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\textbf{\textit{L}^{\infty}\text{-ERROR ESTIMATES FOR A SYSTEM OF QUASI-VARIATIONAL INEQUALITIES}}

M. Boulbrachene\textsuperscript{1}
\textit{Sultan Qaboos University, College of Science, Department of Mathematics, P.O. Box 36, Muscat 123, Sultanate of Oman,}

M. Haiour
\textit{Departement de Mathematiques, Faculte des Sciences, Universite de Annaba, B.P. 12 Annaba 23000 Algeria}

and

S. Saadi\textsuperscript{2}
\textit{The Abdus Salam International Centre for Theoretical Physics, Trieste, Italy.}

\textbf{Abstract}

This paper deals with the numerical analysis of a system of elliptic quasi-variational inequalities. Under a $W^{2,p}(\Omega)$-regularity of the continuous solution, a quasi-optimal $L^{\infty}$-convergence of a piecewise linear finite element method is established, involving a monotone algorithm of Bensoussan- Lions type and standard $L^{\infty}$-error estimates known for elliptic variational inequalities.

MIRAMARE – TRIESTE
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\textsuperscript{1}E-mail: boulbrac@squ.edu.om
\textsuperscript{2}E-mail: samira@ictp.trieste.it
1. Introduction

In this paper we are concerned with the $L^\infty$ convergence of the standard finite element approximation of the system of quasi-variational inequalities (QVIs): Find a vector $U = (u^1, ..., u^M)$ satisfying

$$
\begin{cases}
\quad a^i(u^i, v - u^i) \geq (f^i, v - u^i) \forall v \in H^1_0(\Omega) \\
\quad u^i \leq Mu^i; u^i \geq 0; v \leq Mu^i
\end{cases}
$$

where $\Omega$ is a bounded smooth domain of $\mathbb{R}^N$ with boundary $\partial \Omega$, $a^i(u, v)$ are $M$-bilinear forms defined on $H^1(\Omega) \times H^1(\Omega)$, $(.,.)$ is the inner product in $L^2(\Omega)$ and $f^i$ are $M$ regular functions.

This system arises from the management of energy production problems (see [?] and references therein). In the case studied here, $Mu^i$ represents a "cost function" and the prototype encountered is

$$
Mu^i(x) = k + \inf_{\mu \neq x} \mu^i ; k \text{ is a positive number.}
$$

Naturally, the structure of problem (??) is analogous to that of the classical obstacle problem where the obstacle is replaced by an implicit one depending upon the solution sought. The terminology quasi-variational inequality being chosen is a result of this remark.

The $L^\infty$-error estimate is a challenge not only for practical reasons but also due to its inherent difficulty of convergence in this norm. Moreover, the interest in using such a norm for the approximation of obstacle problems is that they are types of free boundary problems. This fact was validated by the paper of F. Brezzi; C. Caffarelli, [?] and later by that of Nochetto [?], on the convergence of the discrete free boundary to the continuous one.

A lot of results on error estimates for the classical obstacle problems and variational inequalities were achieved in this norm, (cf., e.g [?], [?], [?], [?]). However, very few works concerning quasi-variational inequalities are known on this subject (cf., [?]), particularly the case of systems which is the subject of this paper.

Our primary aim in this paper is to show that problem (??) can be properly approximated by a finite element method which turns out to be quasi-optimally accurate in $L^\infty(\Omega)$. The approximation is carried out by first introducing a monotone iterative scheme of Bensoussan-Lions type which is shown to converge geometrically to the continuous solution. Similarly, using the standard finite element method and the discrete maximum principle, the solution of the discrete system of QVIs is in turn approximated by an analogue discrete monotone iterative scheme and its geometric convergence is given as well. An $L^\infty$-error estimate is then established combining the geometric convergence of both the continuous and discrete iterative schemes with known error estimates for elliptic variational inequalities.

An outline of the paper is as follows: We lay down some necessary notations, assumptions and preliminaries in section 2. We consider the continuous problem and prove some related qualitative properties in section 3. Section 4. deals with the discrete problem for which an analogue study to that of the continuous problem is achieved. Finally, in section 5., we prove a fundamental Lemma and give the main result.

