ON THE FUČÍK SPECTRUM WITH INDEFINITE WEIGHTS

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1 Introduction

This paper is partly concerned with the one-dimensional asymmetric problem with weight

\[
\begin{cases}
Lu = m(t)(au^+ - bu^-) \text{ in } [T_1, T_2], \\
u(T_1) = u(T_2) = 0.
\end{cases}
\]

Here \(Lu := -(p(t)u')' + q(t)u\), \(p, q\) and \(m \in C[T_1, T_2]\), \(p(t) > 0\) on \([T_1, T_2]\), \(q(t) \geq 0\) on \([T_1, T_2]\), \(m(t) \neq 0\) and \(u^\pm := \max\{\pm u, 0\}\). The associated Fučík spectrum is defined as the set \(\Sigma\) of those \((a, b) \in \mathbb{R}^2\) such that (1.1) has a nontrivial solution \(u\).

The description of this spectrum \(\Sigma\) is classical and explicit when \(Lu = -u''\) and there is no weight, i.e. \(m(t) \equiv 1\) (cf. [6], [9]). The same general picture for \(\Sigma\) remains valid when \(L\) is as above and \(m(t) > 0\) on \([T_1, T_2]\) (cf. [8], [5], [12]) : \(\Sigma\) is made of the two lines \(\mathbb{R} \times \lambda_1^m\) and \(\lambda_1^m \times \mathbb{R}\) together with a sequence of hyperbolic like curves in \(\mathbb{R}^+ \times \mathbb{R}^+\) passing through \((\lambda_k^m, \lambda_k^m)\), \(k \geq 2\); one or two such curves emanate from each \((\lambda_k^m, \lambda_k^m)\), and the corresponding solutions of (1.1) along these curves have exactly \(k - 1\) zeros in \([T_1, T_2]\).

Here \((0 < \lambda_1^m < \lambda_2^m < \ldots \to +\infty)\) denotes the sequence of eigenvalues of the associated linear problem

\[
\begin{cases}
Lu = \lambda m(t)u \text{ in } [T_1, T_2], \\
u(T_1) = u(T_2) = 0.
\end{cases}
\]

One of our purposes in this paper is to investigate the situation where the weight function \(m(t)\) in (1.1) changes sign in \([T_1, T_2]\). In that case it is well-known that the eigenvalues in (1.2) form a double sequence : \(-\infty \leftarrow \ldots < \lambda_{-2}^m < \lambda_{-1}^m < 0 < \lambda_1^m < \lambda_2^m < \ldots \to +\infty\). A natural conjecture is then that \(\Sigma\) should be made of the four trivial lines \(\mathbb{R} \times \lambda_1^m, \lambda_1^m \times \mathbb{R}, \mathbb{R} \times \lambda_{-1}^m, \lambda_{-1}^m \times \mathbb{R}\) together with a sequence of hyperbolic like curves in \(\mathbb{R}^+ \times \mathbb{R}^+\) passing through \((\lambda_k^m, \lambda_k^m)\), \(k \geq 2\); one or two such curves emanate from each \((\lambda_k^m, \lambda_k^m)\), and the corresponding solutions of (1.1) along these curves have exactly \(k - 1\) zeros in \([T_1, T_2]\). Here \((0 < \lambda_1^m < \lambda_2^m < \ldots \to +\infty)\) denotes the sequence of eigenvalues of the corresponding linear problem

\[
\begin{cases}
Lu = \lambda m(t)u \text{ in } [T_1, T_2], \\
u(T_1) = u(T_2) = 0.
\end{cases}
\]

More generally we consider the problem

\[
\begin{cases}
Lu = am(t)u^+ - bn(t)u^- \text{ in } [T_1, T_2], \\
u(T_1) = u(T_2) = 0.
\end{cases}
\]

which involves two weights \(m, n \in C[T_1, T_2]\) with \(m(t)\) and \(n(t)\) \(\neq 0\). Let again \(\Sigma\) denote the corresponding Fučík spectrum. Note that in this case, making \(a = b\) in (1.3) does not lead to any obvious elements in \(\Sigma\). Assuming for instance that both \(m(t)\) and \(n(t)\) change sign in \([T_1, T_2]\), we show that beside a trivial part consisting of the 4 lines \(\lambda_1^m \times \mathbb{R}, \lambda_1^m \times \mathbb{R}, \mathbb{R} \times \lambda_1^m, \mathbb{R} \times \lambda_1^m\), \(\Sigma\) is made in each quadrant of a (non zero) odd or infinite number of hyperbolic like curves. These curves can again be classified according to the number of zeros
of the corresponding solutions of (1.3). We also show that all cases can effectively happen with respect to the numbers of these curves: given \( K, L, M, N \in \{0, 1, 2, \ldots, +\infty\} \), there exist weights \( m(t) \) and \( n(t) \) such that \( \Sigma \) exactly contains \( (2K + 1), (2L + 1), (2M + 1) \) and \( (2N + 1) \) hyperbolic like curves in \( \mathbb{R}^+ \times \mathbb{R}^+, \mathbb{R}^- \times \mathbb{R}^-, \mathbb{R}^+ \times \mathbb{R}^- \) and \( \mathbb{R}^- \times \mathbb{R}^+ \) respectively. (Note that here as before some of these curves may be double and are then counted for two, cf. remark 3.9).

Section 3 deals with the two weights problem (1.3) and section 4 with the one weight problem (1.1). In section 5 we investigate for (1.3) the asymptotic behaviour of the first hyperbolic like curves of \( \Sigma \) (i.e. those which lie the closest to the trivial horizontal and vertical lines). It is known that if \( m(t) \) and \( n(t) \) are \( > 0 \) in \( \overline{T_1, T_2} \), then these first curves are asymptotic to the line \( \mathbb{R} \times \lambda(t)^n \) as \( a \to +\infty \) and to the line \( \lambda(t)^m \times \mathbb{R} \) as \( b \to +\infty \) (cf. [12]; cf. also [8], [5] in the one weight problem). As observed in [7], such an asymptotic behaviour is closely connected with the nonuniformity of the antimaximum principle. It turns out that this asymptotic behaviour may be affected by the presence of more general weights. We show in particular that if \( m(t) \) and \( n(t) \) have compact support in \( \overline{T_1, T_2} \), then none of the first curves is asymptotic on any side to the trivial horizontal and vertical lines; the converse implication is also true. In remark 5.6, we briefly comment on the meaning of this result in the context of the antimaximum principle.

Our approach is based on the shooting method and in section 2, which has a preliminary character, we investigate various properties of the "zero-function". Given a nontrivial solution \( u \) of the linear equation \( Lu = am(t)u \), this function sends one zero of \( u \) onto the following zero of \( u \).

We finally mention that results analogous to those in the present paper can also be established for the Neumann problem (cf. [1], [2]). Moreover the study of the Fučík spectrum with indefinite weights in the P.D.E. case has been initiated recently in [3].

## 2 The Zero-function

In this section we consider the zero-function for the linear equation \( Lu = am(t)u \). Here \( L \) is as in the introduction and \( m \in C[T_1, T_2], m(t) \not\equiv 0 \). It will be convenient to extend the coefficients of \( L \) and the weight \( m \) from \( [T_1, T_2] \) to the whole of \( \mathbb{R} \), preserving continuity as well as the inequalities \( p_1 \leq p(t) \leq p_2, 0 \leq q(t) \leq q_2 \) and \( m_1 \leq m(t) \leq m_2 \) for some constants \( p_1, p_2, q_2, m_1, m_2 \) with \( p_1 > 0 \). We will assume below that such an extension has been carried out.

Writing the equation \( Lu = am(t)u \) as a first order system, it follows from standard ODE results (cf. e.g. ch. 5 of [11]) that for each \( s \in \mathbb{R} \), the initial value problem

\[
\begin{align*}
Lu &= am(t)u \quad \text{in } \mathbb{R}, \\
u(s) &= 0, \quad u'(s) = 1/p(s)
\end{align*}
\]

has a unique solution \( u = u(t) = u(t; a, s) \). This solution is a \( C^1 \) function of \( (t, a, s) \in \mathbb{R} \times \mathbb{R} \times \mathbb{R} \); moreover \( u \) and \( (pu') \) have second mixed derivatives with respect to \( t, a \), and also with respect to \( t, s \), which commute and are continuous.
Definition 2.1 The zero-function \( \varphi \) is defined by

\[
\varphi_a(s) := \min\{t \in \mathbb{R} : t > s \text{ and } u(t; a, s) = 0\},
\]

with \( \varphi_a(s) = +\infty \) in case \( u(t) \) does not vanish for any \( t > s \).

