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MODIFIED k-ℓ MODEL AND ITS ABILITY TO SIMULATE
SUPERSONIC AXISSYMMETRIC TURBULENT FLOWS

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Abstract

The k-ℓ turbulence model is a promising two-equation model. In this paper, the k and ℓ model equations were derived from k-kℓ incompressible and one-equation turbulence models. Then the model was modified for compressible and transitional flows, and was applied to simulate supersonic axisymmetric flows over Hollow cylinder flare and hyperboloid flare bodies. The results were compared with the results obtained for the same flows experimentally as well as k-ε, k-ω and Baldwin-Lomax models. It was shown that the k-ℓ model produces good results compared with experimental data and numerical data obtained when other turbulence models were used. It gives better results than k-ω and k-ε models in some cases.

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1. Introduction

Supersonic axisymmetric flows are important in aerodynamics and turbomachinery among other applications. A turbulence model must have the ability to simulate complicated high speed the flows, which contain the recirculation zone, expansion and compression waves and their interaction with other high speed flow phenomena, with a certain degree of accuracy.

In the last thirty years, the majority of research works that have been done in turbulence modeling correspond to k-ε models. The k-ω equation is the second two-equation model that was recently tackled by other researchers, for example see [1].

In this study, the k-ω two-equation turbulence model is derived and tested for the axisymmetric blunt body flows with recompression corner. The turbulence length scale equation is much easier to resolve numerically than the turbulent dissipation and turbulent vorticity equations that were used in the k-ε and k-ω models, respectively. The k-ε model overestimates the skin friction coefficient when applied to the boundary layer flows with adverse pressure gradient. The k-ω equation is sensitive to free stream boundary conditions.

Sarkar [2] and Zemman [3] have modeled the dilatation dissipation and dilatation pressure using DNS results and applied the k-ε and Reynolds stress models to simulate simple free shear flows and obtained good results. The k-ω model is also modified according to Sarkar and Zemman by Wilcox [4]. Wilcox showed that these modifications are not appropriate for flow over a flat plate. Huang et al [5] modified k-ε and k-ω models using Sarkar [2] and Sarkar [3] compressibility correction models. They showed that these modifications are not appropriate for predicting the pure Couette flows.

The k-ω equations have gained much less attention by researchers. Smith [6,7] introduced a transport equation for length scale and applied it to boundary layer flows with success. In this paper, the k-ω equations were first derived from Saeedi et al [8] k-k model. Then they were modified to simulate supersonics flows and transition region.

In this paper k-ω, k-ε, k-ω and Baldwin-Lomax [9] turbulence models, were used to simulate supersonic flow over a hollow cylinder flare, and a hyperbolic-flare body bow shock. The modifications introduced by Sarkar [2] and Zemman [3] for compressibility effects were applied in k-ε and k-ω models. But these modifications were not needed for the k-ω model. The near wall low-Reynolds-number corrections of Chien [10] was applied for the k-ε model. For k-ω equations, the value of k and ω near the wall and the wall functions, which are used for one-equation model [11], is used.

2. Governing Equations

Time-dependent axisymmetric flow Navier-Stokes equations and two-equation turbulence models were used. The final form of the k-ω equations are:

\[
\frac{\partial \rho k}{\partial t} + \frac{\partial \rho k u_k}{\partial x_k} = \rho P_k - c_D \rho \frac{k^{3/2}}{\ell} + \frac{\partial}{\partial x_k} \left[ \left( \mu + \mu_t / \sigma_k \right) \frac{\partial k}{\partial x_k} - \mu \frac{\partial k}{\partial x_i} \frac{\partial k}{\partial x_i} \right] - \mu \frac{k}{\partial x_i} \frac{\partial k}{\partial x_i} 
\]

\[
\frac{\partial \rho \ell}{\partial t} + \frac{\partial \rho \ell u_k}{\partial x_k} = C_L \rho k^{1/2} + \frac{\partial}{\partial x_k} \left[ \left( \mu + \mu_t / \sigma_L \right) \frac{\partial \ell}{\partial x_k} \right] + \rho \ell \frac{\partial u_i}{\partial x_i} + 2\left( \mu + \mu_t / \sigma_L \right) \frac{\partial \ell}{\partial x_i} \frac{\partial k}{\partial x_i} 
\]

where,

\[
\mu = C_M \rho \sqrt{k \ell} , \quad P_k = -u_i \frac{\partial u_i}{\partial x_j} , \quad -u_i u_j = \nu_t \left( \frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i} \right) - \frac{2}{3} \delta_{ij} k
\]
In the supersonic flows, where expansion and compression waves exist, the obtained length scale maybe bigger than the actual length scale, and therefore, the Reynolds shear stress becomes longer than the Reynolds normal stress. The following correction is made to overcome this problem [7].