2. Assumptions, Notations and Preliminaries

We are given functions $a^i_{jk}(x), a^i_k(x), a^i_0(x); 1 \leq i \leq M$, sufficiently smooth such that:

$$
\sum_{1 \leq j,k \leq N} a^i_{jk}(x)\xi_j\xi_k \geq \alpha |\zeta|^2; \zeta \in \mathbb{R}^N; \alpha > 0
$$

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We define the variational forms: for any \( u, v \in H^1(\Omega) \)

\[
a^i(u, v) = \int_\Omega \left( \sum_{1 \leq j, k \leq N} a^i_{jk}(x) \frac{\partial u}{\partial x_j} \frac{\partial v}{\partial x_k} + \sum_{k=1}^N a^i_k(x) \frac{\partial u}{\partial x_k} v + a^i_0(x) uv \right) dx
\]

which we assume to be coercive, i.e.:

\[
a^i(v, v) \geq \gamma \| v \|_{H^1(\Omega)}^2; \quad \gamma > 0
\]

We are also given right hand sides \( f^1, \ldots, f^M \) such that

\[
f^i \in L^\infty(\Omega); \quad f^i \geq 0
\]

**Definition 1.** Let \( f \) be a right-hand side and \( \psi \) an obstacle such that \( \psi \geq 0 \) on \( \partial \Omega \). The following problem is called an elliptic variational inequality (VI): Find \( u \in K \) such that

\[
a(u, v - u) \geq (f, v - u) \quad \forall v \in K
\]

where \( K = \{ v \in H^1_0(\Omega) \text{ such that } v \leq \psi \text{ a.e.} \} \) and \( a(u, v) \) is a bilinear form of the same form as those defined in (??)

**Definition 2.** \( z \in K \) is said to be a subsolution for VI (??) if

\[
a(z, v) \leq (f, v) \quad \forall v \in H^1_0(\Omega), \quad v \geq 0
\]

Let \( X \) denote the set of such subsolutions.

**Theorem 1.** (cf.\cite{[1]},\cite{[2]}) The solution \( u \) of V.I (??) is the maximum element of the set \( X \).

Consider now the following mapping:

\[
\sigma : L^\infty(\Omega) \rightarrow L^\infty(\Omega) \\
\psi \rightarrow \sigma(\psi) = u
\]

where \( u \) is the solution to VI (??).

**Proposition 1.** The mapping \( \sigma \) is increasing and concave.

**Proof.** 1. \( \sigma \) is increasing.

Let \( \psi \) and \( \tilde{\psi} \) in \( L^\infty(\Omega) \) such that \( \psi \leq \tilde{\psi} \). Then \( \sigma(\psi) \leq \psi \leq \tilde{\psi} \). So \( \sigma(\psi) \) is a subsolution for the V.I with obstacle \( \tilde{\psi} \). Then, using Theorem 1, we get the desired result.

2. \( \sigma \) is concave.

Let \( \psi, \tilde{\psi} \) in \( L^\infty(\Omega) \) and \( \theta \in [0; 1] \). Set \( \sigma_\theta = \sigma(\theta \psi + (1 - \theta) \tilde{\psi}) \)

Since \( \sigma(\psi) \leq \psi \) and \( \sigma(\tilde{\psi}) \leq \tilde{\psi} \), it follows that \( \theta \sigma(\psi) + (1 - \theta) \sigma(\tilde{\psi}) \) is a subsolution for V.I with obstacle \( \theta \psi + (1 - \theta) \tilde{\psi} \). Then using Theorem 1, we get the concavity of \( \sigma \).

**Remark 1.** We notice that, under the discrete maximum principle, Theorem 1 and Proposition 1 remain true in the discrete case (cf. \cite{[3]}).
3. The Continuous Problem

3.1. Existence, uniqueness and regularity result. Existence of a unique solution to system (3.1) can be proved, adapting the approach developed in [?] pp. 343-358.

