For example,

$$I_5=\int_0^{\pi/2}d\psi\frac{\sin^2 2\psi}{[1-\gamma^2\sin^2\psi]^3}=$$

$$2\int_0^1 dy\frac{y^{\frac{3}{2}-1}(1-y)^{\frac{3}{2}-1}}{[1-\gamma^2 y]^3}=2B(\frac{3}{2},\frac{3}{2})F(3,\frac{3}{2};3;\gamma^2)$$

$$I_9=\int_0^{\pi/2}d\psi\frac{\sin^4\psi}{[1-\gamma^2\sin^2\psi]^4}$$

$$\frac{1}{2}\int_0^1 dy\frac{y^{\frac{5}{2}-1}(1-y)^{\frac{1}{2}-1}}{[1-\gamma^2 y]^4}=\frac{1}{2}B(\frac{5}{2},\frac{1}{2})F(4,\frac{5}{2};3;\gamma^2)$$

and so on. Here $B(x,y)=\Gamma(x)\Gamma(y)/\Gamma(x+y)$. Of course, some of these integrals are reduced to the elementary functions:

$$I_1=\int_0^{\pi/2}\frac{d\psi}{[1-\gamma^2\sin^2\psi]^3}=\frac{\pi}{16}y(3+2y^2+3y^4),$$

$$I_2=\int_0^{\pi/2}d\psi\frac{\sin^2\psi}{[1-\gamma^2\sin^2\psi]^3}=\frac{1}{2}\frac{d}{d\gamma^2}\int_0^{\pi/2}\frac{d\psi}{[1-\gamma^2\sin^2\psi]^2}=$$

$$\frac{1}{4}\frac{d}{d\gamma^2}\{\frac{1}{1-\gamma^2}[(2-\gamma^2)\frac{1}{\sqrt{1-\gamma^2}}\arctan(\sqrt{1-\gamma^2}\tan\psi)-$$

$$\frac{\gamma^2\sin\psi\cos\psi}{1-\gamma^2\sin^2\psi}]\}_0^{\pi/2}=\frac{\pi}{8}y^3[\frac{3}{2}y^2(1+y^{-2})-1]$$

and

$$I_8=\int_0^{\pi/2}d\psi\frac{\sin^2\psi}{[1-\gamma^2\sin^2\psi]^4}=\frac{1}{3}\frac{d}{d\gamma^2}\int_0^{\pi/2}\frac{d\psi}{[1-\gamma^2\sin^2\psi]^3}=$$

$$\frac{\pi}{16}y^3(\frac{5}{2}y^4+y^2+\frac{1}{2})$$

where $y=R/r$. Thus, after some tedious calculations, the potential (38) acquires the form

$$U_N(r_0\to 0)=U_N^{(1)}+U_N^{(2)} \tag{39}$$

where

$$U_N^{(1)}=\frac{m_{tor}}{8RE(\frac{\pi}{2},\gamma)}\{1-\frac{yr_0^2}{2R^2}+\frac{3}{16}\frac{z^2r^2}{R^2}F(2,\frac{3}{2};3;\gamma^2)+\frac{3\rho^2}{8R^2}\times$$

$$[\frac{1+y^{-2}}{y^3} - 2\tau y^3 + \frac{3}{4}\tau^2 F(2,\frac{5}{2};3;\gamma^2)]\} \tag{40}$$

and

$$U_N^{(2)} = \frac{m_{tori}}{8RE(\frac{\pi}{2},\gamma)}\{\frac{\rho^3}{R^3}[-3y-2y^3-3y^5+3\tau y^3(1+3y^2) - 9\tau^2 F(3,\frac{5}{2};3;\gamma^2)+$$

$$\frac{5}{2}\tau^3 F(3,\frac{7}{2};4;\gamma^2) - \tau^2\frac{z^2}{\rho^2}F(3,\frac{3}{2};3;\gamma^2) + F(3,\frac{5}{2};4;\gamma^2)\frac{1}{2}\frac{z^2\tau^3}{\rho^2}]+$$

$$\frac{3}{16}y^3(1+y^{-2})\frac{r_0^4}{R^4} + \frac{\rho^2 r_0^2}{32R^4}[-109(\frac{3}{2}y + y^3 + \frac{3}{2}y^5 - \tau y^3(1+y^2)+$$

$$\frac{3}{2}\tau^2 F(3,\frac{5}{2};3;\gamma^2)) - 15\frac{z^2\tau^2}{\rho^2}F(3,\frac{3}{2};3;\gamma^2)] + \frac{\rho^4}{R^4}[-127(F(4,\frac{1}{2};1;\gamma^2)-$$

$$\frac{\tau}{4}y^3(5y^4+2y^2+1) + \frac{9}{4}\tau^2 F(4,\frac{5}{2};3;\gamma^2) - \frac{5}{4}\tau^3 F(4,\frac{7}{2};4;\gamma^2)+$$

$$\frac{35}{128}\tau^4 F(4,\frac{9}{2};5;\gamma^2)) - \frac{201}{64}\tau^2\frac{z^2}{\rho^2}(F(4,\frac{3}{2};3;\gamma^2)-$$

$$\tau F(4,\frac{5}{2};4;\gamma^2) + \frac{5}{16}\tau^2 F(4,\frac{7}{2};5;\gamma^2))]\} \tag{41}$$

where $\tau = 1 - y^{-1}$, $y = R/r$.

In all the above cases, the Newtonian and Coulomb potentials of the torus are finite at the origin and possess oscillator behaviors in z - direction. While the oscillator or damping character of the torus potential in ρ - direction, i.e., on the OXY- plane depends upon the structure of the torus, i.e., on the parameter R/r. When the torus is thin ($r \to R$) the oscillator behavior of the torus potential takes place inside some cones defined by the spherical angle θ around z - axis. In this case, when the small radius r of the torus turns to zero the oscillator behaviors of its potentials are dominated in all directions of space at least for the mass distributed on the surface of the torus.

