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THE FSUSY HAMILTONIAN IN CONNECTION WITH THE CHERN-SIMONS GAUGE THEORY

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Abstract
On the two-dimensional lattice, the construction of anyonic operators and its algebras is discussed. Thus, the fractional supersymmetry (FSUSY) and the associated FSUSY Hamiltonian basing on the quonic anyons are recalled. Its particular case for bosonic anyon and fermionic anyon is given. The FSUSY Hamiltonian in connection with the Chern-Simons gauge theory is constructed.

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1 Introduction

In the last twenty years, supersymmetry (SUSY) has been a popular area of research for physicists and mathematicians. It was introduced in relativistic quantum field theories as a generalization of Poincaré symmetry\cite{1, 2}, which can be parametrized by the ordinary space-time coordinates supplemented by Grassmann variables transforming as a spinor. In 1976 Nicolai suggested an analogous generalization for non-relativistic quantum mechanics\cite{3}, where the model consists of one-dimensional lattice with $N$ sites. Furthermore, with the one-dimensional model introduced by Witten\cite{4} in 1981, the SUSY became a major tool in quantum mechanics and mathematical, statistical and condensed matter physics.

As a generalization of this ordinary SUSY, the fractional supersymmetry (FSUSY)\cite{5, 6, 7} was subject of several studies. Some of these studies deal with a complex Grassmann variable. Others\cite{8} with quonic operators which are used to construct the fractional supercharges and the FSUSY Hamiltonian of the quantum system consisting of two quons with different deformation parameters. We gave in paper \cite{9} a new construction of the FSUSY from one system of two quonic anyons of different kinds, where the quonic anyons are particles interpolating between bosonic and fermionic anyons\cite{10, 11}. They are constructed by means of the generalized Jordan-Wigner transformation\cite{12}, which transmutes in this case bosons and fermions into anyons\cite{13, 14, 15, 16, 17} respectively. Yet so, it transmutes quons into quonic anyons. Such as the creation and annihilation anyonic operators for fermionic, bosonic and quonic anyons were constructed respectively in the references \cite{18, 19, 20, 21}.

In the present work, we construct the FSUSY Hamiltonian in connection with the Chern-Simons gauge theory. In this construction we use the definition of anyons indicated by Wilczek\cite{22}, such as an anyon is a charge-flux composition by considering a non-relativistic particle of mass $m$ and electric charge $q$ moving, on two dimensional space $(x, y)$, in the magnetic field $B$ created by an infinitely long and thin solenoid passing through the origin and directed along the $z$-axis.

This paper can be organized as follows: in the second section we briefly give the definition of the anyonic operators for its three aspects and its associated algebras. In the third section, we discuss the notion of FSUSY for quonic anyons defined on the two-dimensional lattice. Finally, in the fourth section, we construct the FSUSY Hamiltonian of the system consisting of one bosonic anyon and one fermionic anyon, where we consider the definition of anyons owing to Wilczek, such as the anyon is the elementary particle associated to Chern-Simons gauge field moving on the two-dimensional space.

2 Anyonic Operators and its Algebras

By means of the generalized Jordan-Wigner transformation\cite{12} the anyonic oscillators can be constructed from three kinds of particles; bosons, fermions and quons (the q-oscillators which interpolate between bosons and fermions in any dimension)\cite{23, 24, 25}. 
For the corresponding transformation there is an essential ingredient. It is the function $\Theta(x, y)$ characterizing the kind of each anyon on the two-dimensional lattice $\Omega$. Let us note here that, in the continuous plane the angle function is a rather familiar angle. Then owing to the references [?, ?, ?] let us give a description of $\Theta(x, y)$.

On the two-dimensional lattice $\Omega$, we choose one cut associated to each site $x$ denoted $\gamma_x$ from minus infinity of $x$-axis and we fix a base point $B$ at the positive infinity of $x$-axis. The angle function is defined as an angle under which the oriented path $Bx$ is seen from a point $y^*$. With $y^*$ the dual point of $y$ in the dual lattice $\Omega^*$, such as $y^* = y + o^*$ and $o^* = (1, 1, \epsilon)$ is its origin ($\epsilon \ll 1$ the lattice spacing). With this type of cuts we fix the path orientation $Bx$ on $\Omega$ and then the direction of the rotation on $\Omega$ by considering the particles defined on each site of the lattice. In this case the function angle is denoted by $\Theta_{\gamma_x}(x, y)$. We can choose another kind of cuts denoted by $\Theta_{\delta_x}(x, y)$. Now, the cuts $\delta_x$ are defined from the positive infinity $x$-axis to the dual point $x^* \in \Omega^*$ ($x^* = x - 0^*$), and the base point will be chosen in the minus infinity of the $x$-axis. These two types of the angle functions satisfy the following relations