Indeed, let $\mathbb{H}^+ = (L^\infty_+(\Omega))^M = \{ V = (v_1^1, ..., v_1^M) \text{ such that } v_i^1 \in L^\infty_+(\Omega) \}$, equipped with the norm:

$$\| V \|_\infty = \max_{1 \leq i \leq M} \| v_i^1 \|_{L^\infty(\Omega)}$$

where $L^\infty_+(\Omega)$ is the positive cone of $L^\infty(\Omega)$. We consider the mapping

$$T: \mathbb{H}^+ \rightarrow \mathbb{H}^+$$

$$W \rightarrow TW = \zeta = (\zeta_1^1, ..., \zeta_M^1)$$

where $\zeta_i^1 = \sigma(Mw_i^1) \in H_0^1(\Omega)$ is solution to the following VI:

$$a_i^1(\zeta_i^1, v - \zeta_i^1) \geq (f_i^1, v - \zeta_i^1) \quad \forall v \in H_0^1(\Omega)$$

$$\zeta_i^1 \leq Mw_i^1 \quad ; \quad v \leq Mw_i^1$$

Problem (3.2) being a coercive VI, thanks to ([?],[?]) has one and only one solution.

Consider now $\bar{\U}^0 = (\bar{\upsilon}^1,0, ..., \bar{\upsilon}^M,0)$, where $\bar{\upsilon}^i,0$ is the solution to the following variational equation:

$$a_i^1(\bar{\upsilon}^i,0, v) = (f_i^1, v) \quad \forall v \in H_0^1(\Omega)$$

Due to (3.2), problem (3.1) has a unique solution. Moreover, $\bar{\upsilon}^i,0 \in W^{2,p}(\Omega); \quad 2 \leq p < \infty$


Proposition 2. Let $\mathcal{C} = \{ W \in \mathbb{H}^+ \text{ such that } 0 \leq W \leq \bar{\U}^0 \}$. Then $T$ maps $\mathcal{C}$ into itself.

Proof. 1. $TW \leq \bar{\U}^0$ for all $W \in \mathbb{H}^+$. (The item is proved by arguments similar to those used in the proof of Proposition 2).
i.e.,
\[ TW \leq U^0 \]

2. \( TW \geq 0, \forall W \in H^+ \).
This follows immediately from standard comparison results in elliptic variational inequalities and the fact that \( f^i \geq 0 \) \( \square \)

**Proposition 3.** \( T \) is increasing on \( H^+ \).

**Proof.** Let \( V = (v^1, \ldots, v^M) \), \( W = (w^1, \ldots, w^M) \) in \( H^+ \) such that:
\[ v^i \leq w^i, \forall i = 1, \ldots, M \]
Then since \( \sigma \) and \( M \) are increasing on \( L^\infty(\Omega) \), it follows that
\[ \sigma(Mv^i) \leq \sigma(Mw^i) \]
Thus \( TV \leq TW \) \( \square \)

**Proposition 4.** \( T \) is concave on \( H^+ \).

**Proof.** Let \( \theta \in [0,1] \). We then have
\[
T(\theta W + (1-\theta)\bar{W}) = \\
[\sigma(M(\theta w^1 + (1-\theta)\bar{w}^1)), \ldots, \sigma(M(\theta w^M + (1-\theta)\bar{w}^M))] \geq \\
[\sigma(\theta Mw^1 + (1-\theta)M\bar{w}^1), \ldots, \sigma(\theta Mw^M + (1-\theta)M\bar{w}^M)] \geq \\
[\theta \sigma(Mw^1) + (1-\theta)\sigma(M\bar{w}^1), \ldots, \theta \sigma(Mw^M) + (1-\theta)\sigma(M\bar{w}^M)]
\]
due to the concaveness of \( \sigma \) \( \square \).

**Proposition 5.** \( T \) is Lipschitz continuous on \( H^+ \) i.e.,
\[
\| TW - T\bar{W} \|_\infty \leq \| W - \bar{W} \|_\infty \forall W, \bar{W} \in H^+
\]