\[
\mu' = \mu \frac{1}{(\alpha^2 + r^2)^{1/2}}
\]

where,

\[
a = 0.9989, \quad r^2 = \frac{3\mu^2 S_{ij} S_{ij}}{8 \rho^2 k^2}
\]

\[
S_j = \left( \frac{\partial U_i}{\partial x_j} + \frac{\partial U_j}{\partial x_i} \right) - \frac{2}{3} \left( \frac{\partial U_k}{\partial x_i} \right) \delta_{ij}
\]

In the transition region, the length scale \( \ell \), was modified as follows according to the Warren et al [11], which were used for the one-equation model.

\[
\ell = C_D \left( [1 - \Gamma](\ell_{TS} + \ell_{SM}) + \Gamma \ell' \right)
\]

where, \( C_D = 0.09 \) and the intermittency coefficient, \( \Gamma \) is [14]:

\[
\Gamma(x) = 1 - \exp(-0.412 \xi^2), \quad \xi = \left( x - x_t \right)/\lambda, \quad Re_\lambda = 9(Re_\alpha)^{0.75}
\]

\( x_t \) is the value of \( x \) at the edge of transition region. \( \ell \) in equation (6) is the turbulent length scale and obtained from \( k-\ell \) equations. \( \ell_{TS} \) is due to the instability disturbances of Tolliman-Shilichting and \( \ell_{SM} \) is due to the acoustic disturbances because of the large amplitude of pressure and temperature.

\[
\ell_{TS} = a \sqrt{\frac{k}{\omega}}, \quad a = 0.04 - 0.06
\]

\[
ostar / U_e^2 = 3.2(Re_\alpha)^{-2/3}
\]

\[
\ell_{SM} = \frac{b}{U_p M_e^3} \sqrt{\frac{kx}{Re_\lambda}}, \quad b = 0.23
\]

where, \( U_e \) and \( M_e \) are free stream velocity and Mach number respectively. \( U_p = 0.94U_e \) is the phase velocity and \( \nu^* \) is viscosity based on a reference temperature.

3. Numerical Solution
The governing equations were transferred into the body fitted coordinate system, and solved numerically by using the Reimann Roe [12] flux splitting method. To prevent entropy deflection around the sonic lines, Harten and Hyman [13] entropy conditions were used. And to prevent numerical oscillations, the minimum of two-gradient limiter is employed, Hirsh [14]. The computational grid is generated algebraically by using the Eiseman method [15] with the two internal surfaces.

4. Results
A computer code is developed and tested with several simple flows to ensure that it works properly [16]. The code was first used to simulate laminar flow around two blunt bodies and several different limiters and entropy conditions were used and stability and convergence of the numerical simulation were examined [16]. The results compared with Kubota [17] and Holden et al [18]. As a result, the minimum two-gradient limiter and the Harten and Hyman
entropy condition, which produced better results for the cases studied, were chosen and used throughout this paper. Supersonic turbulent flows over hollow cylinder flare and hyperboloid-flare bodies were simulated and $k-\ell$, $k-\epsilon$, $k-\omega$ and Baldwin-Lomax models turbulence models were tested.

a. Flow over Hollow Cylinder-Flare

In this flow, the interaction of shock waves and boundary layers are considered. The location of separation point, size of the separation region and rate of heat transfer from the wall are obtained. Fig. (1) shows the corresponding geometry and the inflow properties. The grid size is $160\times60$. The minimum relative size of the cells are $\Delta x_{\text{min}}/r = 2.1\times10^{-4}$ and $\Delta y_{\text{min}}/r = 2.9\times10^{-5}$ and the location of the transition point is 12.5 cm from the edge of the body. To simulate this problem, $k-\ell$, $k-\epsilon$ and Baldwin-Lomax models with near wall modifications are used.