$$
\Theta_{\gamma_x}(x, y) - \Theta_{\gamma_{x'}}(y, x) = \begin{cases} 
\pi, & x > y \\
-\pi, & x < y 
\end{cases},
$$

$$
\Theta_{\delta_x}(x, y) - \Theta_{\delta_{x'}}(y, x) = \begin{cases} 
-\pi, & x > y \\
\pi, & x < y 
\end{cases},
$$

$$
\Theta_{\delta_x}(x, y) - \Theta_{\gamma_{x'}}(x, y) = \begin{cases} 
-\pi, & x > y \\
\pi, & x < y 
\end{cases},
$$

$$
\Theta_{\delta_x}(x, y) - \Theta_{\gamma_{x'}}(y, x) = 0, \forall x, y \in \Omega.
$$

This is because of the ordering and opposite ordering conditions involved by the two kinds of cuts $\gamma_x$ and $\delta_x$. These are described by

$$
x_\delta < y_\delta \Leftrightarrow x_\gamma > y_\gamma \Leftrightarrow x > y \Leftrightarrow \begin{cases} 
x_2 > y_2, \\
x_1 > y_1, x_2 = y_2.
\end{cases}
$$

By indicating two kinds of the angle functions, there are two kinds of anyons defined as

$$
a_i(x_{\alpha}) = D_i(x_{\alpha})c_i(x),
a^\dagger_i(x_{\alpha}) = c_i^\dagger(x)D_i^\dagger(x_{\alpha})
$$

with $\alpha = \gamma, \delta$ and $i \in \mathbb{N}^*$. In this expression $a_i(x_{\alpha})$ and $a^\dagger_i(x_{\alpha})$ are the annihilation and the creation anyonic operators respectively, $c_i(x)$ and $c_i^\dagger(x)$ are the annihilation and the creation operators respectively which can be either bosonic or fermionic or quonic operators denoted respectively by the sets $\{b_i(x), b^\dagger_i(x)\}$, $\{f_i(x), f^\dagger_i(x)\}$ and $\{d_i(x), d^\dagger_i(x)\}$. Thus in equation (3) we introduced the disorder operators $D_i(x_{\alpha})$ which can be expressed by

$$
D_i(x_{\alpha}) = e^{i \sum_{y \neq x} \Theta_{\alpha_x}(x, y)[N_i(y) - \frac{1}{2}]},
$$

$$
3
$$
which satisfy the following equations

\begin{align*}
D_i^1(x_{\alpha})c_i(y) &= e^{i\nu\Theta_{\alpha}(x,y)}c_i(y)D_i^1(x_{\alpha}) \\
D_i^1(x_{\alpha})c_i^\dagger(y) &= e^{-i\nu\Theta_{\alpha}(x,y)}c_i^\dagger(y)D_i^1(x_{\alpha}) \\
D_i(x_{\alpha})c_i(y) &= e^{i\nu\Theta_{\alpha}(x,y)}c_i(y)D_i(x_{\alpha}) \\
D_i(x_{\alpha})c_i^\dagger(y) &= e^{-i\nu\Theta_{\alpha}(x,y)}c_i^\dagger(y)D_i(x_{\alpha}) \\
D_i(x_{\alpha})D_i(y_{\alpha}) &= D_i(y_{\alpha})D_i^1(x_{\alpha}),
\end{align*}

where \( \nu \) is seen as the statistical parameter. \( N_i(x), x \in \Omega, \) is the number operator on the two-dimensional lattice associated to each kind of particles involved in the construction of anyons.