Thus \( \varphi_a \) sends \( s \) onto the following zero of the solution \( u \) of (2.1). Since the zeros of \( u \) are isolated, this definition makes sense. Clearly \( \varphi_a(s) > s \). Note that \( \varphi_a(s) \) is also the first zero following \( s \) of any nontrivial solution of \( Lu = am(t)u, u(s) = 0 \).

Some monotonicity and regularity properties of this function are collected in the following two lemmas. We call \( D \) the domain of \( \varphi \), i.e. \( D := \{(a, s) \in \mathbb{R}^2 : \varphi_a(s) < +\infty\} \).

Lemma 2.2 (i) For each \( a, \varphi_a(s) \) is increasing with respect to \( s \), strictly on \( D \). (ii) For each \( s, \varphi_a(s) \) is decreasing (resp. increasing) with respect to \( a \) for \( a > 0 \) (resp. for \( a \leq 0 \)), strictly on \( D \).

Proof. These monotonicity properties easily follow from the Sturm comparison theorem as given e.g. in ch. 11 of [11]. To apply this theorem in part (ii), it is useful to write first the equation \( Lu = am(t)u \) as

\[
-(p(t)/a)u' + (q(t)/a)u = m(t)u.
\]

Q.E.D.

Lemma 2.3 (i) \( D \) is open and \( \varphi : D \rightarrow \mathbb{R} \) is of class \( C^1 \). (ii) \( \partial \varphi_a(s)/\partial s > 0 \) for \( (a, s) \in D \). (iii) \( \partial \varphi_a(s)/\partial a < 0 \) for \( (a, s) \in D \) with \( a > 0 \). (iv) \( \partial \varphi_a(s)/\partial a > 0 \) for \( (a, s) \in D \) with \( a < 0 \).

Proof. We start with (i). Let \( (a_0, s_0) \in D \). Applying the implicit function theorem, we get open neighbourhoods \( U \) of \( (a_0, s_0) \) and \( V \) of \( \varphi(a_0(s_0)) \) and a \( C^1 \) function \( \tilde{\varphi} : U \rightarrow V \) such that

\[
\tilde{\varphi}(a, s) = t \text{ iff } (t, a, s) \in V \times U \text{ and } u(t; a, s) = 0.
\]

Since \( \tilde{\varphi}(a_0, s_0) = \varphi(a_0(s_0)) > s_0 \), reducing \( U \) if necessary, we can assume \( \tilde{\varphi}(a, s) > s_0 \) on \( U \). The definition of \( \varphi(s) \) then implies \( \varphi_a(s) \leq \tilde{\varphi}(a, s) \). Consequently \( U \subset D \) and \( D \) is open.

We will now show that \( \varphi \equiv \tilde{\varphi} \) near \( (a_0, s_0) \), which will conclude the proof of (i). Assume by contradiction the existence of a sequence \( (a_i, s_i) \rightarrow (a_0, s_0) \) such that

\[
s_i < \varphi_{a_i}(s_i) < \tilde{\varphi}(a_i, s_i).
\]

For a subsequence, \( \varphi_{a_i}(s_i) \rightarrow r \) which satisfies \( r \in [s_0, \varphi(a_0(s_0))] \) and \( u(r; a_0, s_0) = 0 \). This implies \( r = s_0 \) or \( r = \varphi(a_0(s_0)) \). By (2.2) we can apply Rolle’s theorem to \( u(t; a_i, s_i) \) between either \( s_i \) and \( \varphi_{a_i}(s_i) \), or \( \varphi_{a_i}(s_i) \) and \( \tilde{\varphi}(a_i, s_i) \). We deduce that either \( u'(s_0; a_0, s_0) = 0 \) or \( u'(\varphi(a_0(s_0)); a_0, s_0) = 0 \), a contradiction.

We now turn to the proof of (ii), (iii), (iv). These properties easily follow by derivating the relation \( u(\varphi_a(s); a, s) \equiv 0 \) once we know that

\[
[\partial u(t; a, s)/\partial t]_{t=\varphi_a(s)} < 0, \tag{2.3}
\]

\[
[\partial u(t; a, s)/\partial s]_{t=\varphi_a(s)} > 0, \tag{2.4}
\]

\[
[\partial u(t; a, s)/\partial a]_{t=\varphi_a(s)} < 0 \text{ if } a > 0 \text{ and } > 0 \text{ if } a < 0. \tag{2.5}
\]
(2.3) is clear from the definition of \( \varphi_a(s) \). To prove (2.4) we define \( v(t) := \partial u(t; a, s)/\partial s \). Derivating the relation \( u(s; a, s) \equiv 0 \) with respect to \( s \), we get \( v(s) = -1/p(s) \neq 0 \). Derivating the equation (2.1) with respect to \( s \), we get \( L u = a m(t) v \). So \( v(\varphi_a(s)) \neq 0 \) and, since \( v(s) < 0 \), the Sturm comparison theorem implies \( v(\varphi_a(s)) > 0 \), i.e. (2.4). To prove (2.5) we define \( w(t) := \partial u(t; a, s)/\partial a \). Since \( u(s; a, s) \equiv 0 \), we get \( w(s) = 0 \). Derivating the equation (2.1) with respect to \( a \), we get \( L w = a m(t) w + (L u)/a \). Multiplying by \( u \) and integrating from \( s \) to \( \varphi_a(s) \), we obtain, after two integrations by part,
\[
\int_{p(\varphi_a(s))w(\varphi_a(s))u'(\varphi_a(s)) = \frac{1}{a} \int_{a}^{\varphi_a(s)} p(t)(u')^2 + \frac{1}{a} \int_{a}^{\varphi_a(s)} q(t)u'^2,
\]
which implies (2.5). Q.E.D.

The behaviour of \( \varphi_a(s) \) as \( a \to \pm \infty \) will be important in our study. Let us define
\[
\alpha^> := \inf\{t > s : m(t) > 0\},
\alpha^<= \inf\{t > s : m(t) < 0\},
\]
with the usual convention that \( \inf \phi = +\infty \). In the simplest cases, \( \alpha^> \) (resp. \( \alpha^- \)) represents the lower bound of the first positive (resp. negative) bump of \( m(t) \) situated at the right of \( s \). Clearly \( \alpha^> \geq s \) and \( \alpha^- \leq s \).

**Lemma 2.4** For each \( s \in \mathbb{R} \), \( \varphi_a(s) \to \alpha^> \) as \( a \to +\infty \) and \( \varphi_a(s) \to \alpha^- \) as \( a \to -\infty \).

**Proof.** We will consider only the case \( a \to +\infty \) (the other case can be reduced to this one by considering \(-m\)). We will also assume \( \alpha^> < +\infty \) (otherwise the conclusion is trivial).

Assume first that \( s \) is such that \( m(s) > 0 \). Given \( \epsilon > 0 \) sufficiently small, \( m \) is \( > 0 \) on \([s, s + \epsilon]\), and consequently there exists \( a_\epsilon \) such that \( a m(t) \geq p^2(\pi/\epsilon)^2 + q_2 \) on \([s, s + \epsilon]\) for all \( a \geq a_\epsilon \), where \( p_2 \) and \( q_2 \) are defined at the beginning of section 2. We will compare on \([s, s + \epsilon]\) our equation \( L u = a m(t) u \) with the equation
\[
-p_2 v'' = p_2(\pi/\epsilon)^2 v.
\]
Since \( \sin(\pi(t - s)/\epsilon) \) is a solution of the latter which vanishes at \( s \) and \( s + \epsilon \), the Sturm comparison theorem implies that \( s < \varphi_a(s) \leq s + \epsilon \). The conclusion of the lemma then follows since \( \alpha^> = s \) in the case under consideration.

We now turn to the general case. Let \( \epsilon > 0 \). By the definition of \( \alpha^> \), we can find \( s_\epsilon \in ]\alpha^- + \epsilon, \alpha^> + \epsilon[ \) such that \( m(s_\epsilon) > 0 \). The case already treated then implies that for \( a \) sufficiently large,
\[
\varphi_a(s) < \varphi_a(s_\epsilon) \leq s_\epsilon + \epsilon \leq \alpha^- + 2\epsilon.
\]
On the other hand one has
\[
\alpha^> < \varphi_a(s).
\]
Indeed, if (2.8) does not hold, then \( m \) is \( \leq 0 \) on \([s, \varphi_a(s)]\). Comparing on this interval our equation \( L u = a m u \) with the equation \(-p_1 v'' = 0v\), where \( p_1 \) is defined at the beginning of section 2, we deduce from the Sturm comparison theorem that any solution \( v \) of the latter equation must have a zero in \([s, \varphi_a(s)]\), which is clearly false. Combining (2.7) and (2.8) finally yields the conclusion of the lemma. Q.E.D.
Lemma 2.5 For each $s \in \mathbb{R}$, $\varphi_a(s) \to +\infty$ as $a \to 0$.