**Proof.** Let \( W = (w^1, \ldots, w^M) ; \bar{W} = (\bar{w}^1, \ldots, \bar{w}^M) \) and \( \delta = (\delta^1, \ldots, \delta^M) \) such that
\[
\delta^i = \| w^i - \bar{w}^i \|_{L^\infty(\Omega)}
\]
Now setting
\[
\Phi = \| \delta \|_\infty
\]
the monotonicity property of \( T \) implies that
\[
TW \leq T(\bar{W} + \delta) \leq \\
[\sigma(M(\delta^1 + \bar{w}^1)), \ldots, \sigma(M(\delta^i + \bar{w}^i)), \ldots, \sigma(M(\delta^M + \bar{w}^M))] = \\
[\sigma(\delta^1 + M\bar{w}^1), \ldots, \sigma(\delta^i + M\bar{w}^i), \ldots, \sigma(\delta^M + M\bar{w}^M)]
\]
\[ = [\sigma(M\bar{w}^1), \ldots, \sigma(M\bar{w}^i), \ldots, \sigma(M\bar{w}^M) + (\delta^1, \ldots, \delta^M)]
\]
thus,
\[
TW \leq T\bar{W} + \delta
\]
Similarly, interchanging the roles of \( W \) and \( \bar{W} \), one can also get
This completes the proof. \(\square\)

**Remark 2.** The discrete version of Proposition 5 will play an important role in the finite element error analysis part of this work.

**Remark 3.** We notice that the solutions of system (??) correspond to fixed points of mapping \(T\), that is \(U = TU\). In this view it is natural to consider the following iterative scheme.

### 3.1.2. A Continuous Iterative Scheme of Bensoussan-Lions Type.

Starting from \(U^0\) defined in (??) (resp. \(U^0 = (0, \ldots, 0)\)), we define the sequences below:

\[
U^{n+1} = TU^n; \quad n = 0, 1, \ldots
\]

(resp.)

\[
U^{n+1} = TU^n; \quad n = 0, 1, \ldots
\]

The convergence analysis of such an iterative scheme rests upon the following results.

**Lemma 1.** Let \(0 < \lambda < \inf \left\{ \frac{k}{k ||U^0||_\infty} : \lambda \right\} \). Then, we have \(T(0) \geq \lambda U^0\)

**Proof.** The proof is very similar to that of (??, p.351) \(\square\)

**Proposition 6.** Let \(\gamma \in [0, 1]\) such that

\[
W - \bar{W} \leq \gamma W, \quad \forall W, \bar{W} \in \mathbb{R}
\]

Then, under conditions of Lemma 1, we have

\[
TW - T\bar{W} \leq \gamma(1 - \lambda)TW
\]

**Proof.** From (??), we have \((1 - \gamma)W \leq \bar{W}\). Then, applying Proposition 4, we get

\[
(1 - \gamma)TW + \gamma T(0) \leq T[(1 - \gamma)W + \gamma 0] \leq T\bar{W}
\]

and, due to Lemma 1, the desired result follows. \(\square\)

### 3.1.3. Convergence of the Continuous Iterative Scheme.

**Theorem 2.** (cf. [?] p.453). Under conditions of Propositions 2-4, the sequences \((U^n)\) and \((\bar{U}^n)\) are monotone and well defined in \(C\). Moreover, they converge respectively from above and below to the unique solution of system (??).

### 3.1.4. Regularity of the Solution of System (??).

**Lemma 2.** ([?]) For any \(i = 1, 2, \ldots, M\)

\[
\max_{n \geq 0} (\|w^i_n\|_{W^{2,p}(\Omega)}, \|w^i_n\|_{W^{2,p}(\Omega)}) \leq C; \quad 2 \leq p < \infty
\]

**Theorem 3.** ([?] p.453). Assume \(a^i_{jk}(x)\) in \(C^{1,\alpha}(\bar{\Omega})\), \(a^i(x)\), \(a^i_0(x)\) and \(f^i\) in \(C^{0,\alpha}(\bar{\Omega})\). Then \((u^1, \ldots, u^M) \in (W^{2,p}(\Omega))^M\); \(2 \leq p < \infty\).
3.2. Rate of Convergence of The Continuous Iterative Scheme.