V. MOTION OF A PARTICLE IN THE TORUS POTENTIAL

A. Harmonic Oscillations near the Torus

From the previous section we have considered that, not far from the torus, its potential has oscillator character and therefore the force is given by

$$\mathbf{F} = -k\mathbf{r} \tag{42}$$

where the parameter k is different for z - direction and is the same for x - and y - directions. First of all, we consider motion of the particle in two dimensions in the potential force of the torus. Eq. (42) can be rewritten in polar coordinates into the components

$$F_x = -kr\cos\varphi = -kx$$

$$F_y = -kr\sin\varphi = -ky \tag{43}$$

Then the equations of motion takes the form

$$\ddot{x} + \omega_0^2 x = 0$$

$$\ddot{y} + \omega_0^2 y = 0 \tag{44}$$

The standard solutions are

$$x(t) = A\cos(\omega_0 t - \alpha)$$

$$y(t) = B\cos(\omega_0 t - \beta) \tag{45}$$

where $\omega_0 = \sqrt{k/m}$ and the parameter k is given in the previous sections and depends on the structure of the torus, i.e., it is a function of variable r/R. From solution (45) we can see that in the torus potential the motion of the particle is one of simple harmonic oscillation in each of the two directions, both oscillations having the same frequency but possibly, differing in phase. The equation for the trajectory of the particle is derived by eliminating the time t between the two equations (45). Let us take the standard method

$$y(t) = B\cos[\omega_0 t - \alpha + (\alpha - \beta)] =$$

$$B\cos(\omega_0 t - \alpha)\cos(\alpha - \beta) - B\sin(\omega_0 t - \alpha)\sin(\alpha - \beta)$$

Since $\cos(\omega_0 t - \alpha) = x/A$ and therefore

$$Ay - Bx\cos\delta = -B\sqrt{A^2 - x^2}\sin\delta \tag{46}$$

where $\delta = \alpha - \beta$ and upon squaring this equation becomes

$$A^2y^2 - 2ABxy\cos\delta + B^2x^2\cos^2\delta =$$

$$A^2B^2\sin^2\delta - B^2x^2\sin^2\delta$$

so that

$$B^2x^2 - 2ABxy\cos\delta + A^2y^2 = A^2B^2\sin^2\delta \tag{47}$$

Depending on the parameters δ and A, B we can obtain from this equation (47) straight lines, circles and ellipses for the trajectory of the particle. For example, if δ is set equal to $\pm\pi/2$ one gets

$$\frac{x^2}{A^2} + \frac{y^2}{B^2} = 1, \quad \delta = \pm\pi/2 \tag{48}$$

Further, if the amplitudes are equal, $A = B$, then

$$x^2 + y^2 = A^2; \quad \delta = \pm\pi/2, \quad A = B \tag{49}$$

that is the equation of a circle. Moreover, if the phase δ vanishes, then

$$B^2x^2 - 2ABxy + A^2y^2 = 0; \quad \delta = 0.$$

This is the equation of a straight line:

$$y = \frac{B}{A}x, \quad \delta = 0. \tag{50}$$

Similarly, the phase $\delta = \pm\pi$ gives the straight line of opposite slope:

$$y = -\frac{B}{A}x, \quad \delta = \pm\pi. \tag{51}$$

Thus forms of the path of the particle moving near the torus are very rich and interesting. We know that the frequencies for the motion in the $x-$ and z-directions or in the y- and z-directions are different, so that Eqs.(45) become

$$x(t) = A\cos(\omega_x t - \alpha)$$

$$z(t) = B\cos(\omega_z t - \beta) \tag{52}$$

or

$$y(t) = B\cos(\omega_y t - \alpha)$$

$$z(t) = C\cos(\omega_z t - \gamma) \tag{53}$$

Now the path of the motion is no longer an ellipse, but is a Lissajous curve (for example, see Marion, 1965 [7]). Such a curve will be closed if the motion repeats itself at regular intervals of time. This will be possible only if the frequencies ω_x and ω_z(or ω_y and ω_z) are commensurable, i.e., if $\omega_x \ / \ \omega_z$ or $\omega_y \ / \ \omega_z$ is a rational fraction. As seen above, it is almost impossible for the torus case, and therefore it seems that the ratio of the frequencies is not a rational fraction, the curve will be open: that is, the moving particle will never pass twice through the same point with the same velocity. Therefore, after a sufficiently long time has elapsed, the curve will pass arbitrary close to any given point lying within the rectangle $2A \times 2B$ or $2B \times 2C$ and will therefore "full" the rectangle .

In the general case, three-dimensional motion of the particle in the torus field can be analyzed in a similar manner.

B. A Particle Constrained to Move on the Surface of the Torus

Let us consider a particle of the mass m that is constrained to move on the surface of the torus. We distinguish two different coordinate systems. One of which belongs to inside torus and another coordinate system with origin at the center of mass of the torus, as above is fixed. Let us consider motion of the particle in the first coordinate system, the defining equation of which is

$$x^2 + y^2 = (\frac{R-r}{2})^2. \tag{54}$$

The particle is to be subject to a force directed toward the origin (in torus system of reference) and proportional to the distance of the particle from the origin

$$\mathbf{F} = -k_i \mathbf{r}$$

for x and y- directions $k_x = k_y = k$ and $k_z \neq k$ for z- direction. The potential energy is

$$U = \frac{1}{2}k(x^2 + y^2) + \frac{1}{2}k_z z^2 = \frac{1}{2}k((\frac{R-r}{2})^2 + \epsilon z^2)$$

where $\epsilon = k_z/k$. Moreover, $z = (\frac{R+r}{2})\varphi$, where φ is the polar angle in the fixed system of reference.

In the torus cylindrical coordinates the square of the velocity of the particle is

$$v^2 = \dot{\rho}^2 + \rho^2\dot{\alpha}^2 + \dot{z}^2 \tag{55}$$

where $\rho^2 = (\frac{R-r}{2})^2$ is a constant so that the kinetic energy is

$$T = \frac{1}{2}m[(\frac{R-r}{2})^2\dot{\alpha}^2 + (\frac{R+r}{2})^2\dot{\varphi}^2]. \tag{56}$$

We may now write the Lagrangian as

$$L = T - U = \frac{1}{2}m[(\frac{R-r}{2})^2\dot{\alpha}^2 + \dot{z}^2] - \frac{1}{2}k[(\frac{R-r}{2})^2 + z^2]. \tag{57}$$

The generalized coordinates are α and $z = \frac{R+r}{2}\varphi$ and the generalized momenta are

$$p_\alpha = \frac{\partial L}{\partial \dot{\alpha}} = m(\frac{R-r}{2})^2\dot{\alpha} \tag{58}$$

and

$$p_z = \frac{\partial L}{\partial \dot{z}} = m(\frac{R+r}{2})\dot{\varphi}. \tag{59}$$