Proof. We consider only the case $a \geq 0$ (the case $a \leq 0$ can be reduced to this one by considering $-m$). Let $R > 0$. There exists $a_R > 0$ such that $am(t) \leq p_1(\pi/R)^2$ for $0 \leq a \leq a_R$, where $p_1$ is defined at the beginning of section 2. We compare on $[s, s + R]$ our equation $Lu = am(t)u$ with the equation

$$-p_1 v'' = p_1(\pi/R)^2 v.$$  

Since $\sin(\pi(t - s)/R)$ is a solution of the latter which has $s$ and $s + R$ as consecutive zeros, the Sturm comparison theorem implies $\varphi_a(s) > s + R$. Q. E. D.

To conclude this section we return to our given interval $[T_1, T_2]$. Clearly the restriction of the function $\varphi_a$ to the set $\{s \in [T_1, T_2]; \varphi_a(s) \in [T_1, T_2]\}$ does not depend on the extension of the data carried out at the beginning of this section.

If $m^+(t) \not\equiv 0$ on $[T_1, T_2]$, then $\alpha^+(T_1)$ defined in (2.6) is $< T_2$, and it follows from lemmas 2.2, 2.3, 2.4 and 2.5 that $\varphi_a(T_1) = T_2$ for exactly one value of $a > 0$. Clearly this value of $a$ is equal to the first positive eigenvalue $\lambda_1^m$ of (1.2). Similarly, if $m^-(t) \not\equiv 0$ on $[T_1, T_2]$, then $\varphi_a(T_1) = T_2$ for exactly one value of $a < 0$, which is equal to $\lambda_2^m$.

The zero-function $\varphi_a$ clearly depends on the weight $m(t)$ and we will from now on denote it by $\varphi^m_a$.

3 The two weights problem

In this section we consider problem (1.3), with $L$ as in the introduction and $m, n \in C[T_1, T_2]$, $m(t)$ and $n(t) \not\equiv 0$.

Since the zeros of a nontrivial solution of (1.3) are isolated, one can classify these solutions according to the number of their zeros. This yields the following description of the Fučik spectrum $\Sigma$:

Proposition 3.1 We have

$$\Sigma = C_1^+ \cup C_1^- \cup C_2^+ \cup C_2^- \cup C_3^+ \cup C_3^- \cup \ldots$$

where

$$C_k^+ \ (\text{resp. } C_k^-) = \{(a, b) \in \mathbb{R}^2; \ (1.3) \ has \ a \ solution \ u \ with \ k - 1 \ zeros \ in \ [T_1, T_2] \ and \ u'(T_1) > 0 \ (\text{resp. } u'(T_1) < 0)\}.$$  

Moreover, if $(a, b) \in C_k^+$ (resp. $C_k^-$), then all nontrivial solutions of (1.3) with $u'(T_1) > 0$ (resp. $u'(T_1) < 0$) are multiple one of the other.

These sets $C_k^+$ and $C_k^-$ can themselves be described in terms of the zero-functions $\varphi^m_a$ and
\[ C_1^+ = \{(a, b) \in \mathbb{R}^2 : \varphi_{a}^m(T_1) = T_2\}, \]
\[ C_2^+ = \{(a, b) \in \mathbb{R}^2 : \varphi_{b}^n(T_1) = T_2\}, \]
\[ C_3^+ = \{(a, b) \in \mathbb{R}^2 : \varphi_{a}^m(\varphi_{b}^n(T_1)) = T_2\}, \]
\[ C_1^- = \{(a, b) \in \mathbb{R}^2 : \varphi_{a}^m(T_1) = T_2\}, \]
\[ C_2^- = \{(a, b) \in \mathbb{R}^2 : \varphi_{b}^n(T_1) = T_2\}, \]
\[ C_3^- = \{(a, b) \in \mathbb{R}^2 : \varphi_{a}^m(\varphi_{b}^n(T_1)) = T_2\}, \]...

\[(3.1)\]

Of course, as we will see later, some of these sets may be empty. Note that \( C_1^+ \) is made of one or two vertical lines \( \lambda_1^m \times \mathbb{R} \) and \( \lambda_{-1}^m \times \mathbb{R} \) (depending on whether \( m(t) \) is \( \geq 0 \), \( \leq 0 \) or changes sign). Similarly for \( C_1^- \) with the horizontal lines \( \mathbb{R} \times \lambda_1^n \) and \( \mathbb{R} \times \lambda_{-1}^n \). It will be convenient to denote by \( \Sigma^* \) the set \( \Sigma \) without these (two, three or four) trivial lines.

The monotonicity properties of the zero-function imply that if for instance \( m(t) \) and \( n(t) \) change sign, then \( \Sigma^* \) is contained in the four quadrants

\[
(\lfloor \lambda_1^m, +\infty \rfloor \times \lfloor \lambda_1^n, +\infty \rfloor) \cup (\lfloor -\infty, \lambda_{-1}^m \rfloor \times \lfloor -\infty, \lambda_{-1}^n \rfloor) \]
\[
\cup (\lfloor -\infty, \lambda_{-1}^m \rfloor \times \lfloor +\infty, \lambda_1^n \rfloor) \cup (\lfloor +\infty, \lambda_1^m \rfloor \times \lfloor -\infty, \lambda_{-1}^n \rfloor).
\]

In the case where for instance \( m(t) \) changes sign but \( n(t) \) is, say, \( \leq 0 \), then \( \Sigma^* \) is contained in the two quadrants

\[
(\lfloor -\infty, \lambda_{-1}^m \rfloor \times -\infty, \lambda_{-1}^n \rfloor) \cup (\lfloor \lambda_1^m, +\infty \rfloor \times \lfloor -\infty, \lambda_{-1}^n \rfloor).
\]

If \( m(t) \) and \( n(t) \) do not change sign, then \( \Sigma^* \) is contained in one quadrant.

**Remark 3.2** The sets \( C_k^+ \) and \( C_k^- \) depend on the weights : \( C_k^+ = C_k^+ (m, n) \) and \( C_k^- = C_k^- (m, n) \). Clearly

\[(a, b) \in C_k^+ (m, n) \iff (a, -b) \in C_k^- (m, -n) \iff \ldots \]

and similarly for \( C_k^- \). It follows that by changing the sign of the weights, the study of the Fučík spectrum in \( \mathbb{R} \times \mathbb{R} \) for the weights \( (m, n) \) can be reduced to the study of the intersection with \( \mathbb{R}^+ \times \mathbb{R}^+ \) of the Fučík spectrum for the weights \( (\pm m, \pm n) \). We will denote below \( C_k^+ \cap (\mathbb{R}^+ \times \mathbb{R}^+) \) by \( C_k^{++} \), and similarly for \( C_k^- \) and for the other quadrants. So, for instance, \( C_k^{<+} := C_k^- \cap (\mathbb{R}^- \times \mathbb{R}^+) \) and one has

\[ C_k^{<+} (m, n) = \{(a, b) : (-a, b) \in C_k^{++} (-m, n)\} \].

As a consequence of this remark, we will often limit ourselves below to the study of the intersection of \( \Sigma^* \) with \( \mathbb{R}^+ \times \mathbb{R}^+ \).

It will be convenient to introduce a notation for the functions which appear in the description (3.1) of the sets \( C_k^+, C_k^- \). We will denote by \( \Phi_k^+ (a, b) \) (resp. \( \Phi_k^- (a, b) \)) the composition of \( k \) alternating functions \( \varphi_{a}^m \) and \( \varphi_{b}^n \) starting with \( \varphi_{a}^m \) (resp. \( \varphi_{b}^n \)). So, for instance,

\[ \Phi_k^+ (a, b)(s) = \varphi_{a}^m (\varphi_{b}^n (\varphi_{a}^m (s))), \quad \Phi_k^- (a, b)(s) := \varphi_{b}^n (\varphi_{a}^m (\varphi_{b}^n (s))). \]
When considering these functions, we will generally assume that, as in section 2, the data have been extended from $[T_1, T_2]$ to the whole of $\mathbb{R}$. So, with this notation
\[ C_k^{++} = \{(a, b) \in \mathbb{R}^+ \times \mathbb{R}^+: \Phi_k^>(a, b)(T_1) = T_2\} \] (3.2)
and similarly for $C_k^{<+}$ and for the other quadrants. These functions $\Phi_k^>$ and $\Phi_k^<$ enjoy properties which are rather similar to those of the zero-function.

The following lemma will be used repeatedly. It characterizes the nonemptiness of the sets $C_k^{++}, C_k^{<+}$ in terms of $\Phi_k^>, \Phi_k^<$ respectively.

