A Z₃-graded harmonic oscillator and its coherent states

M. El Baz
Faculté des Sciences, Département de Physique, LPT-ICAC,
Av. Ibn Battouta, B.P. 1014, Agdal, Rabat, Morocco,

and

Y. Hassouni
Faculté des Sciences, Département de Physique, LPT-ICAC,
Av. Ibn Battouta, B.P. 1014, Agdal, Rabat, Morocco

The Abdus Salam International Centre for Theoretical Physics, Trieste, Italy.

Abstract

A Z₃-graded harmonic oscillator is introduced. Its coherent states are also constructed to allow a deep study of it.

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1E-mail: moreagl@yahoo.co.uk
2Regular Associate of the Abdus Salam ICTP. E-mail: Y-hassou@fsr.ac.ma
1 Introduction

Recently Kerner [Kerner (1993), Kerner (1996), Kerner (2000)] investigated on the use of $Z_3$-graded structures instead of $Z_2$-graded ones in physics. This leads to some interesting results when gauge theories are constructed using these structures.

In this letter we construct a $Z_3$-graded harmonic oscillator together with the coherent states associated to it. This will prove useful when the physical meaning of such structures is under discussion.

We begin by reviewing the $Z_3$-graded Grassman algebra and its properties. In section 3 we introduce what we shall call a $Z_3$-graded harmonic oscillator through the so-called k-fermions. In section 4 we discuss the relationship between the Grassman algebra of section 2 and the harmonic oscillator of section 3. This will allow us in section 5 to construct the coherent states for this system without any ambiguities.

2 $Z_3$-graded Grassman algebra

$Z_3$ is the cyclic group of three elements. It can be represented in the complex plane as multiplication by the primary cubic root of unity $j = e^{2\pi i / 3}$, $j^2$ and $j^3 = 1$. The analogue of the $Z_2$-graded Grassman algebra can be introduced as follows [Kerner (1996)]:

It is an associative algebra spanned by $N$ generators $\xi_a; \ a = 0, 1, \ldots, N$, between which, only ternary relations exists. By this we mean that the binary products of any such elements are considered independent [Kerner (1993)] ($\xi_a\xi_b$ is independent of $\xi_b\xi_a$ where $a, b = 0, 1, \ldots, N$).

Instead of this, the analogue of the anti-commutation in the $Z_2$-graded case is given by the following ternary relations:

\[ \xi_a\xi_b\xi_c = j\xi_b\xi_c\xi_a = j^2\xi_c\xi_a\xi_b \quad a, b, c = 0, 1, \ldots, N \]  \hspace{1cm} (1)

From this, two important properties follows automatically [Kerner (1996)]:

- The cubic power (or higher) of any generator must vanish

\[ (\xi_a)^3 = 0 \]  \hspace{1cm} (2)

- Any product of four or more generators must also vanish

\[ \xi_a\xi_b\xi_c\xi_d = 0 \quad a, b, c, d = 0, 1, \ldots, N \]  \hspace{1cm} (3)

At this level one remarks that there's no symmetry between grade-1 elements (the $\xi$'s) and grade-2 elements (the $\xi\xi$'s). Normally these elements should play a symmetric role with regard to $j$ and $j^2$. This symmetry is restored in the most natural way, by adding $N\ grade$-2 generators $\bar{\xi}_a$ (the duals of $\xi_a$). These added elements satisfy the same relations as the $\xi$'s, but with $j^2$ instead of $j$

\[ \bar{\xi}_a\bar{\xi}_b\bar{\xi}_c = j^2\bar{\xi}_b\bar{\xi}_c\bar{\xi}_a \]  \hspace{1cm} (4)
and their binary product with the $\xi$'s:

$$\xi_a \xi_b = j \xi_b \xi_a$$

(5)

Now, by requiring that the grades adds (modulo 3) up to multiplication and that grade-0 elements commute with all the other elements, and that grade-1 with grade-2 elements satisfy the same relation as in (5), then many additional terms must vanish (terms like $\xi_a \xi_b \xi_c$ for example). The algebra then, contains elements of the form[Kerner (1996)]:

Grade-0 : $I, \, \xi \xi, \, \xi \xi \xi, \, \xi \xi \xi \xi$

Grade-1 : $\xi, \, \xi \xi$

Grade-2 : $\xi, \, \xi \xi$

and its dimension is: $D = \frac{3 + 4N + 9N^2 + 2N^3}{3}$

For additional properties and proofs see [Kerner (1996)].