The hamiltonian H is just the total energy expressed in terms of the variables α, p_α, z and p_z; but α does not occur explicitly, so that

$$H(z, p_\alpha, p_z) = T + U = \frac{p_\alpha^2}{2m(\frac{R-r}{2})^2} + \frac{p_z^2}{2m} + \frac{1}{2}k_z z^2 \tag{60}$$

where the constant term $\frac{1}{2}k(\frac{R-r}{2})^2$ has been suppressed. The equations of motion are therefore found from the standard canonical equations:

$$\dot{p}_\alpha = -\frac{\partial H}{\partial \alpha} = 0 \tag{61}$$

$$\dot{p}_z = -\frac{\partial H}{\partial z} = -k_z z \tag{62}$$

$$\dot{\alpha} = \frac{\partial H}{\partial p_\alpha} = \frac{p_\alpha}{m(\frac{R-r}{2})^2} \tag{63}$$

$$\dot{z} = \frac{\partial H}{\partial p_z} = \frac{p_z}{m}. \tag{64}$$

Eqs. (63) and (65) just duplicate Eqs. (58) and (59) . Equation (61) yields

$$p_\alpha = m(\frac{R-r}{2})^2\dot{\alpha} = const$$

so that the angular momentum about the z-axis is a constant of the motion; this result is assured since the z-axis is the symmetry axis of the problem. Combining Eqs. (59) and (62) we find

$$\ddot{z} + \omega_z^2 z = 0, \tag{65}$$

where

$$\omega_z^2 = k_z/m.$$

Therefore, the motion in the z-direction is simple harmonic.

Now it is interesting to study the above motion of the particle in the fixed coordinate system, where its coordinates are given by

$$x = L\cos\varphi$$
$$y = L\sin\varphi$$
$$z = (\frac{R-r}{2})\sin\alpha \tag{66}$$

where $L = r + (R-r)\sin^2\frac{\alpha}{2}$ as above. In this case, equations of the motion of the particle take the form

$$m(\frac{R-r}{2}\cos\alpha\ddot{\alpha} - \frac{R-r}{2}\sin\alpha\dot{\alpha}^2) = -k_z\frac{R-r}{2}\sin\alpha \tag{67}$$

$$\begin{aligned} m\{\dot{\varphi}^2[-\cos\varphi(r+(R-r)\sin^2\frac{\alpha}{2})] - \ddot{\varphi}\sin\varphi(r+(R-r)\sin^2\frac{\alpha}{2}) + \\ \cos\alpha\cos\varphi(R-r)\frac{\dot{\alpha}^2}{2} + \frac{1}{2}(R-r)\sin\alpha\cos\varphi\ddot{\alpha} - \\ (R-r)\sin\varphi\sin\alpha\dot{\varphi}\dot{\alpha}\} = -k\cos\varphi(r+(R-r)\sin^2\frac{\alpha}{2}), \end{aligned} \tag{68}$$

$$m\{\dot{\varphi}^2[-\sin\varphi(r+(R-r)\sin^2\frac{\alpha}{2})] + \ddot{\varphi}\cos\varphi(r+(R-r)\sin^2\frac{\alpha}{2}) +$$

$$\sin\varphi\cos\alpha(R-r)\frac{\dot{\alpha}^2}{2} + \frac{1}{2}\sin\varphi\sin\alpha(R-r)\ddot{\alpha} + (R-r)\cos\varphi\sin\alpha\dot{\alpha}\dot{\varphi}\} =$$

$$-k\sin\varphi(r+(R-r)\sin^2\frac{\alpha}{2}). \tag{69}$$

Multiplying Eq. (68) by $\sin\varphi$ and Eq. (69) by $\cos\varphi$ and subtracting the resulting equations from each other, one gets

$$m\{\ddot{\varphi}L+(R-r)\sin\alpha\dot{\varphi}\dot{\alpha}\}=0. \tag{70}$$

Similarly, multiplying Eq. (68) by $\cos\varphi$ and Eq. (69) by $\sin\varphi$ and adding the obtained equations, the result reads

$$m\{-\dot{\varphi}^2L+\cos\alpha(R-r)\frac{\dot{\alpha}^2}{2}+\frac{1}{2}\sin\alpha(R-r)\ddot{\alpha}\}=-kL. \tag{71}$$

A similar action for Eqs. (67) and (71) gives

$$m\{-\dot{\varphi}^2L\cos\alpha+(R-r)\frac{\dot{\alpha}^2}{2}\}=-k\cos\alpha L+k_z\frac{R-r}{2}\sin^2\alpha. \tag{72}$$

Eqs. (70) and (72) are the equations of motion of the particle that is constrained to move on the surface of the torus in the fixed coordinate system. It turns out that these two equations (70) and (72) are integrated by means of elementary functions. Indeed, from Eq. (70) one gets

$$\frac{d}{dt}\{\ln[\dot{\varphi}(r+(R-r)\sin^2\frac{\alpha}{2})^2]\}=0$$

and therefore

$$\dot{\varphi}=L^{-2}e^d,\quad L=r+(R-r)\sin^2\frac{\alpha}{2} \tag{73}$$

where d is a constant which is defined by an initial condition of the problem. Substituting solution (73) into Eq. (72) yields

$$t(\alpha)=\int\frac{d\alpha}{\sqrt{(k_z/m)\sin^2\alpha-(k/m)\frac{2L}{R-r}\cos\alpha+\frac{2\cos\alpha\cdot e^{2d}}{(R-r)L^3}}}=\pm t+C, \tag{74}$$

where C is an integration constant.

This integral may (formally, at least) be inverted to obtain $\alpha(t)$, which in turn, may be substituted into Eq. (73) to yield $\varphi(t)$. Since the angles α and φ completely specify the orientation of the particle moving in the surface of torus, the results for $\alpha(t)$ and $\varphi(t)$ constitute a complete solution for the problem.

VI. DYNAMICS OF THE RIGID TORUS

A. The Inertia Tensor of the Torus

The main characteristics of rigid bodies are the inertia tensor and by means of which we can construct dynamics of bodies. It is interesting to calculate the inertia tensor of the torus. We now turn to this problem. By general definition, the torus as a continuous distribution of matter with mass volume density $\sigma(\mathbf{r})$ possesses the inertia tensor, which is given by

$$I_{ij} = \int_{V_t} \sigma(\mathbf{r})[\delta_{ij} \sum\nolimits_k x_k^2 - x_i x_j] dv \tag{75}$$

where $dv = dx_1 dx_2 dx_3$ is the element of volume at the position defined by the vector $\mathbf{r}$, and V_t is the volume of the torus.

First, let us consider the case when mass of the particle is distributed on the surface of the torus with the uniform surface density $\lambda(\mathbf{r}) = \lambda$.