Fig. (2) shows pressure contours and streamlines in the region near the corner. These results were obtained for the $k-\ell$ model. It shows the locations of separation point, reattachment point, separated region and shock waves. The simulated surface pressure distributions obtained by three different models and also corresponding experimental results obtained by Joulot [19] are shown in Fig. (3). It shows that the modifications made in $k-\ell$ equations based on the Warren et al [11] ideas, produce good results and predict the location of separation point and distribution of pressure well.

Fig. (4) shows the distribution of Stanton number obtained by using three different models and compared with Joulot experimental [19] and numerical, Paciorri et al [20] results. Passori, et al used the Spalart Allmaras [21] turbulence model in their numerical simulations. As shown in Fig. (4), all of the three models predict the Stanton number well at the transition region and the separation point. But the Baldwin-Lomax model gives less accurate results as expected.

The difference between the numerical results and the experimental ones may be due to the shortcoming of the turbulence models. But it could also be due to the three-dimensionality and unsteadiness nature of the problem, numerical method especially with the use of limiters, grid configuration and the large value of the $y^+$ on the first node near the wall at the separation region.

b. Flow Over Hyperboloid Flare Body

The geometry, numerical grid and flow properties are shown in Fig. (5). The numerical grid was produced algebraically [15], and it is $160\times60$. The flow was simulated using three different turbulence models. In the two-equation models, the near wall modifications are applied. Fig. (6) shows the temperature contours for $k-\ell$ model. The shock and expansion waves are well predicted. The distribution of pressure and Stanton number are shown in Figs. (7) and (8) and compared with the experimental results [22]. The location of separation point when $k-\ell$, $k-\epsilon$ and Baldwin-Lomax models were used, are 0.0371, 0.0411 and 0.0388 respectively. All of the models produce relatively accurate results with regard to the locations of separation and reattachment points. The pressure distribution at the separation region is predicted more accurately by the $k-\ell$ model and it is least accurate when $k-\epsilon$ is used. However, all models predict the maximum value of pressure and its location accurately. All the models predict Stanton number with some degree of errors. The inaccuracy of Stanton number could be due to similar reasons mentioned at the end of part a of this paper.

6. Conclusions

In this work, a modified version of the $k-\ell$ model is introduced. Then turbulent models, $k-\ell$, $k-\epsilon$, $k-\omega$ and Baldwin-Lomax models were examined and compared by using two different
supersonic flows with different flow characteristics and phenomena. The aim was to examine the k-$\ell$ model, which has gained less attraction among researchers, and compare it with other models. It was shown that this version of the k-$\ell$ model produces as good results as the k-$\varepsilon$ model and in some cases it produces even better results. It seems that if more effort is put into the k-$\ell$ model, it will become a much better two-equation model.

In the flow over axisymmetric bodies, the comparison of k-$\ell$ and k-$\varepsilon$ results shows that these models produce rather accurate results. Although the Baldwin-Lomax model is a simple zero-equation model, it produces reasonable results.

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References

**Table**

<table>
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<tr>
<th>Parameter</th>
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<td>$Pr$</td>
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**Fig. 1** Hollow cylinder flare geometry

**Fig. 2** Streamlines and pressure contours for hypersonic flow in Hollow cylinder flare problem

**Fig. 3** Hollow cylinder flare: wall pressure distribution

**Fig. 4** Hollow cylinder flare: Stanton numbers distribution
6.85 \quad M_{\infty}
600 \quad T_0, K
3.25 \times 10^6 \quad P_0, Pa
300 \quad T_w, K
14.8 \times 10^6 \quad Re, m^{-1}
0.72 \quad Pr
0.9 \quad Pr_i
160 \times 60 \quad Grid

Fig. 5 Grid for hyperboloid-flare problem

Fig. 6 Hyperboloid-flare: Temperature contours (n=30)

Fig. 7 Hyperboloid-flare: distribution of pressure coefficient

Fig. 8 Hyperboloid-flare: distribution of Stanton number