We can define in the general case \( N_i(x) \) by

\begin{align*}
\delta_i(x)d_i(x) &= [N_i(x)]_{\pm}, \\
d_i(x)d_i^\dagger(x) &= [N_i(x) + 1]_{\pm},
\end{align*}

i.e. for quons characterized by the deformation parameter \( q_i \) which take \( \pm1 \) for bosons and fermions respectively. In equation (6) we have the notation \([x]_{\pm} = \frac{q_i-1}{q_i+1}\). Thus the quonic operators satisfy the following oscillator algebra

\begin{align*}
\{d_i(x), d_j(y)\} &= \delta_{ij}\delta(x,y) \\
\{d_i(x), d_j^\dagger(y)\} &= 0 \quad \forall x, y, \forall i, j \\
\{d_i^\dagger(x), d_j^\dagger(y)\} &= 0 \quad \forall x, y, \forall i, j \\
[N_i(x), d_j(y)] &= -\delta_{ij}\delta(x,y)d_i(x) \\
[N_i(x), d_j^\dagger(y)] &= \delta_{ij}\delta(x,y)d_i^\dagger(x)
\end{align*}

with

\begin{align*}
(d_i(x))^{\ell_i} &= (d_i^\dagger(x))^{\ell_i} = 0, \quad \ell_i \geq 2,
\end{align*}

where the deformation parameter is supposed to be a root of unity; i.e. \( q_i^{\ell_i} = 1 \) so \( q_i = e^{i\frac{2\pi}{\ell_i}} \). In the particular case of \( q_i = +1 \) we will have the bosonic algebra as follows

\begin{align*}
[b_i(x), b_j^\dagger(y)] &= \delta_{ij}\delta(x,y) \\
[b_i(x), b_j(y)] &= 0 \quad \forall x, y, \forall i, j \\
[b_i^\dagger(x), b_j^\dagger(y)] &= 0 \quad \forall x, y, \forall i, j \\
[N_i(x), b_j(y)] &= -\delta_{ij}\delta(x,y)b_i(x) \\
[N_i(x), b_j^\dagger(y)] &= \delta_{ij}\delta(x,y)b_i^\dagger(x)
\end{align*}

where \( N_i(x) = b_i^\dagger(x)b_i(x) \) is the number operator for the bosons. Also for \( q_i = -1 \) the fermionic algebra will be

\begin{align*}
\{f_i(x), f_j^\dagger(y)\} &= \delta_{ij}\delta(x,y) \\
\{f_i(x), f_j(y)\} &= 0 \quad \forall x, y, \forall i, j \\
\{f_i^\dagger(x), f_j^\dagger(y)\} &= 0 \quad \forall x, y, \forall i, j \\
[N_i(x), f_j(y)] &= -\delta_{ij}\delta(x,y)f_i(x) \\
[N_i(x), f_j^\dagger(y)] &= \delta_{ij}\delta(x,y)f_i^\dagger(x)
\end{align*}

In the relations (9) and (10) we define the Dirac function as

\begin{align*}
\delta(x,y) &= \begin{cases} 
1 & \text{if } x = y \\
0 & \text{if } x \neq 0.
\end{cases}
\end{align*}
Owing to the relations (7) as a general case of bosonic and fermionic operators, it is easy to prove that the operators of (3) obey the following anyonic algebra

\[
[a_i(x_i), a_j(y_j)]_{q_i p^{-1}} = 0, \quad x > y
\]

\[
[a_i^\dagger(x_i), a_j^\dagger(y_j)]_{q_i p^{-1}} = 0, \quad x > y
\]

\[
[a_i(x_i), a_j^\dagger(y_j)]_{q_i} = 0, \quad x > y
\]

\[
[a_i(x_i), a_j^\dagger(y_j)]_{q_i} = 1
\]

\[
[a_i(x_i), a_j(y_j)] = 0, \quad \forall i \neq j, \forall x, y \in \Omega
\]

\[
[a_i^\dagger(x_i), a_j^\dagger(y_j)] = 0, \quad \forall i \neq j, \forall x, y \in \Omega
\]

\[
[a_i(x_i), a_j^\dagger(y_j)] = 0, \quad \forall i \neq j, \forall x, y \in \Omega
\]

The same results can be obtained for the kind \(\delta\) by exchanging \(p\) by \(p^{-1}\). For different kinds of anionic operators, we obtain the following commutation relations

\[
[a_i(x_i), a_j(y_j)]_{q_{ij}} = 0, \forall x, y \in \Omega
\]

\[
[a_i^\dagger(x_i), a_j^\dagger(y_j)]_{q_{ij}} = \delta_{ij} \delta(x, y) p^{-\sum_{x < y} \sum_{y > x} |N_i(x) - |} (13)
\]

with \(p = e^{i\nu\pi}\). Thus, according to (8) it is easy to see that

\[
(a_i(x_\alpha))^\ell_i = (a_i^\dagger(x_\alpha))^\ell_i = 0 (14)
\]

which generalizes the hard-core condition in the work [?] for \(\ell_i = 2\); i.e., for the fermionic anyons.