The coordinates of the vector $\mathbf{r}$ are

$$x_1 = L\cos\varphi, \quad x_2 = L\sin\varphi, \quad x_3 = \frac{R-r}{2}\sin\alpha, \tag{76}$$

where $L = r + \frac{R-r}{2}(1-\cos\alpha) = r + (R-r)\sin^2\frac{\alpha}{2}$, as before, then, by definition

$$I_{33} = \oint_\Sigma ds\lambda(\mathbf{r})[x_1^2 + x_2^2 + x_3^2 - x_3^2] = \lambda \oint_\Sigma d\alpha d\varphi(\frac{R-r}{2})\sqrt{L^2+h^2}L^2, \tag{77}$$

where $h = \frac{R-r}{2}\sin\alpha$. Other components of the inertia tensor of the torus are defined as

$$\begin{aligned}
I_{22} &= \oint_\Sigma ds\lambda(\mathbf{r})[x_1^2 + x_3^2] = \lambda \oint_\Sigma ds[L^2\cos^2\varphi + \frac{(R-r)^2}{4}\sin^2\alpha] \\
I_{11} &= \oint_\Sigma ds\lambda(\mathbf{r})[x_2^2 + x_3^2] = \lambda \oint_\Sigma ds[L^2\sin^2\varphi + \frac{(R-r)^2}{4}\sin^2\alpha] \\
I_{12} &= -\oint_\Sigma ds\lambda(\mathbf{r})x_1x_2 = -\lambda\oint_\Sigma dsL^2\cos\varphi\sin\varphi \\
I_{13} &= -\oint_\Sigma ds\lambda(\mathbf{r})x_1x_3 = -\lambda\oint_\Sigma dsL(\frac{R-r}{2})\cos\varphi\sin\alpha \\
I_{23} &= -\oint_\Sigma ds\lambda(\mathbf{r})x_2x_3 = -\lambda\oint_\Sigma dsL(\frac{R-r}{2})\sin\varphi\sin\alpha
\end{aligned} \tag{78}$$

Using the results of the previous section and changing the interaction variable $\alpha = 2\psi + \pi$, one gets

$$I_{33} = 4\pi\lambda R(R-r)\int_0^{\pi/2} d\psi\sqrt{1-\gamma^2\sin^2\psi}(r^2 + 2r(R-r)\cos^2\psi + (R-r)^2\cos^4\psi) =$$

$$m_{tori}\{r^2 + \frac{2}{3}\frac{r(R-r)}{\gamma^2}[2 - \frac{r^2}{R^2}(1+O)] +$$

$$\frac{(R-r)^2}{15\gamma^4}[3\gamma^4 + 7\gamma^2 - 2 + 2\frac{r^2}{R^2}(1-3\gamma^2)O]\} \tag{79}$$

Similar calculations read

$$I_{22} = \frac{1}{2}I_{33} + \frac{(R-r)^2}{15\gamma^4}m_{tori}[2(\gamma^4 - \gamma^2 + 1) - \frac{r^2}{R^2}(2-\gamma^2)O], \tag{80}$$

$$I_{11} = \frac{1}{2}I_{33} + \frac{(R-r)^2}{15\gamma^4}m_{tori}[2(\gamma^4 - \gamma^2 + 1) - \frac{r^2}{R^2}(2-\gamma^2)O], \tag{81}$$

where $O = F(\frac{\pi}{2}, \gamma)/E(\frac{\pi}{2}, \gamma)$, and

$$I_{12} = I_{13} = I_{23} = 0.$$

Thus, the inertia tensor of the torus is diagonal

$$I_{ij} = I_i \delta_{ij} = \begin{pmatrix} I_1 & 0 & 0 \\ 0 & I_2 & 0 \\ 0 & 0 & I_3 \end{pmatrix},$$

where $I_1 = I_{11}$, $I_2 = I_{22}$ and $I_3 = I_{33}$. Then its angular momentum is simple

$$L_i = \sum\nolimits_j I_i \delta_{ij} \omega_j = I_i \omega_i, \tag{82}$$

where $\vec{\omega}$ is the angular velocity of the torus, and the rotational kinetic energy is

$$T_{rot} = \frac{1}{2} \sum\nolimits_{ij} I_i \delta_{ij} \omega_i \omega_j = \frac{1}{2} \sum\nolimits_i I_i \omega_i^2.$$

Second, we would like to evaluate the inertia tensor of the torus when mass of the particle is distributed uniformly in its volume. Here, we can perform similar calculations to obtain I'_{ij}. Their explicit forms are:

$$I'_3 = I'_{33} = \int_{V_t} dv \sigma(\mathbf{r}) [L^2 \cos^2 \varphi + L^2 \sin^2 \varphi] =$$

$$2\pi\sigma R(R-r)^2 \int_0^{\pi/2} d\psi \sin 2\psi \sqrt{1 - \gamma^2 \sin^2 \psi} [r^2 + 2r(R-r)\cos^2 \psi + (R-r)^2 \cos^4 \psi] =$$

$$m'_{tori} \{ r^2 + \frac{2rR}{5\gamma^2(1+g+g^2)} [2g^5 - 5g^2 + 3] +$$

$$\frac{R^2(1-g)}{35\gamma^4(1+g+g^2)} [15\gamma^4 - 8g^7 - 4g^2(3 - 5g^2)] \} \tag{83}$$

and

$$I'_{11} = I'_{22} = \frac{1}{2} I'_{33}, \tag{84}$$

where $g = \frac{r}{R} = y^{-1}$. Here nondiagonal elements of the inertia tensor of the torus are also absent.