**Lemma 3.3** Let $k \geq 2$. Then
\[ C_k^{++} \neq \emptyset \iff \lim_{a,b \to +\infty} \Phi_k^>(a,b)(T_1) < T_2, \]
\[ C_k^{<+} \neq \emptyset \iff \lim_{a,b \to +\infty} \Phi_k^<(a,b)(T_1) < T_2. \]

**Proof.** Note that the limits above exist since, by lemma 2.2, $\Phi_k^>(a,b)(T_1)$ and $\Phi_k^<(a,b)(T_1)$ are separately decreasing with respect to $(a,b) \in \mathbb{R}^+ \times \mathbb{R}^+$. Let us prove the statement relative to $C_k^{++}$ (the other one is proved similarly). The implication $\Rightarrow$ follows from (3.2) and the fact that $\Phi_k^>(a,b)(T_1)$ is separately strictly decreasing on its domain in $\mathbb{R}^+ \times \mathbb{R}^+$. The converse implication follows from the continuity of $\Phi_k^>(a,b)(T_1)$ and the fact that, by lemma 2.5, $\Phi_k^>(a,b) \to +\infty$ as $a$ or $b \to 0$ Q. E. D.

Our first main result in this section shows that $\Sigma^*$ is made of hyperbolic like curves of class $C^1$.

**Theorem 3.4** Let $k \geq 2$ and assume $C_k^{++}$ nonempty. Then there exist $\nu_k^> \geq \lambda_1^u$, $\mu_k^> \geq \lambda_1^u$ and a strictly decreasing $C^1$ function $f_k^>$ from $]\nu_k^>, +\infty[$ onto $]\mu_k^>, +\infty[$, with $(f_k^>')'(x) < 0$ for $x \in ]\nu_k^>, +\infty[$, such that
\[ C_k^{++} = \{(x, f_k^>(x)) : x \in ]\nu_k^>, +\infty[\}. \]

Similar result for $C_k^{<+}$

**Proof.** Let us prove the result relative to $C_k^{++}$ (the other one is proved similarly). Call
\[ I := \{a > 0 : \exists b > 0 \text{ with } \Phi_k^>(a,b)(T_1) = T_2\}. \]
Since $C_k^{++} \neq \emptyset$, $I \neq \emptyset$. Moreover, by the strict monotonicity of $\Phi_k^>$, for any $a \in I$, there exists only one $b > 0$ such that $\Phi_k^>(a,b)(T_1) = T_2$, which we denote by $f_k^>(a)$. Applying the implicit function theorem together with lemma 2.3 to the relation $\Phi_k^>(a,b)(T_1) = T_2$, one deduces that $I$ is open and that $f_k^>$ is of class $C^1$, with $(f_k^>')'(x) < 0$. Moreover, using the monotonicity and continuity properties of $\Phi_k^>$, one easily verifies that $I$ is an interval. It remains to prove that $I$ is unbounded from above, i.e. of the form $]\nu_k^>, +\infty[$, and that $f_k^>(x) \to +\infty$ as $x \to \nu_k^>$. To verify the unboundedness one observes that if $a \in I$ and $\bar{a} > a$, then
\[ \Phi_k^>(\bar{a}, f_k^>(a))(T_1) < \Phi_k^>(a, f_k^>(a))(T_1) = T_2; \]
since, by lemma 2.5, $\Phi_k^>(\bar{a}, b)$ is $> T_2$ for some (small) $b > 0$, the conclusion that $\bar{a} \in I$ follows by continuity. Now if $f_k^>(x) \to L < +\infty$ as $x \to \nu_k^>$, then, by continuity, we get
$\Phi_k^+(\nu_k^+, L)(T_1) = T_2$, so that $\nu_k^+ \in I$, which contradicts the fact that $I$ is open. Finally the inequalities $\nu_k^+ \geq \lambda_k^m$ and $\mu_k^+ \geq \lambda_k^n$ follow from the general properties of $\Sigma$ mentioned at the beginning of section 3. Q. E. D.

The next proposition gives some information on how these hyperbolic like curves are situated one with respect to the other.

**Proposition 3.5** Let $k \geq 2$. Assume that $C_{k+1}^{++}$ (or $C_{k+1}^<$) is nonempty. Then $C_k^{++}$ and $C_k^<$ are both nonempty. Moreover the curves $f_k^+$ and $f_k^<$ are both strictly below the curve $f_{k+1}^+$ (or $f_{k+1}^<$).

**Proof.** Let us assume $C_{k+1}^{++}$ nonempty (the argument is similar if it is $C_{k+1}^<$ which is nonempty). By lemma 3.3, this hypothesis means

$$\lim_{a, b \to +\infty} \Phi_{k+1}^+(a, b)(T_1) < T_2.$$ (3.3)

Let us suppose for instance $k$ even. Then

$$\Phi_{k+1}^+(a, b)(T_1) = \varphi_{\alpha}^m[\Phi_k^+(a, b)(T_1)] > \Phi_k^+(a, b)(T_1)$$

and consequently, by (3.3), we get

$$\lim_{a, b \to +\infty} \Phi_k^+(a, b)(T_1) < T_2,$$

which implies, by lemma 3.3, $C_k^{++} \neq 0$. Similar argument for $k$ odd. Let us now prove that $C_k^<$ is also non empty. The argument in fact is similar to the preceding one and is now based on the relations

$$\Phi_{k+1}^+(a, b)(T_1) = \Phi_k^<(a, b)[\varphi_{\alpha}^m(T_1)] > \Phi_k^<(a, b)(T_1).$$

It remains to see that $f_k^+$ and $f_k^<$ both lie strictly below $f_{k+1}^+$. Suppose by contradiction that $f_k^+(x) \geq f_{k+1}^+(x)$ for some $x > \max(\nu_k^+, \nu_{k+1}^+)$. Then

$$T_2 = \Phi_{k+1}^+(x, f_{k+1}^+(x))(T_1) \geq \Phi_{k+1}^-(x, f_k^+(x))(T_1)$$

$$> \Phi_k^+(x, f_k^+(x))(T_1) = T_2,$$

a contradiction. Similar argument for $f_k^<$. Q. E. D.

The rest of this section is mainly concerned with some results on the number of these hyperbolic like curves. Theorem 3.6 below is the converse of the first part of proposition 3.5.

**Theorem 3.6** Suppose $m^+(t) \neq 0$ and $n^+(t) \neq 0$ on $]T_1, T_2[$. Then $C_2^{++}$ or $C_2^<$ is nonempty. Moreover if for some $k \geq 2$, $C_k^{++}$ and $C_k^<$ are both nonempty, then $C_{k+1}^{++}$ or $C_{k+1}^<$ is nonempty.

Combining with remark 3.2, we get the following

**Corollary 3.7** If both $m(t)$ and $n(t)$ change sign, then $\Sigma^*$ contains at least one hyperbolic like curve in each quadrant.
Corollary 3.8 Suppose $m^+(t) \neq 0$ and $n^+(t) \neq 0$. Then either (i) $C_{k^+}^{>++}$ and $C_{k^+}^{<++}$ are nonempty for all $k$, or (ii) for some $k_0 \geq 1$, $C_{k^+}^{>++}$ and $C_{k^+}^{<++}$ are nonempty for all $k \leq k_0$. $C_{k^0+1}^{>++}$ or $C_{k^0+1}^{<++}$ is nonempty, the other one being empty, and $C_{k^+}^{>++}$ and $C_{k^+}^{<++}$ are empty for all $k > k_0 + 2$.

So, if $m(t)$ and $n(t)$ both change sign, then each quadrant contains a (non zero) odd or infinite number of nonempty sets $C_k^>$, $C_k^<$ with $k \geq 2$. We will see at the end of this section that all cases can effectively happen (cf. proposition 3.12).

Remark 3.9 It may happen that $C_{k^+}^{>++} = C_{k^+}^{<++}$. For instance on $[-1,+1]$, if $p(-t) = p(t)$, $q(-t) = q(t)$, $m(-t) = m(t)$ and $n(-t) = n(t)$, then $C_{k^+}^{>++} = C_{k^+}^{<++}$ for all $k$. On the other hand, in the one weight problem on $[0,2\pi]$, if $m(t) = \sin t$ on $[0,\pi]$ and $m(t) = 0$ on $[\pi,2\pi]$, then $C_{2^+}^{>++} \neq C_{2^+}^{<++}$. In fact theorem 5.1 implies that $C_{2^+}^{>++}$ and $C_{2^+}^{<++}$ have different asymptotic behaviours.