**Proposition 7.** Let the conditions of Proposition 6 hold. Then we have

\[ \| U^n - U \|_\infty \leq (1 - \lambda)^n \| U^0 \|_\infty \]  

(3.8)

\[ \| U^n - U \|_\infty \leq (1 - \lambda)^n \| U^0 \|_\infty \]  

(3.9)

**Proof.** By Theorem 2, we have:

\[ 0 \leq U \leq \bar{U}^0 \]

so

\[ 0 \leq \bar{U}^0 - U \leq \bar{U}^0 \]

Then, applying (??) with \( \gamma = 1 \), we get

\[ 0 \leq T\bar{U}^0 - TU \leq (1 - \lambda)T\bar{U}^0 \]

and by (??)

\[ 0 \leq \bar{U}^1 - U \leq (1 - \lambda)\bar{U}^1 \]

Now, using (??) again with \( \gamma = 1 - \lambda \) it follows that

\[ 0 \leq T\bar{U}^1 - TU \leq (1 - \lambda)(1 - \lambda)T\bar{U}^1 \]

i.e.,

\[ 0 \leq \bar{U}^2 - U \leq (1 - \lambda)^2\bar{U}^2 \]

and inductively

\[ 0 \leq \bar{U}^n - U \leq (1 - \lambda)^n\bar{U}^0 \leq (1 - \lambda)^n\bar{U}^0 \]

We prove estimation (??) as estimation (??). \( \square \)

4. The Discrete Problem

Let \( \Omega \) be decomposed into triangles and let \( \tau_h \) denote the set of all those elements; \( h > 0 \) is the mesh size. We assume the family \( \tau_h \) is regular and quasi-uniform.

Let \( \mathcal{V}_h \) denote the standard piecewise linear finite element space, \( \mathcal{A}^i, 1 \leq i \leq M \) be the matrices with generic coefficients \( a^i(\varphi_i, \varphi_s) \), where \( \varphi_s, s = 1, 2, \ldots m(h) \) are the nodal basis functions. Let also \( r_h \) be the usual interpolation operator.

**The Discrete Maximum Principle Assumption (dmp):** We assume that the matrices \( \mathcal{A}^i \) are M-matrices. (cf.[?]).

The discrete system of QVIs is then defined as follows: Find \( U_h = (u_h^1, \ldots, u_h^N) \in (\mathcal{V}_h)^M \) such that

\[ \begin{cases} 
  a^i(u_h^i, v - u_h^i) \geq (f^i, v - u_h^i) \quad \forall v \in \mathcal{V}_h \\
  u_h^i \leq r_h M u_h^i \quad ; u_h^i \geq 0 \quad ; v \leq r_h M u_h^i 
\end{cases} \]

(4.1)
4.1. **Existence and Uniqueness.** Existence and uniqueness of a solution to system (??) can be shown similarly to that of the continuous case provided the discrete maximum principle (dmp) is satisfied. Indeed, the idea for proving that consists of associating with the system (??) the following discrete fixed point mapping:

\[
T_h : \mathbb{H}^+ \longrightarrow (\mathbb{V}_h)^M
\]

where \(c_h = \sigma_h(Mw)\) is the solution of the following discrete VI:

\[
\begin{cases}
    a^i(c_h,v - c_h) \geq (f^i,v - c_h) \quad \forall v \in \mathbb{V}_h \\
    c^i_h \leq r_h M w^i, \quad v \leq r_h M w^i
\end{cases}
\]

**Remark 4.** Under the discrete maximum principle (d.m.p) the mapping \(T_h\) possesses analogous properties to that of mapping \(T\) (see Propositions 2-5). The proofs of such properties will not be given as they are very similar to those of the continuous case.

4.1.1. **Some Properties of The Mapping \(T_h\).** Let \(U^0_h = (u^0_h, ..., u^0_h)\) be the discrete analogue to the solution of problem (??):

\[
a_i(U^0_h, v) = (f^i,v) \quad \forall v \in \mathbb{V}_h
\]

Then we have the discrete analogues to Propositions 2-5.

**Proposition 8.** \(T_h\) maps \(\mathbb{C}_h\) into itself where \(\mathbb{C}_h = \{W \in (L^\infty(\Omega))^M \text{ such that } 0 < W \leq U^0_h\}\)

**Proposition 9.** \(T_h\) is increasing, concave on \(\mathbb{H}^+\).

**Proposition 10.** \(T_h\) is Lipschitz continuous on \(\mathbb{H}^+\), i.e.,

\[
\|T_hW - T_h\tilde{W}\|_\infty \leq \|W - \tilde{W}\|_\infty \quad \forall W, \tilde{W} \in \mathbb{H}^+
\]

**Remark 5.** It is not hard to see that the solution of system of QVIs (??) is a fixed point of \(T_h\), that is \(U_h = T_h U_h\). Therefore, as in the continuous problem, one can define the following discrete iterative scheme.