B. Euler's Equations for the Rigid Torus

Since the inertia tensor of the torus is diagonal, its rotational kinetic energy T is

$$T = \frac{1}{2} \sum\nolimits_i I_i \omega_i^2. \tag{85}$$

If we choose the Eulerian angles (φ, θ, ψ) as the generalized coordinates, then the Lagrange equation for the coordinate ψ is

$$\frac{\partial T}{\partial \psi} - \frac{d}{dt}\frac{\partial T}{\partial \dot{\psi}}, \tag{86}$$

which may be expressed as

$$\sum_i \frac{\partial T}{\partial \omega_i}\frac{\partial \omega_i}{\partial \psi} - \frac{d}{dt}\sum_i \frac{\partial T}{\partial \omega_i}\frac{\partial \omega_i}{\partial \dot{\psi}}. \tag{87}$$

If we differentiate the components of the $\vec{\omega}$ which is expressed by the Eulerian angles:

$$\begin{aligned}
\omega_1 &= \dot{\varphi}_1 + \dot{\theta}_1 + \dot{\psi}_1 = \dot{\varphi}\sin\theta\sin\psi + \dot{\theta}\cos\psi, \\
\omega_2 &= \dot{\varphi}_2 + \dot{\theta}_2 + \dot{\psi}_2 = \dot{\varphi}\sin\theta\cos\psi - \dot{\theta}\sin\psi, \\
\omega_3 &= \dot{\varphi}_3 + \dot{\theta}_3 + \dot{\psi}_3 = \dot{\varphi}\cos\theta + \dot{\psi}
\end{aligned} \tag{88}$$

with respect to ψ and $\dot{\psi}$ we have

$$\begin{aligned}
\frac{\partial \omega_1}{\partial \psi} &= \dot{\varphi}\sin\theta\cos\psi - \dot{\theta}\sin\psi = \omega_2, \\
\frac{\partial \omega_2}{\partial \psi} &= -\dot{\varphi}\sin\theta\sin\psi - \dot{\theta}\cos\psi = -\omega_1, \\
\frac{\partial \omega_3}{\partial \psi} &= 0,
\end{aligned} \tag{89}$$

and,

$$\begin{aligned}
\frac{\partial \omega_1}{\partial \dot{\psi}} = \frac{\partial \omega_2}{\partial \dot{\psi}} &= 0, \\
\frac{\partial \omega_3}{\partial \dot{\psi}} &= 1.
\end{aligned} \tag{90}$$

From Eq. (85) we also have

$$\frac{\partial T}{\partial \omega_i} = I_i \omega_i. \tag{91}$$

Therefore, Eq. (87) reads

$$I_1\omega_1\omega_2 + I_2\omega_2(-\omega_1) - \frac{d}{dt}I_3\omega_3 = 0$$

or,

$$(I_1 - I_2)\omega_1\omega_2 - I_3\dot{\omega}_3 = 0. \tag{92}$$

Since the designation of any particular axis as the x_3 -axis is entirely arbitrary, Eq. (92) may be permuted to obtain relations for $\dot{\omega}_1$ and $\dot{\omega}_2$. By making use of the permutation symbol, we may write in general

$$(I_i - I_j)\omega_i\omega_j - \sum_k I_k\dot{\omega}_k\epsilon_{ijk} = 0. \tag{93}$$

These three equations are Euler's equations for the rigid torus for the case of force-free motion.

According to the standard procedure in order to obtain Euler's equations for the case of motion of the torus in a force field, one can use the fundamental relation for the torque N in the classical mechanics dynamics:

$$(\frac{\partial \mathbf{L}}{\partial t})_{fixed} = \mathbf{N}, \tag{94}$$

where symbol "fixed" has been explicitly appended to $\dot{\mathbf{L}}$ since their relation is obtained from Newtonian equation and is therefore valid only in an inertial frame of reference. In the fixed and the body coordinate systems there is connection

$$(\frac{\partial \mathbf{L}}{\partial t})_{fixed} = (\frac{\partial \mathbf{L}}{\partial t})_{body} + \vec{\omega} \times \vec{L} \tag{95}$$

or

$$(\frac{\partial \mathbf{L}}{\partial t})_{body} + \vec{\omega} \times \vec{L} = \mathbf{N} \tag{96}$$

The component of this equation along the body x_3 -axis is

$$\dot{L}_3 + \omega_1 L_2 - \omega_2 L_1 = N_3. \tag{97}$$

But since in this coordinate system the inertia tensor of the torus is diagonal, we have from Eq. (82)

$$L_i = I_i \omega_i,$$

so that

$$I_3 \dot{\omega}_3 - (I_1 - I_2)\omega_1 \omega_2 = N_3$$

or, in general

$$(I_i - I_j)\omega_i \omega_j - \sum\nolimits_k (I_k \dot{\omega}_k - N_k)\epsilon_{ijk} = 0, \tag{98}$$

which are the desired Euler's equations for the motion of the torus in a force field.

Notice that the motion of the rigid torus depends on its structure only through the three numbers I_1, I_2 and I_3.

VII. FORCE-FREE MOTION OF THE RIGID TORUS

Let us consider the rigid torus with $I_1 = I_2 \neq I_3$, then the force-free Euler's equations (Eq. (93)) become

$$\begin{aligned} (\Lambda_{12} - I_3)\omega_2\omega_3 - \Lambda_{12}\dot{\omega}_1 &= 0, \\ (I_3 - \Lambda_{12})\omega_3\omega_1 - \Lambda_{12}\dot{\omega}_2 &= 0, \\ I_3\dot{\omega}_2 &= 0. \end{aligned} \tag{99}$$

Here Λ_{12} has been substituted for both I_1 and I_2. Since for force-free motion, the center of mass of the torus is either at rest or in uniform motion with respect to the fixed or inertial frame of reference, one can, without loss of generality, specify that the center of mass of the torus is at rest and located at the origin of the fixed coordinate system. This is a standard requirement.

It is natural that a third of Eqs. (99) is reduced

$$\omega_3(t) = const. \tag{100}$$

Then the other two equations in (99) may be written as

$$\begin{aligned}\dot{\omega}_1 &= -[\frac{I_3 - \Lambda_{12}}{\Lambda_{12}}\omega_3]\omega_2,\\ \dot{\omega}_2 &= -[\frac{I_3 - \Lambda_{12}}{\Lambda_{12}}\omega_3]\omega_1.\end{aligned} \tag{101}$$

Seeing that the terms in the brackets are equal and composed of constants, one can define

$$\Omega \equiv \frac{I_3 - \Lambda_{12}}{\Lambda_{12}}\omega_3, \tag{102}$$

so that

$$\begin{aligned}\dot{\omega}_1 + \Omega\omega_2 &= 0,\\ \dot{\omega}_2 - \Omega\omega_1 &= 0.\end{aligned} \tag{103}$$

Multiplying the second equation by i and adding it to the first one, we have

$$(\dot{\omega}_1 + i\dot{\omega}_2) - i\Omega(\omega_1 + i\omega_2) = 0 \tag{104}$$

or, defining $\Gamma = \omega_1 + i\omega_2$, then

$$\dot{\Gamma} - i\Omega\Gamma = 0, \tag{105}$$

solution of which is

$$\Gamma(t) = Ae^{i\Omega t}. \tag{106}$$

Thus

$$\omega_1 + i\omega_2 = A\cos\Omega t + iA\sin\Omega t$$

and therefore

$$\begin{aligned}\omega_1(t) &= A\cos\Omega t,\\ \omega_2(t) &= A\sin\Omega t.\end{aligned} \tag{107}$$

Since $\omega_3 = const$, and the value of $\vec{\omega}$ is also constant:

$$|\vec{\omega}| = \omega = \sqrt{\omega_1^2 + \omega_2^2 + \omega_3^2} = \sqrt{A^2 + \omega_3^2} = const.$$

Equations (107) are the parametric equations of a circle, so that the projection of the vector $\vec{\omega}$ onto the $x_1 - x_2$ plane describes a circle with time, as shown in Fig.2.