Proof of Theorem 3.6. Let us first consider $C_{2^+}^{>++}$ and $C_{2^+}^{<++}$. We will use the notation (2.6) as well as a similar one for the weight $n(t) : \beta_k^+ := \inf \{ t > s : n(t) > 0 \}$. Since $m^+(t) \neq 0$ and $n^+(t) \neq 0$, we have $\alpha^+_1 < T_2$ and $\beta^+_1 < T_2$. Two cases are distinguished : (i) $\alpha^+_1 \leq \beta^+_1$, or (ii) $\beta^+_1 < \alpha^+_1$. Consider case (i). By the definition of $\beta^+_1$,

$$n^+(t) \neq 0 \quad \text{on} \quad \left[ \beta^+_1, \beta^+_1 + \epsilon \right].$$

for any $\epsilon > 0$. Since, by lemma 2.4, $\varphi^m_\alpha(T_1)$ converges to $\alpha^+_1 \leq \beta^+_1$, as $a \to +\infty$, we deduce from (3.4) and lemma 2.4 that

$$\lim_{a,b \to +\infty} \varphi^m_\alpha(T_1) \leq \beta^+_1 < T_2.$$ 

Lemma 3.3 then implies $C_{2^+}^{>++} \neq \emptyset$. In case (ii), a similar argument yields $C_{2^+}^{<++} \neq \emptyset$.

Let us now turn to the study of $C_{k^+}^{>++}$ and $C_{k^+}^{<++}$, assuming $k \geq 2$, $C_{k^+}^{>++} \neq \emptyset$ and $C_{k^+}^{<++} \neq \emptyset$. This hypothesis means, by lemma 3.3, that

$$\alpha := \lim_{a,b \to +\infty} \Phi_k^>(a,b)(T_1) < T_2 \quad \text{and} \quad \beta := \lim_{a,b \to +\infty} \Phi_k^<(a,b)(T_1) < T_2.$$ 

We again distinguish two cases : (i) $\alpha \leq \beta$ or (ii) $\beta < \alpha$. Consider case (i) and let us first assume $k$ odd. So

$$\Phi_k^>(a,b)(T_1) = \varphi^m_\beta \left[ \Phi_{k-1}^<(a,b)(T_1) \right] \to \beta \quad \text{as} \quad a, b \to +\infty.$$ 

This implies that

$$n^+(t) \neq 0 \quad \text{on} \quad \left[ \beta, \beta + \epsilon \right].$$

for any $\epsilon > 0$. Indeed, for $a, b$ sufficiently large, one has $\beta < \varphi^m_\beta \left[ \Phi_{k-1}^<(a,b)(T_1) \right] < \beta + \epsilon$, which implies $n^+(t) \neq 0$ on $\left[ \Phi_{k-1}^<(a,b)(T_1), \beta + \epsilon \right]$, and (3.5) follows. Now, $\Phi_k^>(a,b)(T_1)$ converges to $\alpha \leq \beta$ as $a, b \to +\infty$, and so we deduce from (3.5) and lemma 2.4 that

$$\lim_{a,b \to +\infty} \varphi^m_\beta \left[ \Phi_k^>(a,b)(T_1) \right] \leq \beta < T_2.$$ 

Consequently, by lemma 3.3, $C_{k^+}^{>++} \neq \emptyset$. A similar argument for $k$ even would also lead to $C_{k^+}^{<++} \neq \emptyset$. And in case (ii) one would get $C_{k^+}^{<++} \neq \emptyset$. Q. E. D.

We will now give a sufficient (and almost necessary) condition on the weights $m(t)$ and $n(t)$ in order that $\Sigma^* \cap (IR^+ \times IR^+)$ contains an infinite number of hyperbolic like curves.
Theorem 3.10 If \( m^+(t) \cdot n^+(t) \neq 0 \) on \([T_1, T_2]\), then \( C_k^{++} \) and \( C_k^{<+} \) are nonempty for all \( k \). Conversely suppose that the set where \( m(t) \) (or \( n(t) \)) is \( > 0 \) is made of a finite union of intervals. Under this hypothesis, if \( C_k^{++} \) and \( C_k^{<+} \) are nonempty for all \( k \), then \( m^+(t) \cdot n^+(t) \neq 0 \) on \([T_1, T_2]\).

Proof. If \( m^+(t) \cdot n^+(t) \neq 0 \), then one can find \( T_1 < t_1 < t_2 < T_2 \) such that \( m(t) \) and \( n(t) \) are both \( > 0 \) on \([t_1, t_2]\). Lemma 2.4 then implies that for each \( k \), \( \Phi_k(a,b)(t_1) \to t_1 \) as \( a,b \to +\infty \). By monotonicity we deduce that for each \( k \), the limits of \( \Phi_k(a,b)(T_1) \) and \( \Phi_k(a,b)(T_2) \) are \( \leq t_1 < T_2 \) as \( a,b \to +\infty \). Lemma 3.3 then yields the conclusion.

Conversely let us assume that \( C_k^{++} \) and \( C_k^{<+} \) are nonempty for all \( k \) and that, say, \( m(t) \) is \( > 0 \) on \( I_1 \cup I_2 \cup \ldots \cup I_r \) and \( \leq 0 \) outside this set, where \( I_i \) is an interval of extremities \( s_i, t_i \), with \( T_1 \leq s_1 < t_1 < s_2 < t_2 \leq \ldots \leq s_r < t_r \leq T_2 \).

Suppose by contradiction that \( m^+(t) \cdot n^+(t) \equiv 0 \) on \([T_1, T_2]\). So \( n(t) \leq 0 \) on each \([s_i, t_i]\). This implies that \( \varphi_a^m(T_1) \geq s_1, \varphi_b^m(s_i) \geq t_i, \varphi_a^m(t_i) \geq s_{i+1}, \varphi_a^m(t_r) \geq T_2 \) for all \( a,b \geq 0 \) and each \( i = 1, \ldots, r \). Consequently \( \Phi_k(a,b)(T_1) \geq T_2 \) for \( k \geq 2r + 2 \) and for all \( a,b \geq 0 \). Lemma 3.3 then implies \( C_k^{<+} = \emptyset \) for \( k \geq 2r + 2 \), a contradiction. Similar argument if it is \( n(t) \) which is \( > 0 \) on a finite union of intervals. Q. E. D.

Remark 3.11 Some hypothesis on the bumps of the weights is needed in the second part of theorem 3.10. For example if \([T_1, T_2] = [0, \pi] \) and \( m(t) = -n(t) = \sin(t) \cdot \sin(1/t) \), then \( m^+(t) \cdot n^+(t) \equiv 0 \) although \( C_k^{++} \) and \( C_k^{<+} \) are nonempty for all \( k \). This latter fact follows from lemma 3.3 since here, \( \Phi_k(a,b)(0) \) and \( \Phi_k(a,b)(0) \to 0 \) as \( a,b \to +\infty \). More general hypothesis on the weights are considered in [1] which guarantee the validity of the second part of theorem 3.10.

To conclude this section, we show that all cases can effectively happen with respect to the numbers of nonempty sets \( C_k^{+}, C_k^{-} \) with \( k \geq 2 \) in the various quadrants.

Proposition 3.12 Given \( K, L, M, N \) in \( \{0, 1, 2, \ldots, +\infty\} \), there exist weights \( m(t), n(t) \), which both change sign, such that the total number of nonempty sets \( C_k^{++} \) and \( C_k^{<+} \) (resp. \( C_k^{<--} \) and \( C_k^{><-} \), \( C_k^{<+<} \) and \( C_k^{><+} \), \( C_k^{<+>} \) and \( C_k^{><<} \)) with \( k \geq 2 \) is equal to \( 2K + 1 \) (resp. \( 2L + 1, 2M + 1, 2N + 1 \)).

Proof. The proof is based on the observation that the number of nonempty sets \( C_k^{+} \) and \( C_k^{-} \) with \( k \geq 2 \) in \( \mathbb{R}^+ \times \mathbb{R}^+ \) (resp. \( \mathbb{R}^- \times \mathbb{R}^-, \mathbb{R}^+ \times \mathbb{R}^-, \mathbb{R}^- \times \mathbb{R}^+ \)) only depends on the relative position of the positive (resp. negative, positive, negative) bumps of \( m \) and of the positive (resp. negative, negative, positive) bumps of \( n \). A positive bump of \( m \) which intersects a positive bump of \( n \) leads, by theorem 3.10, to an infinity of nonempty sets \( C_k^{<} \) and \( C_k^{<} \) with \( k \geq 2 \) in \( \mathbb{R}^+ \times \mathbb{R}^+ \). A succession of \( K + 2 \) alternating positive bumps of \( m \) and positive bumps of \( n \) (i.e., for instance, \( m(t) = \sin(t) \) and \( n(t) = \sin(t + \pi) \)) on \([0, (K + 2)\pi]\) leads to exactly \( 2K + 1 \) nonempty sets \( C_k^{+} \) and \( C_k^{-} \) with \( k \geq 2 \) in \( \mathbb{R}^+ \times \mathbb{R}^+ \) (a simple consequence of lemmas 2.4 and 3.3). This latter construction of alternating bumps will be used repeatedly below.