4.1.2. **A Discrete Iterative Scheme of Bensoussan-Lions Type.** Starting from \(U^0_h\) solution of (??) (resp. from \(U^0_h = (0, ..., 0)\)), one can compute

\[
U^{n+1}_h = T_h U^n_h \quad n = 0,1,...
\]

(resp.)

\[
U^{n+1}_h = T_h U^n_h \quad n = 0,1,...
\]

By analogy to the solution of system, one can prove the convergence of the discrete iterative scheme to the solution of system (??), using the following intermediate results whose proofs are very similar to those of Lemma 1 and Proposition 6 respectively.

**Lemma 3.** Let \(0 < \lambda < \inf(\frac{k}{\|U^0_h\|}, 1)\). Then under the d.m.p, we have \(T_h(0) \geq \lambda U^0_h\).

**Proposition 11.** Let \(\gamma \in [0, 1]\) such that

\[
W - \tilde{W} \leq \gamma W, \quad \forall W, \tilde{W} \in \mathbb{C}
\]

Then we have

\[
T_h W - T_h \tilde{W} \leq \gamma(1 - \lambda)T_h W
\]

**Theorem 4.** Under the "dmp" and conditions of Propositions 8, 10, 11, the sequences \((\mathring{U}_h^n)\) and \((\mathring{U}_h^n)\) are monotone and well defined in \(\mathbb{C}_h\). Moreover, they converge respectively from above and below to the unique solution of system (??).
Proof. Very similar to that of Theorem 2.

4.1.3. Rate of Convergence of The Discrete Iterative Scheme.

Proposition 12.

\begin{align*}
\| U^n_h - U_h \|_\infty &\leq (1 - \lambda)^n \| U^0_h \|_\infty \\
\| \bar{U}^n_h - U_h \|_\infty &\leq (1 - \lambda)^n \| \bar{U}^0_h \|_\infty
\end{align*}

Proof. It is exactly the same as that of Proposition 7. \qed

5. The Finite Element Error Analysis

This section is devoted to demonstrate that the proposed method is quasi-optimally accurate in \( L^\infty (\Omega) \). For this purpose, we need first to introduce an auxiliary sequence of discrete variational inequalities (VIs) and next prove a fundamental lemma.

From now on \( C \) will denote a constant independent of both \( h \) and \( n \).

5.1. An Auxiliary Sequence of Discrete VIs. We introduce the following discrete sequence

\begin{equation}
\begin{cases}
\bar{U}^{n+1}_h = T_h \tilde{U}^n_h; & n = 0, 1, ... \\
\text{with } \bar{U}^0_h = \tilde{U}^0_h
\end{cases}
\end{equation}

where \( \tilde{U}^0_h \) is defined in (??) and for any \( n \geq 1 \), \( \tilde{u}^{i,n}_h \) is a solution to following discrete variational inequality:

\begin{equation}
\begin{cases}
a^i (\tilde{u}^{i,n+1}_h, v - \tilde{u}^{i,n+1}_h) \geq (f^i, v - \tilde{u}^{i,n+1}_h) \forall v \in \mathcal{V}_h \\
\tilde{u}^{i,n+1}_h \leq r_h M \bar{u}^{i,n}_h, \quad v \leq r_h M \bar{u}^{i,n}_h
\end{cases}
\end{equation}

\( \bar{U}^n = (\bar{u}^{1,n}_h, \ldots, \bar{u}^{M,n}_h) \) being the sequence defined by (??). Again, thanks to [?], (??) has one and only one solution.

We notice that \( \tilde{u}^{i,n}_h \) solution of (??) represents the standard finite element approximation of \( \bar{u}^{i,n}_h \). Therefore, using the regularity result provided by Lemma 2 and next adapting [?], we have the following uniform error estimate.

Theorem 5. Under conditions of Lemma 2, we have

\begin{equation}
\| \bar{U}^n_h - \bar{U}^n_h \|_\infty \leq C h^2 \log h^2
\end{equation}

Proof. The proof is just an adaptation of [?]. \qed

The following lemma will play a crucial role in proving the main result.