The case $N = 1$ was considered in detail in [Chung (1994)], however the author there considered also binary relations between elements of the same grade, and the arguments for this are not too convincing. In the following sections we are going to investigate on the same case, however, without imposing such binary relations. This will be more consistent with what we announced previously based on Kerner’s works [Kerner (1993), Kerner (1996), Kerner (2000)]

3 \textbf{Z}_3\text{-graded Harmonic Oscillator}

The most natural way to introduce a $Z_3$-graded harmonic oscillator is through the so-called $k$-fermionic oscillators. These in turn can be obtained as a special case of the deformed harmonic oscillator algebra generated by the operators $a, a^+, N$. These operators satisfy the following commutation relations:

$$aa^+ - qa^+ a = q^{-N}$$

$$Na - aN = -a$$

$$Na^+ - a^+ N = a^+$$

$$q^N a^+ = a^+ q^{N+1}$$

$$q^N a = aq^{N-1}$$

(6)

where $q$ is an arbitrary complex parameter of deformation.

Now when $q$ is the primary $k^\text{th}$-root of unity (i.e. $q = e^{\frac{2\pi i}{k}}$), one can prove that the annihilation and creation operators $a$ and $a^+$ are nilpotent of degree $k$ [Mansour (1996)]

$$ (a)^k = 0 \; , \; (a^+)^k = 0 $$

(7)

This means that no more than $k - 1$ $k$-fermions are allowed to occupy the same state. This clearly generalizes the Pauli exclusion principle.
Comparing this with (2), leads to interpret this oscillator as a $Z_3$-graded oscillator for $k = 3$ (a $Z_k$-graded oscillator in general).

The Fock space representation of this algebra is given by:

$$
a|n> = \sqrt{|n|}|n-1>
$$
$$a^+|n> = \sqrt{|n+1|}|n+1>
$$
$$N|n> = n|n>
$$

where $|n>; n = 0, 1,..., k$ is the usual Fock space orthonormal basis.

and

$$[n] = \frac{q^n - q^{-n}}{q - q^{-1}}$$

At this point we stress the fact that for the case we are considering ($q$ root of unity) $a^+$ is really the hermitian conjugate of $a$. This reflects itself in the representation (8), where $[n] = \bar{[n]}$ which is not the case for a generic $q$. One then has to introduce two other operators in the algebra, a creator (hermitian conjugate to $a$) and an annihilator (hermitian conjugate to $a^+$), but this, we don’t have to, here.

4 $Z_3$-graded grassman variables and the $Z_3$-graded harmonic oscillator:

Before beginning with the construction of coherent states associated to the $Z_3$-graded harmonic oscillator we should define the behavior of the $Z_3$ Grassmanian variables introduced in section 2 with the $Z_3$ harmonic oscillator’s operators introduced in section 3.

We shall begin by unifying the notations: so we use $q$ instead of $j$ used in section 2.