To carry through the details of the proof, we will distinguish several cases according to the number of quadrants having to contain an infinite number of sets \( C_k^{+} \) and \( C_k^{-} \) with \( k \geq 2 \).
Case 1: four “infinite” quadrants, i.e. $K = L = M = N = +\infty$. Applying theorem 3.10 and remark 3.2, one easily verifies that this situation occurs for instance on $[0, 2\pi]$ if we take $m(t) = \sin t$ and $n(t) = \sin(t + \pi/2)$.

Case 2: one “finite” quadrant, three “infinite” quadrants. Applying remark 3.2, one sees that it suffices to construct an example with $K$ given in $\{0, 1, 2, \ldots \}$ and $L = M = N = +\infty$. Using theorem 3.10 and lemmas 2.4 and 3.3, one easily verifies that this situation occurs for instance on $[0, (K + 2)\pi]$ if we take $m(t) = \sin t$ on $[0, 2\pi]$, $n(t) = -\sin 2t/3$ on $[0, 3\pi/2]$, $n(t) = -\sin 2(t - 3\pi/2)$ on $[3\pi/2, 2\pi]$, and then, on $[2\pi, (K + 2)\pi]$, $K$ alternating positive bumps for $m$ and positive bumps for $n$ (starting with $m$).

Case 3: two “finite” quadrants, two “infinite” quadrants. Applying remark 3.2, one sees that it suffices to construct an example with $K, L$ given in $\{0, 1, 2, \ldots \}$ and $M = N = +\infty$, an example with $K, M$ given in $\{0, 1, 2, \ldots \}$ and $L = N = +\infty$, and an example with $K, N$ given in $\{0, 1, 2, \ldots \}$ and $L = M = +\infty$. We will describe below the construction relative to $K, L$ finite (and $K \geq L, L$ even). The other cases can be treated along similar lines. The desired situation occurs for instance on $[0, (K + 2)\pi]$ if we take $m(t) = \sin t$ on $[0, (L + 2)\pi]$, $n(t) = -\sin t$ on $[0, (L + 2)\pi]$, and then, on $[(L + 2)\pi, (K + 2)\pi]$, $K - L$ alternating positive bumps for $m$ and positive bumps for $n$ (starting with $m$).

Case 4: three “finite” quadrants, one “infinite” quadrant. Applying remark 3.2, one sees that it suffices to construct an example with $K = +\infty$ and $L, M, N$ given in $\{0, 1, 2, \ldots \}$. We will describe below the construction for $L, M, N$ even $\neq 0$. The other cases can be treated along similar lines. The desired situation occurs for instance if we take $m(t) = n(t) = \sin t$ on $[0, \pi]$, and then $L$ alternating negative bumps for $m$ and $n$ (starting with $m$), followed by $N$ alternating positive bumps for $n$ and negative bumps for $m$ (starting with $n$), followed by $M$ alternating positive bumps for $m$ and negative bumps for $n$ (starting with $m$).

Case 5: four “finite” quadrants, i.e. $K, L, M, N$ given in $\{0, 1, 2, \ldots \}$. We will describe below the construction for $K, L, M, N$ even $\neq 0$. The other cases can be treated along similar lines. The desired situation occurs for instance if we take $K$ alternating positive bumps for $m$ and $n$ (starting with $m$), followed by $M$ alternating positive bumps for $m$ and negative bumps for $n$ (starting with $m$), followed by one positive bump for $m$, followed by $L$ alternating negative bumps for $m$ and $n$ (starting with $m$), followed finally by $N$ alternating positive bumps for $n$ and negative bumps for $m$ (starting with $n$). Q. E. D.

**Example 3.13** If we take a positive bump for $m$, followed by a positive bump for $n$, followed by a negative bump for $m$, followed by a negative bump for $n$, (i.e., for instance, $m(t) = (\sin t)^+, n(t) = (\sin t)^- \in [0, 2\pi]$, and $m(t) = -(\sin t)^+, n(t) = -(\sin t)^- \in [2\pi, 4\pi]$), then $\Sigma^*$ contains exactly one hyperbolic like curve in each quadrant.

**Proposition 3.12** Concerns weights which both change sign. Similar arguments lead to the following two propositions.

**Proposition 3.14** Given a pair $(A, B)$ of quadrants different from the diagonal pairs $(\mathbb{R}^+ \times \mathbb{R}^+, \mathbb{R}^- \times \mathbb{R}^-)$ or $(\mathbb{R}^+ \times \mathbb{R}^-, \mathbb{R}^- \times \mathbb{R}^+)$, and given $R, S \in \{0, 1, 2, \ldots, +\infty\}$, there exist weights $m(t)$ and $n(t)$, one of them which changes sign and the other one which does not, such that the total number of nonempty sets $C^+_k$ and $C^-_k$ with $k \geq 2$ in the quadrant $A$ (resp. $B$) is equal to $2R + 1$ (resp. $2S + 1$), while no such sets appear in the other two quadrants.
Proposition 3.15 Given a quadrant and \( R \in \{0,1,2,\ldots, +\infty\} \), there exist weights \( m(t) \) and \( n(t) \), which do no change sign, such that the total number of nonempty sets \( C_k^+ \) and \( C_k^- \) with \( k \geq 2 \) in the given quadrant is \( 2R + 1 \), while no such sets appear in the other three quadrants.

4 The one weight problem

We now consider problem (1.1) with \( L \) as in the introduction and \( m \in C[T_1,T_2] \), \( m(t) \neq 0 \).

In this case the Fučík spectrum \( \Sigma \) is symmetric with respect to the diagonal \( a = b \) in the \((a,b)\) plane. In fact replacing \( u \) by \(-u\) in (1.1) shows that \( C_k^+ \) (resp. \( C_k^- \)) is symmetric of \( C_k^+ \) (resp. \( C_k^- \)). Moreover \((\lambda_k^m, \lambda_k^n) \in C_k^+ \cap C_k^- \) and \((\lambda_k^m, \lambda_k^n) \in C_k^- \cap C_k^+ \) for all \( k \) (depending of course whether \( m \) is \( \geq 0 \), \( \leq 0 \) or changes sign); in particular the corresponding sets \( C_k^+ \) and \( C_k^- \) in \( R^+ \times R^- \) and - or \( R^- \times R^+ \) are nonempty. As in section 3, all the sets \( C_k^+, C_k^-, C_k^-, C_k^+, C_k^+, C_k^- \), \( C_k^+, C_k^- \) with \( k \geq 2 \), when nonempty, are hyperbolic like curves (cf. theorem 3.4).

The case where the weight \( m(t) \) does not change sign is simpler: if \( m \geq 0 \) (resp. \( m \leq 0 \)), then \( \Sigma^* \) is made of the infinite sequence of hyperbolic like curves \( C_k^+ \) and \( C_k^- \) (resp. \( C_k^-, C_k^+ \)), \( k = 2,3,\ldots \).

From now on in this section we will assume that

\[ m(t) \text{ changes sign in } ]T_1,T_2[. \] (4.1)

Under this assumption, \( \Sigma^* \) contains the infinite sequence of hyperbolic like curves

\[ C_k^+, C_k^-, C_k^-, C_k^+, k = 2,3,\ldots \]

Moreover, by remark 3.2 and theorem 3.6, hyperbolic like curves also appear in \( R^+ \times R^- \) and \( R^- \times R^+ \). These additional curves in \( R^+ \times R^- \) are symmetric of those in \( R^- \times R^+ \), and their distribution is given by corollary 3.8: either (i) \( C_k^+ \) and \( C_k^- \) are nonempty for all \( k \), or (ii) for some \( k_0 \geq 1 \), \( C_k^+ \) and \( C_k^- \) are nonempty for all \( k \leq k_0 \), \( C_{k_0+1}^+ \) or \( C_{k_0+1}^- \) is nonempty, the other one being empty, and \( C_k^+ \) and \( C_k^- \) are empty for all \( k \geq k_0 + 2 \).

The rest of this section is concerned with showing that the number of these additional curves in \( R^+ \times R^- \) and \( R^- \times R^+ \) is directly related to the number of changes of sign of the weight \( m(t) \). First we have to make precise this notion of number of changes of sign.