Lemma 4.

\begin{equation}
\| \bar{U}^n - \bar{U}^n_h \|_\infty \leq \sum_{p=0}^n \| \bar{U}^p - \bar{U}^p_h \|_\infty
\end{equation}

Proof. We proceed by induction. Indeed, we know that

\( \bar{U}^1 = T \bar{U}^0, \bar{U}^1_h = T_h \bar{U}^0, \bar{U}^1_h = T_h \bar{U}^0 \)

Then

\begin{align*}
\| \bar{U}^1 - \bar{U}^1_h \|_\infty \leq \\
\| \bar{U}^1 - \bar{U}^1_h \|_\infty + \| \bar{U}^1_h - \bar{U}^1_h \|_\infty \leq
\end{align*}
So, since $T_h$ is Lipschitz continuous, we have
\[
\| U^1 - \tilde{U}_h^1 \|_\infty + \| T_h U^0 - T_h \tilde{U}_h^0 \|_\infty
\]
Now assume that
\[
\| U^{n-1} - \tilde{U}_h^{n-1} \|_\infty \leq \sum_{p=0}^{n-1} \| U^p - \tilde{U}_h^p \|_\infty
\]
Then,
\[
(5.5) \quad \| U^n - \tilde{U}_h^n \|_\infty \leq \| U^n - \tilde{U}_h^n \|_\infty + \| \tilde{U}_h^n - U^n - T_h U^{n-1} - T_h \tilde{U}_h^{n-1} \|_\infty
\]
Using again the Lipschitz continuity of $T_h$, it follows that (5.7) is less than or equal to
\[
(5.6) \quad \| U^n - \tilde{U}_h^n \|_\infty + \| T_h U^{n-1} - T_h \tilde{U}_h^{n-1} \|_\infty \leq \sum_{p=0}^{n} \| U^p - \tilde{U}_h^p \|_\infty
\]
This completes the proof of Lemma 4. \qed

Now guided by Lemma 4, Propositions 7, 12 and Theorem 5, we are in a position to demonstrate our main result.

5.2. $L^\infty$ -Error Estimate for the system of QVIs (1.1).

Theorem 6.
\[
(5.6) \quad \| U - U_h \|_\infty \leq C h^2 \text{Log} h^3
\]
\[
(5.7) \quad \| U - U_h \|_{1,\infty} \leq C h \text{Log} h^6
\]
where:
\[
\| U \|_{1,\infty} = \max_{1 \leq i \leq M} \| u^i \|_{W^{1,\infty}(\Omega)}
\]
Proof. Using estimations (??), (??) and (??) we have:

\[
\| U - U_h \|_\infty \leq \| U - \bar{U}^n \|_\infty + \| \bar{U}^n - \bar{U}_h^n \| + \| \bar{U}_h^n - U_h \|_\infty \\
\leq \| U - \bar{U}^n \|_\infty + \sum_{p=0}^{n} \| \bar{U}^p - \bar{U}_h^p \|_\infty + \| \bar{U}_h^p - U_h \|_\infty \leq \\
\| U - \bar{U}^n \|_\infty + \| \bar{U}^0 - \bar{U}_h^0 \|_\infty + \sum_{p=1}^{n} \| \bar{U}^p - \bar{U}_h^p \|_\infty + \| \bar{U}_h^n - U_h \|_\infty \\
\leq C h^2 | \log h |^{3/2} + n.C h^2 | \log h |^2 + (1 - \lambda)^n \| \bar{U}^0 \|_\infty \\
+ (1 - \lambda)^n \| \bar{U}_h^0 \|_\infty \\
\]

where we have also used standard uniform error estimate due to J. Nitsche (cf.[?]). Finally, letting

\[(1 - \lambda)^n = h^2 \]

we get the desired result.

The \( W^{1,\infty} \) - error estimate (??) follows immediately from the standard inverse inequality (cf. [?]). \( \square \)

6. Conclusion

1. We have established a convergence order in the \( L^\infty \) norm for a coercive system of Quasi-Variational Inequalities. A future paper will be devoted to the noncoercive case for which a different approach will be developed and analyzed.

2. It is also important to notice that the error estimate obtained in this paper contains an extra power in \( \log h \) than expected. We believe that this is due to the approach followed.

3. The same approach may also be extended to other important problems, such as system of QVIs related to games theory [?].

References