The x_3 -axis is the symmetry axis of the torus, so we find that the angular velocity vector $\vec{\omega}$ revolves or precesses about the torus x_3 -axis with a constant angular frequency Ω. Thus, to an observer in the torus coordinate system $\vec{\omega}$ traces out a cone about the torus symmetry axis.

For force-free motion of the torus, the angular momentum vector **L** is stationary in the fixed coordinate system and is constant in time. The rotational kinetic energy of the torus is also constant

$$T_{rot} = \frac{1}{2}\vec{\omega} \cdot \vec{L} = const. \tag{108}$$

Since $\mathbf{L} = const$, $\vec{\omega}$ must move in such a manner that its projection on the stationary angular momentum vector is constant. Thus, $\vec{\omega}$ precesses about the x_3 -axis and makes a constant angle with the vector **L**. If one can stipulate that the x'_3 -axis in the fixed coordinate system coincide with **L**, then to an observer in the fixed system $\vec{\omega}$ traces out a cone about the fixed x'_3-axis. The situation is then described as in Figure 3, by one cone rolling on another, such that $\vec{\omega}$ precesses about the x_3 -axis in the torus system and the x'_3 -axis (or **L**) in the fixed system.

The rate at which $\vec{\omega}$ precesses about the torus symmetry axis is given by (102)

$$\Omega = \frac{I_3 - \Lambda_{12}}{\Lambda_{12}}\omega_3.$$

For the thin torus (when $r \to R$) $\Lambda_{12} \sim I_3$, then Ω becomes very small compared with ω_3.

VIII. THE MOTION OF THE TORUS WITH ONE POINT FIXED BY MEANS OF TWO MASSLESS RIGID THREADS

This problem is exactly the same as the motion of a symmetrical top with one point fixed, dynamics of which is described in Marion (1965) [7]. We expound here only the main conclusions about this problem.

For the motion of the torus it is able to separate the kinetic energy into translational and rotational parts by taking the center of mass of the torus to be the origin of the rotating or torus coordinate system. Alternatively if it is possible to choose the origins of the fixed and the torus coordnate systems to coincide, then the translational kinetic energy will vanish, since $\mathbf{V} = \dot{\mathbf{R}} = 0$. Such a choice is quite convenient for the discussion of the torus motion. In the translational coordinate system in z -direction, i.e., in the symmetric direction of the torus its inertia tensor becomes always diagonal. Indeed, according to Steiner's parallel-axis theorem the inertia tensor of a body in two different coordnate systems origins of which are separated by translational vector **a** is given by

$$J_{ij} = I_{ij} + M[a^2\delta_{ij} - a_i a_j],$$

where I_{ij} is the inertia tensor of a body in a coordinate system with origin at the center of mass. Let $\mathbf{h} = (h_x, h_y, h_z) = (0, 0, h)$ be translational vector in z -direction, then the inertia tensor of the torus in this coordinate system is

$$J_{11} = I_{11} + Mh^2, \quad J_{22} = I_{11} + Mh^2,$$

$$J_{33} = I_{33}, \quad J_{12} = I_{13} = I_{23} = 0,$$

where I_{11}, I_{22} and I_{33} are defined by the formulas (79) - (81), (83) and (84). The Euler angles for this situation are shown in Figure 4. The x_3' -axis corresponds to the vertical and the x_3 -(torus)axis is chosen to be the symmetry axis of the torus. The distance from the fixed tip to the center of mass is h and the mass of the torus is M_{tori}. For the torus $J_1 = J_2 \equiv \Lambda'_{12}$. Then the kinetic energy is given by

$$T = \frac{1}{2}\sum_i J_i\omega_i^2 = \frac{1}{2}\Lambda'_{12}(\omega_1^2 + \omega_2^2) + \frac{1}{2}J_3\omega_3^2 \tag{109}$$

According to Eqs. (88) one gets

$$\begin{aligned} \omega_1^2 = (\dot\varphi\sin\theta\sin\psi + \dot\theta\cos\psi)^2 = \\ \dot\varphi^2\sin^2\theta\sin^2\psi + 2\dot\varphi\dot\theta\sin\theta\sin\psi\cos\psi + \dot\theta^2\cos^2\psi, \\ \omega_2^2 = (\dot\varphi\sin\theta\cos\psi - \dot\theta\sin\psi)^2 = \\ \dot\varphi^2\sin^2\theta\cos^2\psi - 2\dot\varphi\dot\theta\sin\theta\sin\psi\cos\psi + \dot\theta^2\sin^2\psi, \end{aligned} \tag{110}$$

so that

$$\omega_1^2 + \omega_2^2 = \dot\varphi^2\sin^2\theta + \dot\theta^2. \tag{111}$$

Also,

$$\omega_3^2 = (\dot\varphi\cos\theta + \dot\psi)^2. \tag{112}$$

Then, the kinetic energy acquires the form

$$T = \frac{1}{2}\Lambda'_{12}(\dot\varphi^2\sin^2\theta + \dot\theta^2) + \frac{1}{2}J_3(\dot\varphi\cos\theta + \dot\psi)^2. \tag{113}$$

Since the potential energy is $M_{tori}gh\cos\theta$, the Lagrangian becomes

$$L = \frac{1}{2}\Lambda'_{12}(\dot\varphi^2\sin^2\theta + \dot\theta^2) + \frac{1}{2}J_3(\dot\varphi\cos\theta + \dot\psi)^2 - M_{tori}gh\cos\theta. \tag{114}$$

The Lagrangian is cyclic in both the φ- and ψ - coordinates. The momenta conjugate to these coordinates are therefore constants of the motion:

$$P_\varphi = \frac{\partial L}{\partial\dot\varphi} = (\Lambda'_{12}\sin^2\theta + J_3\cos^2\theta)\dot\varphi + J_3\dot\psi\cos\theta = const, \tag{115}$$

$$P_\psi = \frac{\partial L}{\partial\dot\psi} = J_3(\dot\psi + \dot\varphi\cos\theta) = const. \tag{116}$$