Then with this construction of anyonic algebra in the general case basing on the quonic algebra we pointed out to define in the work [?] one version of FSUSY through the anyons constructed from the quons.

3 N=2 FSUSY Through Quonic Anyons

In this section we will recall the construction of the FSUSY from one system consisting of two quonic anyons (\(q_i\)-anyons, \(i = 1, 2\)) [7]. The two anyons were constructed from quons of different deformation parameters. Also we considered two different kinds of anyons to construct the supercharges of our N=2 FSUSY.

On the two-dimensional lattice \(\Omega\), the supercharges are introduced as follows

\[
Q_+(x) = a_i^\dagger(x_i) a_2(x_\phi)
\]

\[
Q_-(x) = a_1(x_\delta) a_2^\dagger(x_i)
\]

with the nilpotency condition, according to the equality (14),

\[
(Q_\pm(x))^\ell_2 = 0
\]
by supposing \( \ell_2 < \ell_1 \).

By introducing the hermitian conjugate of the generators \( Q_{\pm}(x) \) as

\[
Q_{\pm}^\dagger(x) = a_2(x_\delta)a_1(x_\gamma),
\]

the FSUSY Hamiltonian operator on each site \( x \) in \( \Omega \) corresponding to the discussed system can be given by the following operation

\[
q_1Q_+(x)Q_-(y) + q_1^{-1}Q_-(y)Q_+^\dagger(x) - q_2Q_-(y)Q_+(x) - q_2^{-1}Q_+^\dagger(x)Q_-(y) = \delta(x,y)H(x). \tag{18}
\]

In a straightforward calculation we can obtain

\[
[N_1(x)]_{q_1^{-1}} = q_1^{1-N_1(x)}[N_1(x)]_{q_1}. \tag{19}
\]

Consequently, owing to equations (17), (18) and (19) the FSUSY Hamiltonian operator on \( x \in \Omega \) can be written as

\[
H(x) = [P^{-1}q_1^{-N_1(x)} + Pq_1][N_1(x)]_{q_1} - [P^{-1}q_2^{-N_2(x)} + Pq_2][N_2(x)]_{q_2},
\]

where the operator \( P \) is defined as

\[
P = \sum_{x < y} - \sum_{x > y} ||N_1(x) + N_2(x) - 1|| . \tag{21}
\]

In order to strengthen this result we assume in this work one study of a particular case of its extremities. That is, we will discuss the \( N = 2 \) FSUSY for one system consisting of one bosonic anyon and one fermionic anyon, from which we will pass to write the corresponding Hamiltonian operator. Otherwise, in the following section, we will give the FSUSY Hamiltonian in connection with the Chern-Simons gauge theory, in order to make the properties of the fractional statistics of the corresponding system appear.

Let us assume the following limits

\[
q_1 = 1, \quad q_2 = -1 \tag{22}
\]

and the nilpotency condition becomes

\[
(Q_{\pm}(x))^2 = 0. \tag{23}
\]

Then the FSUSY Hamiltonian operator will be

\[
H(x) = [P^{-1} + P][N_b(x)] - [P^{-1}(-1)^{-N_f(x)}] - P[N_f(x)],
\]

with

\[
N_b(x) = N_1(x) = a_1^\dagger(x_\delta)a_1(x_\alpha) = B^\dagger(x)B(x)
\]

\[
N_f(x) = N_2(x) = a_2^\dagger(x_\delta)a_2(x_\alpha) = F^\dagger(x)F(x).
\]

\[
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\( N_b(x) \) and \( N_f(x) \) are respectively the bosonic and the fermionic number operators defined on the site \( x \in \Omega \). Thus \( B^\dagger(x) \) (\( F^\dagger(x) \)) and \( B(x) \) (\( F(x) \)) are the creation and annihilation bosonic (fermionic) operators respectively.