Definition 4.1 Let \( s \in ]T_1,T_2[. \) We say that \( s \) is a simple point of change of sign of \( m \) if there exists \( T_1 < s' < s < T_2 \) such that (i) \( m \leq 0 \) on \( ]s' - \epsilon, s[ \) for all \( 0 < \epsilon < \epsilon_0 \), \( m \equiv 0 \) on \( ]s', s[ \) and \( m \equiv 0 \) on \( ]s, s + \epsilon[ \) for all \( 0 < \epsilon < \epsilon_0 \), or (ii) \( m \geq 0 \) on \( ]s' - \epsilon, s'[ \) for all \( 0 < \epsilon < \epsilon_0 \), \( m \equiv 0 \) on \( ]s', s[ \) and \( m \equiv 0 \) on \( ]s, s + \epsilon[ \) for all \( 0 < \epsilon < \epsilon_0 \).

Definition 4.2 Let \( s \in ]T_1,T_2[. \) We say that \( s \) is a multiple point of change of sign of \( m \) if either (i) \( s > T_1 \) and \( m^+ \) and \( m^- \) are \( \neq 0 \) on \( ]s - \epsilon, s[ \) for any \( \epsilon > 0 \), or (ii) \( s < T_2 \) and \( m^+ \) and \( m^- \) are \( \neq 0 \) on \( ]s, s + \epsilon[ \) for any \( \epsilon > 0 \).

Definition 4.3 If \( m \) admits at least one multiple point of change of sign, then we say that the number of changes of sign of \( m \) is \( +\infty \). If \( m \) does not admit any multiple point of change of sign, then it is easily verified that, under (4.1), \( m \) admits a nonzero finite number \( N \) of
simple points of change of sign. In this case we say that the number of changes of sign of \(m\) is \(N\).

**Theorem 4.4** Assume (4.1). Let \(N \in \{1, 2, \ldots, +\infty\}\) be the number of changes of sign of \(m\). Then the total number of nonempty sets \(C_{k}^{++}\) and \(C_{k}^{<+}\) with \(k \geq 2\) is equal to \(2N - 1\).

**Proof.** Consider first the situation where \(N = +\infty\) and let \(s \in [T_1, T_2]\) be a multiple point of change of sign of \(m\). If \(s < T_2\), then one easily verifies, using lemma 2.4, that for any \(k = 2, 3, \ldots\),

\[
\lim_{a \to +\infty} \Phi_{k}^{>}(a, b)(T_1) \leq s \quad \text{and} \quad \lim_{a \to +\infty} \Phi_{k}^{<}(a, b)(T_1) \leq s.
\]

If \(s = T_2\), then one gets that the above limits are \(< s\). So, in any case, lemma 3.3 implies that \(C_{k}^{++}\) and \(C_{k}^{<+}\) are nonempty for all \(k\). Consider now the situation where \(N\) is finite. Then there exist

\[
s_1 = T_1 < s_2 < \ldots < s_{N+1} < s_{N+2} = T_2
\]

such that either \(m\) is \(>0\) on \([s_1, s_2]\), \(<0\) on \([s_2, s_3]\), \(>0\) on \([s_3, s_4]\), \ldots, or \(m\) is \(\leq 0\) \([s_1, s_2]\), \(>0\) on \([s_2, s_3]\), \(\leq 0\) on \([s_3, s_4]\), \ldots. Let us deal with the first case with, say, \(N\) even, so that \(m\) is \(\geq 0\) on \([s_{N+1}, s_{N+2}]\) (the other cases are treated similarly). Using lemma 2.4, one gets that

\[
\lim_{a \to +\infty} \Phi_{N+1}^{>}(a, b)(T_1) < T_2, \quad \lim_{a \to +\infty} \Phi_{N+2}^{>}(a, b)(T_1) = T_2,
\]

\[
\lim_{a \to +\infty} \Phi_{N}^{<}(a, b)(T_1) < T_2, \quad \lim_{a \to +\infty} \Phi_{N+1}^{<}(a, b)(T_1) = T_2.
\]

Consequently, lemma 3.3 implies that \(C_{k}^{++}\) and \(C_{k}^{<+}\) are nonempty for \(k = 2, \ldots, N\), that \(C_{N+1}^{++}\) is nonempty, that \(C_{N+1}^{<+}\) is empty, and that all \(C_{k}^{++}\) and \(C_{k}^{<+}\) are empty for \(k \geq N + 2\). The conclusion follows. Q. E. D.

Theorem 4.4 implies a result analogous to proposition 3.12: given \(N \in \{1, 2, \ldots, +\infty\}\), there exists a weight \(m(t)\) which changes sign such that \(\mathbb{R}^{+} \times \mathbb{R}^{-}\) exactly contains \(2N - 1\) nonempty sets \(C_{k}^{++}\) and \(C_{k}^{<+}\) with \(k \geq 2\).

**Remark 4.5** It may happen that \(C_{k}^{++} = C_{k}^{<+}\) (and then of course, by symmetry, \(C_{k}^{<+} = C_{k}^{<+}\)). This is the case for instance for all \(k\) even if \([T_1, T_2] = [-1, +1]\), \(p(-t) = p(t)\), \(q(-t) = q(t)\) and \(m(-t) = m(t)\). In fact, in this example, one also has \(C_{k}^{++} = C_{k}^{++}\) and \(C_{k}^{<+} = C_{k}^{<+}\) for all \(k\) even. On the other hand, if \([T_1, T_2] = [0, 2\pi]\) and \(m(t) = \sin t\), then \(C_{2}^{++} \neq C_{2}^{<+}\). In fact, theorem 5.1 implies that these two curves have different asymptotic behaviours.

### 5 Asymptotic behaviour of the first curves

In this section we return to the two weights problem (1.3) with \(L\) as in the introduction and \(m, n \in C[T_1, T_2]\), \(m(t)\) and \(n(t)\) \(\neq 0\). Our purpose is to investigate the asymptotic behaviour of the first hyperbolic like curves \(C_{2}^{++}, C_{2}^{<+}, C_{2}^{<+}, C_{2}^{<+}, C_{2}^{<+}, C_{2}^{<+}, C_{2}^{<+}, C_{2}^{<+}\). We recall that at least one such curve appears in \(\mathbb{R}^{+} \times \mathbb{R}^{+}\) (resp. \(\mathbb{R}^{-} \times \mathbb{R}^{-}\), \(\mathbb{R}^{+} \times \mathbb{R}^{-}\), \(\mathbb{R}^{-} \times \mathbb{R}^{-}\))
if (and only if) \( m^+ \neq 0 \) and \( n^- \neq 0 \) (resp. \( m^- \neq 0 \) and \( n^+ \neq 0 \), \( m^+ \neq 0 \) and \( n^- \neq 0 \) and \( n^+ \neq 0 \)). In particular, in the one weight problem (1.1), at least one such curve appears in each quadrant if \( m \) changes sign.

Some notations are needed to state our result. Let us define

\[
T_{1m}^+ := \inf\{t \in [T_1, T_2[ : m(t) > 0\}, \\
T_{2m}^+ := \sup\{t \in [T_1, T_2[ : m(t) > 0\}, \\
T_{1m}^- := \inf\{t \in [T_1, T_2[ : m(t) < 0\}, \\
T_{2m}^- := \sup\{t \in [T_1, T_2[ : m(t) < 0\},
\]

and similarly for \( n : T_{1n}^+, T_{2n}^+, T_{1n}^-, T_{2n}^- \). Note that some of these quantities may not have sense in \([T_1, T_2[\) : for instance \( T_{1m}^+ \) and \( T_{2m}^+ \) are only defined if \( m^+ \neq 0 \), and similarly for the others. We will also denote by \( \lambda_1(m, [t_1, t_2[) \) (resp. \( \lambda_{-1}(m, [t_1, t_2[) \)) the first positive (resp. negative) eigenvalue of \( L \) with Dirichlet boundary condition on the interval \([t_1, t_2[\) for the weight \( m \). Again here this quantity makes sense only if \( m \) has a nontrivial positive (resp. negative) part on \([t_1, t_2[\).

Theorem 5.1

(i) If \( C_{2m}^{++} \neq \emptyset \), then \( C_{2m}^{++} \) is asymptotic to the lines \( \mathbb{R} \times \lambda_1(n, ]T_{1m}^+, T_{2m}^+[) \) and \( \lambda_1(m, ]T_{1m}^+, T_{2m}^+[) \) \times \mathbb{R}.

(ii) If \( C_{2m}^{++} \neq \emptyset \), then \( C_{2m}^{++} \) is asymptotic to the lines \( \mathbb{R} \times \lambda_1(n, ]T_{1m}^+, T_{2m}^+[) \) and \( \lambda_1(m, ]T_{1m}^+, T_{2m}^+[) \) \times \mathbb{R}.