Equations (115) and (116) may be solved for $\dot\varphi$ and $\dot\psi$ in terms of θ. From Eq. (116) one gets

$$\dot{\psi} = \frac{P_\psi - J_3 \dot{\varphi} \cos\theta}{J_3} \tag{117}$$

and substituting this expression into Eq. (115) we define

$$(\Lambda'_{12} \sin^2\theta + J_3 \cos^2\theta)\dot{\varphi} + (P_\varphi - J_3\dot{\varphi}\cos\theta)\cos\theta = P_\varphi$$

or

$$(\Lambda'_{12} \sin^2\theta)\dot{\varphi} + P_\psi \cos\theta = P_\varphi$$

so that

$$\dot{\varphi} = \frac{P_\varphi - P_\psi \cos\theta}{\Lambda'_{12} \sin^2\theta}. \tag{118}$$

Inserting this quantity into Eq. (117) reads

$$\dot{\psi} = \frac{P_\psi}{J_3} - \frac{(P_\varphi - P_\psi \cos\theta)\cos\theta}{\Lambda'_{12}\sin^2\theta} \tag{119}$$

Assuming that our system is conservative and therefore the total energy is a constant of the motion

$$E = \frac{1}{2}\Lambda'_{12}(\dot{\varphi}^2 \sin^2\theta + \dot{\theta}^2) + \frac{1}{2}J_3\omega_3^2 + M_{tori}gh\cos\theta = const. \tag{120}$$

Making use of Eq. (112) one can write Eq. (116) as

$$P_\psi = J_3\omega_3 = const \tag{121}$$

or

$$J_3\omega_3^2 = \frac{P_\psi^2}{J_3} = const. \tag{122}$$

Therefore, not only is E a constant of the motion, but so is $E - \frac{1}{2}J_3\omega_3^2$; we denote this quantity by E':

$$E' = E - \frac{1}{2}J_3\omega_3^2 = \frac{1}{2}\Lambda'_{12}(\dot{\varphi}^2\sin^2\theta + \dot{\theta}^2) + M_{tori}gh\cos\theta = const. \tag{123}$$

Substituting into this equation the expression for $\dot{\varphi}$ [Eq. (118)], we have

$$E' = \Lambda'_{12}\dot{\theta}^2 + \frac{(P_\varphi - P_\psi\cos\theta)^2}{2\Lambda'_{12}\sin^2\theta} + M_{tori}gh\cos\theta \tag{124}$$

or

$$E' = \frac{1}{2}\Lambda'_{12}\dot{\theta}^2 + N(\theta), \tag{125}$$

where $N(\theta)$ is an "effective potential" given by

$$N(\theta) = \frac{(P_\varphi - P_\psi\cos\theta)^2}{2\Lambda'_{12}\sin^2\theta} + M_{tori}gh\cos\theta. \tag{126}$$

Equation (125) may be solved to yield $\Gamma(\theta)$:

$$\Gamma(\theta) = \int \frac{d\theta}{\sqrt{(2/\Lambda'_{12})[E' - N(\theta)]}}. \tag{127}$$

This integral may be solved to obtain $\theta(t)$, which, in turn, may be substituted into Eqs. (118) and (119) to give $\varphi(t)$ and $\psi(t)$. Since the Euler angles θ, φ, ψ completely specify the orientation of the torus, the results for $\theta(t)$, $\varphi(t)$ and $\psi(t)$ constitute a complete solution for the problem.

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Figure Captions

Figure 1. The surface differential element (protuberant trapezium) on the circular torus, center of which belongs to the point N.

Figure 2. The projection of the angular vector velocity $\vec{\omega}$ of the torus onto the $x_1 - x_2$ plane describes a circle with time.

Figure 3. One cone rolling on another, such that the angular velocity $\vec{\omega}$ of the torus precesses about the x_3 - axis in the torus system and the x_3' (or bfL) in the fixed system.

Figure 4. The Euler angles for the motion of the torus with one point is held fixed by means of two massless rigid threads.