According to [Chung (1994)]: $|0>$ is of grade-0, $|1>$ of grade-2 while $|2>$ is of grade-1. This, together with equations (10) or (11) (see below) leads us to interpret $a^+$ as a grade-2 and $a$ as a grade-1. From (8) we obtain

$$a^+|0> = |1>
$$
$$a|1> = |0>
$$

or

$$a^+ a^+|0> = \sqrt{2}|2>
$$
$$a|2> = \sqrt{2}|1>
$$

Now, using the conventions cited at the end of section 2, one is led to the following relations:

$$\xi a^+ = qa^+ \xi
$$
$$\bar{\xi} a = \bar{q}a \bar{\xi}
$$
To be consistent with what we announced in section 2 no relations are imposed on the products $\xi a$ and $a\xi$ (the same stands for $\bar{\xi}a^+$ and $a^+\bar{\xi}$). Furthermore, no analogue of the ternary relation (1) can be imposed on products of the form $a\xi a^+$, this is due to the non-conventional commutation relations in (6). One should, in order to compute such entities, use relations (6) and (12).

We notice that in [Chung (1994)] the authors took $\xi$ and $\bar{\xi}$ to commute with $a$ and $a^+$. This is not necessarily true, since even in the fermionic case (i.e. $Z_2$ case) that we are supposed to generalize, the $\xi$’s anti-commute with the $a$’s!

5 Coherent States

When investigating on the possibilities that are allowed in order to construct coherent states for the $Z_3$-graded harmonic oscillator introduced in section 3 the only possibility is given by:

$$|\xi\rangle = f(a^+\xi)|0\rangle$$

where $f(a^+\xi) = 1 + a^+\xi - qa^+\xi a^+\xi$ generalizes the function $(1 + a^+\xi)$ in the fermionic case [Ohnuki (1978)].

Using (12) this state can be rewritten as:

$$|\xi\rangle = |0\rangle + q^2|1\rangle - q\sqrt{2}|2\rangle$$

In what follows we shall demonstrate that these states are indeed coherent states for the $Z_3$ harmonic oscillator.

First of all, using (6), (8), (12) and the rules cited at the end of section 4, it’s easy to see that the states (13) are indeed eigenstates of the annihilation operator

$$a|\xi\rangle = \xi|\xi\rangle$$

One can also compute the scalar product of two such states using the same relations and the orthonormality of the Fock space basis, the result is then:

$$<\bar{\xi}_1|\xi_2> = 1 + q^2\bar{\xi}_1\xi_2 - q\sqrt{2}<\bar{\xi}_1|\xi_2>$$

where

$$<\bar{\xi}_1| = 0, q < 1|\xi_1 - q^2\sqrt{2} < 2|\bar{\xi}_1|$$

A resolution of the identity is also possible in terms of the states (13) or equally(14). In fact, since the three eigenvectors $|0\rangle$, $|1\rangle$ and $|2\rangle$ form an orthonormal basis, the identity operator may be expressed as

$$I = |0\rangle <0| + |1\rangle <1| + |2\rangle <2|$$
and using the integrals defined by Majid [Majid (1991)]:

\[
\begin{align*}
\int d\xi^1 &= \int d\xi^2 \xi = 0 \\
\int d\xi^2 &= \int d\xi^2 \xi = 0 \\
\int d\xi^2 \xi^2 &= \int d\xi^2 \xi^2 = 1
\end{align*}
\] (19)

The identity operator, in terms of the \(|\xi>\)'s is given by:

\[
\int d\xi \ d\xi \ h(\xi) \ |\xi> <\xi| = I
\] (20)

where the weight function is given by

\[
h(\xi) = -q + q^2 \xi + \xi \xi \xi
\] (21)

This completes the proof of the fact that the states (13) are indeed coherent states [Klauder (1985)].

We believe that these results constitute an important brick in our quest on searching for possible physical application of the \(Z_3\)-graded structures. In fact, seeing the \(Z_3\)-graduation as a generalization of the conventional non-commutative geometries [Kerner (1996), Kerner (2000)], we can use these results to construct the generalized quantum plane associated to it, and then investigate on its properties.

Another area where we can use these results is on the so-called fractional supersymmetry. One should construct the fractional supersymmetry associated to these structures.

We deem that results concerning these two points will be soon available in a forthcoming work.

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**References**