(iii) If \( C_{2m}^{--} \neq \emptyset \), then \( C_{2m}^{--} \) is asymptotic to the lines \( \mathbb{R} \times \lambda_{-1}(n, ]T_{1m}^-, T_{2m}^-[) \) and \( \lambda_{-1}(m, ]T_{1m}^-, T_{2m}^-[) \) \times \mathbb{R}.

(iv) If \( C_{2m}^{--} \neq \emptyset \), then \( C_{2m}^{--} \) is asymptotic to the lines \( \mathbb{R} \times \lambda_{-1}(n, ]T_{1m}^-, T_{2m}^-[) \) and \( \lambda_{-1}(m, ]T_{1m}^-, T_{2m}^-[) \) \times \mathbb{R}.

(v) If \( C_{2m}^{++} \neq \emptyset \), then \( C_{2m}^{++} \) is asymptotic to the lines \( \mathbb{R} \times \lambda_{-1}(n, ]T_{1m}^-, T_{2m}^-[) \) and \( \lambda_{-1}(m, ]T_{1m}^-, T_{2m}^-[) \) \times \mathbb{R}.

(vi) If \( C_{2m}^{++} \neq \emptyset \), then \( C_{2m}^{++} \) is asymptotic to the lines \( \mathbb{R} \times \lambda_{-1}(n, ]T_{1m}^-, T_{2m}^-[) \) and \( \lambda_{-1}(m, ]T_{1m}^-, T_{2m}^-[) \) \times \mathbb{R}.

(vii) If \( C_{2m}^{--} \neq \emptyset \), then \( C_{2m}^{--} \) is asymptotic to the lines \( \mathbb{R} \times \lambda_1(n, ]T_{1m}^+, T_{2m}^+[) \) and \( \lambda_1(m, ]T_{1m}^+, T_{2m}^+[) \) \times \mathbb{R}.

(viii) If \( C_{2m}^{--} \neq \emptyset \), then \( C_{2m}^{--} \) is asymptotic to the lines \( \mathbb{R} \times \lambda_1(n, ]T_{1m}^+, T_{2m}^+[) \) and \( \lambda_1(m, ]T_{1m}^+, T_{2m}^+[) \) \times \mathbb{R}.

Remark 5.2 In (i) above, the hypothesis \( C_{2m}^{++} \neq \emptyset \) means that

\[
\varphi_a^b (\varphi_a^m(T_i)) = T_2 \quad \text{for some } a, b > 0.
\]

This implies that \( n^+ \neq 0 \) on \( ]T_{1m}^+, T_{2m}^+[\), and consequently the eigenvalue \( \lambda_1(n, ]T_{1m}^+, T_{2m}^+[) \) which appears in (i) is well-defined. (5.1) also implies that \( m^+ \neq 0 \) on \( ]T_{1m}^+, T_{2m}^+[\), and consequently the eigenvalue \( \lambda_1(m, ]T_{1m}^+, T_{2m}^+[) \) which appears in (i) is well-defined. Similar observation for the other curves.

15
Proof of Theorem 5.1. By using remark 3.2, one sees that the statements relative to the curves in $\mathbb{R}^- \times \mathbb{R}^-$, $\mathbb{R}^+ \times \mathbb{R}^-$ and $\mathbb{R}^- \times \mathbb{R}^+$ can be deduced from those relative to the curves in $\mathbb{R}^+ \times \mathbb{R}^+$. Dealing with $\mathbb{R}^+ \times \mathbb{R}^+$, we will only consider $C_2^{++}$ ($C_2^{++}$ can be treated similarly).

Let $(a, b) \in \mathbb{R}^+ \times \mathbb{R}^+$. Then

\[(a, b) \in C_2^{++} \iff \varphi_a^n(\varphi_a^m(T_1)) = T_2 \iff \varphi_a^m(T_1) = (\varphi_a^n)^{-1}(T_2)\] (5.2)

If $a \to +\infty$ in (5.2), then, by lemma 2.4, $\varphi_a^m(T_1) \to T_1^{m^+}$, and we deduce from part (ii) of lemma 5.3 below that $b \to \bar{b}$ with $\varphi_b^m(T_1^{m^-}) = T_2$, i.e. $\bar{b} = \lambda_1(n, [T_1^{m^+}, T_2])$. If $b \to +\infty$ in (5.2), then, by part (i) of lemma 5.3 below, $\varphi_b^m(T_2) \to T_2^{n^+}$, and we deduce from part (ii) of lemma 5.3 below that $a \to \bar{a}$ with $\varphi_a^m(T_1) = T_2^{n^+}$, i.e. $\bar{a} = \lambda_1(m, [T_1, T_2^{n^+}])$. Q. E. D.

Lemma 5.3 (i) $(\varphi_a^m)^{-1}(T_2) \to T_2^{m^+}$ as $a \to +\infty$. (ii) Let $\varphi_a^m(x) = y$ with $a > 0$. If $x \to \bar{x}$ and $y \to \bar{y}$ with $\bar{x} \neq \bar{y}$, then $a \to \text{some } \bar{a} > 0$ which verifies $\varphi_a^m(\bar{x}) = \bar{y}$.

Proof. Part (i) follows from lemma 2.4 by reversing the time. Part (ii) concerns the continuity of the function $(x, y) \to a > 0$ defined by the relation $\varphi_a^m(x) = y$. The implicit function theorem combined with (2.5) and an argument based on Rolle’s theorem as in the proof of lemma 2.3 imply that this function is of class $C^1$. Q. E. D.

Part (ii) of lemma 5.3 can also be seen as a result on the continuous dependence of $\lambda_1(m, [x, y])$ with respect to the domain $[x, y]$.

Corollary 5.4 If $m$ and $n$ both have compact support in $[T_1, T_2]$, then none of the first hyperbolic like curves is asymptotic on any side to the trivial horizontal and vertical lines. The converse implication also holds.

Proof. Assume that $m$ and $n$ have compact support and, to fix the ideas that both $m$ and $n$ change sign (similar argument in the other cases). Then the quantities $T_1^{m^+}, T_1^{m^-}, T_1^{n^+}, T_1^{n^-}$ (resp. $T_2^{m^+}, T_2^{m^-}, T_2^{n^+}, T_2^{n^-}$) are all well-defined and $> T_1$ (resp. $< T_2$). It follows that the intervals like $[T_1^{m^+}, T_2^{m^-}, T_1^{m^-}, T_2^{m^+}, T_1^{n^+}, T_2^{n^-}, T_1^{n^-}, T_2^{n^+}]$ which appear in the statement of theorem 5.1 are all strictly smaller than $[T_1, T_2]$. By the strict monotonocity dependence of the first eigenvalues with respect to the domain, we deduce that the positive (resp. negative) first eigenvalue which appear in the statement of theorem 5.1 are all $> (resp. <)$ than the corresponding positive (resp. negative) first eigenvalues on $[T_1, T_2]$. The conclusion of Corollary 5.4 follows. Consider now the converse implication and suppose, by contradiction, that, for instance, the support of $m$ hits $T_1$. Then the support of $m^+$ or the support of $m^-$ hits $T_1$. Suppose it is that of $m^+$. This implies that $T_1^{m^+} = T_1$ and also that at least $C_2^{++}$ or $C_2^{+-}$ is $\neq \emptyset$. Suppose it is $C_2^{++}$. Part (i) of theorem 5.1 then implies that $C_2^{++}$ is asymptotic to $\mathbb{R} \times \lambda_1(n, [T_1, T_2]) = \mathbb{R} \times \lambda_1^n$, a contradiction. Similar argument in the other cases. Q. E. D.

Corollary 5.5 Consider the one weight problem (1.1) with $m \geq 0$. Then there exists $\epsilon > 0$ such that $\Sigma^*$ is contained in $[\lambda_1^m + \epsilon, +\infty[ \times \lambda_1^m + \epsilon, +\infty[ \iff m$ has compact support in $[T_1, T_2]$.

Remark 5.6 When there is no weight, it was observed in [7], [4], under various boundary conditions, that a strong connexion (qualitative and quantitative) exists between on one side
the asymptotic behaviour of the first curves of $\Sigma^*$ and on the other side the uniformity or nonuniformity of the antimaximum principle: the existence of an $\epsilon > 0$ as in corollary 5.5 corresponds to the uniformity of the antimaximum principle, and moreover the largest $\epsilon$ admissible corresponds exactly to the largest interval of uniformity. In this context corollary 5.5 should be compared with the recent result of [10] where it is shown that in the Dirichlet problem, whatever the weight, the antimaximum principle is always non uniform. It follows that the connexion referred to above does not hold anymore in the presence of a weight with compact support, even $\neq 0$.

References