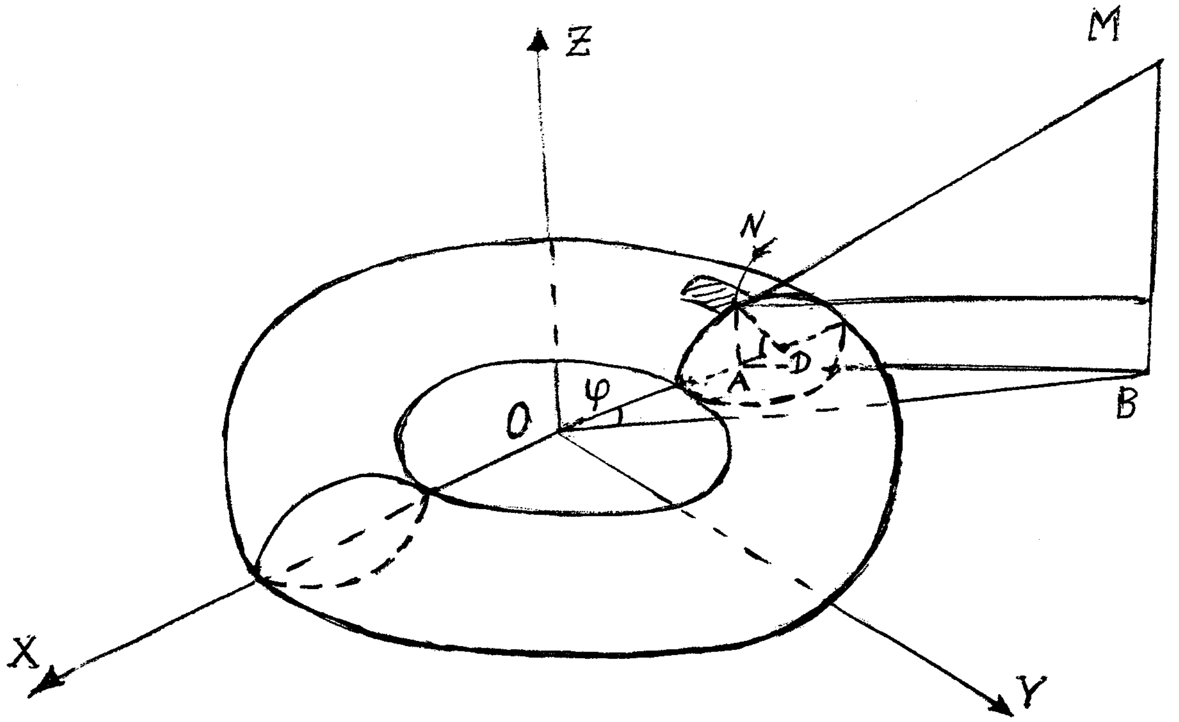


Figure 1

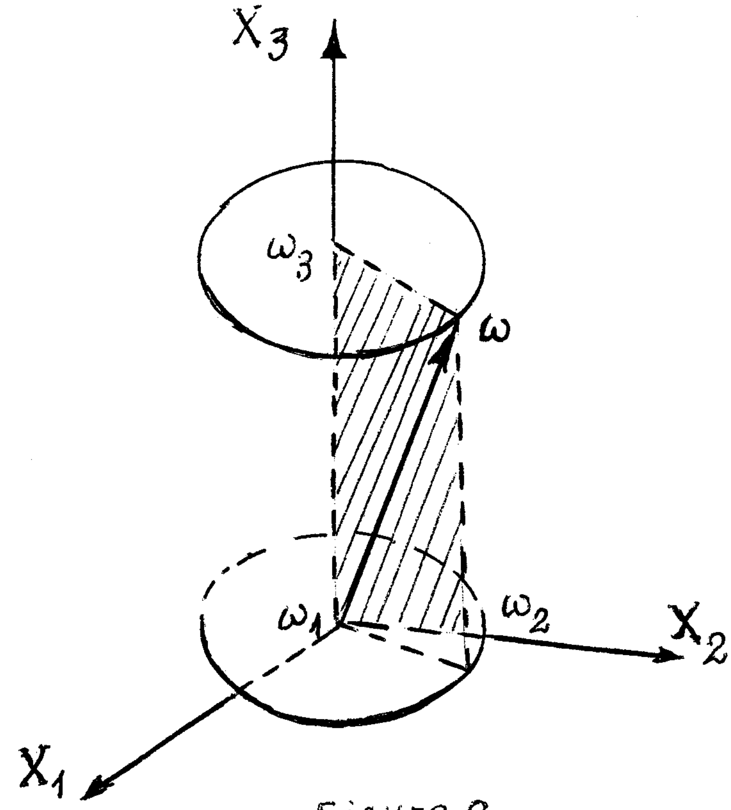


Figure 2.

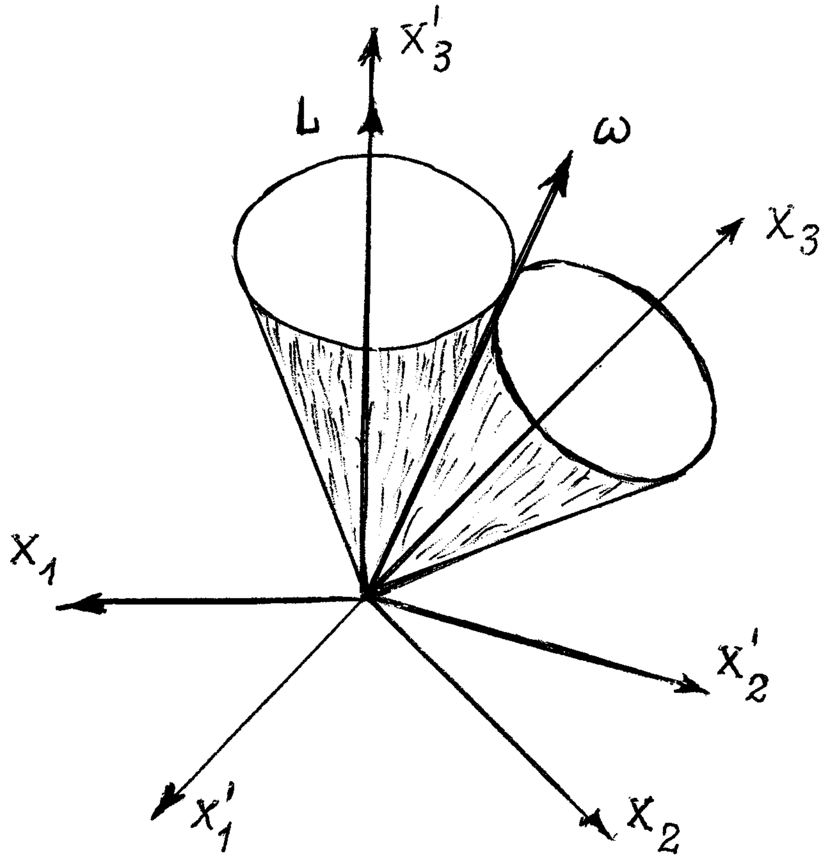
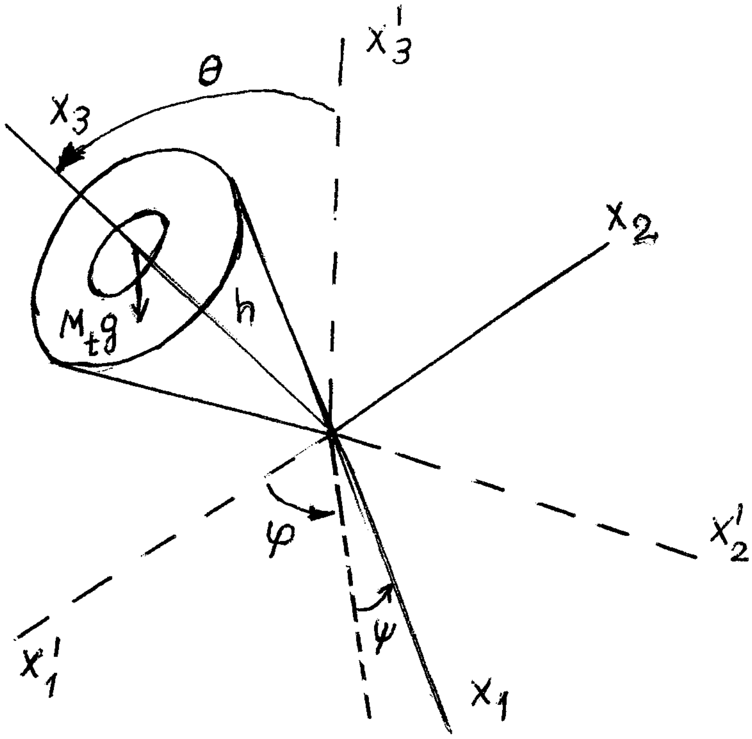


Figure 3.



Line of nodes

Figure 4.