Thus, namely
\[
|n_{xb}, n_{xf} > = |n_{xb} > \otimes |n_{xf} >
\]
an element of Fock space associated to our two particles system \{bosonic anyon, fermionic anyon\}. We will have
\[
N_b(x)|n_{xb}, n_{xf} > = n_{xb}|n_{xb}, n_{xf} >, \\
N_f(x)|n_{xb}, n_{xf} > = n_{xf}|n_{xb}, n_{xf} >,
\]
\( n_{xb} \) and \( n_{xf} \) are the numbers of bosonic anyons and fermionic anyons respectively on \( x \), with
\[
n_{xb} \in \mathbb{N}, \quad n_{xf} = 0, 1.
\]

In our SUSY, we assumed that we have only one bosonic and one fermionic anyon on the site \( x \). Then \( n_{xf} = 1 \). In this case we remark that
\[
(-1)^{-N_f(x)} \Rightarrow -1
\]
and the corresponding Hamiltonian will be
\[
H(x) = [P^{-1} + P][N_b(x) + N_f(x)].
\]

So we have only one bosonic anyon and one fermionic anyon on the site \( x \). For the other sites there are no particles. Then the eigenvalues of the number operators \( N_b(z) \) and \( N_f(z) \) are zero (\( z \neq x \))
\[
n_{zb} = 0 = n_{zf},
\]
for this case the operator \( P \) (see Eq.(21)goes to identity
\[
P \Rightarrow 1
\]
and equation (30) becomes
\[
H(x) = 2[B^\dagger(x)B(x) + F^\dagger(x)F(x)],
\]
such as
\[
[B(x), B^\dagger(x)] = 1, \quad \{F(x), F^\dagger(x)\} = 1.
\]
This FSUSY Hamiltonian operator can be written as the sum
\[
H(x) = H_b(x) + H_f(x),
\]
such as \( H_b(x) \) is the Hamiltonian of the bosonic anyon and \( H_f(x) \) is the Hamiltonian of the fermionic anyon, with
\[
H_b(x) = 2[B^\dagger(x)B(x) + \frac{1}{2}], \\
H_f(x) = 2[F^\dagger(x)F(x) - \frac{1}{2}].
\]
By considering the local operators with repeated use of comultiplication, we define the global operators as follows

\[ H = \sum_{x \in \Omega} 1 \otimes \ldots \otimes 1 \otimes H(x) \otimes 1 \otimes \ldots \otimes 1 \]

\[ B = \sum_{x \in \Omega} 1 \otimes \ldots \otimes 1 \otimes B(x) \otimes 1 \otimes \ldots \otimes 1 \]

\[ B^\dagger = \sum_{x \in \Omega} 1 \otimes \ldots \otimes 1 \otimes B^\dagger(x) \otimes 1 \otimes \ldots \otimes 1 \]  
(37)

\[ F = \sum_{x \in \Omega} 1 \otimes \ldots \otimes 1 \otimes F(x) \otimes 1 \otimes \ldots \otimes 1 \]

\[ F^\dagger = \sum_{x \in \Omega} 1 \otimes \ldots \otimes 1 \otimes F^\dagger(x) \otimes 1 \otimes \ldots \otimes 1. \]

Then

\[ H = 2 |B^\dagger B + F^\dagger F|. \]  
(38)

The two particles are always defined on the two-dimensional lattice. Consequently, the notion of the fractional statistics will be hidden in the definition of the operators \( B, B^\dagger, F \) and \( F^\dagger \). It will be described in the introduction of the Chern-Simons field.

4 The FSUSY Hamiltonian

Following Wilczek\(^7\), an anyon is a composite of (fictitious) electric charge \( q \) and magnetic flux \( \Phi \). Let us note here that these fictitious charge and flux have nothing to do with the ordinary electro-magnetism. But rather a new gauge field is introduced purely as a mathematical device, called a Chern-Simons gauge field. This kind of field given by the (fictitious) vector potential

\[ A^\ell(x) = \frac{\Phi}{2\pi} \epsilon_{ij} \frac{x^j}{|x|^2}, \quad i = 1, 2. \]  
(39)

can realize a flux tube attached on a point particle. With \( x = (x_1, x_2) \) on the plane (if we tend the lattice spacing to zero), the corresponding flux \( \Phi \) is given by

\[ \oint A^\ell dx^\ell = \Phi \]  
(40)

for any path winding around the associated particle in anti-clockwise direction. We add that there is no classical force since the magnetic field vanishes everywhere away from the origin where the point particle is localized.

Let us note here that we can continue our investigation on the plane instead of the two-dimensional lattice without any ambiguities.

With the intention of making the fractional property of the statistics and the SUSY of our two different particles system appear, we are going to use the complex notation to simplify the calculation in investigating the Hamiltonian of one bosonic anyon. We can write

\[ z = x^1 + ix^2, \quad \partial = \frac{1}{2}(\partial_1 - i\partial_2) \]

\[ \bar{z} = x^1 - ix^2, \quad \bar{\partial} = \frac{1}{2}(\partial_1 + i\partial_2) \]  
(41)
For the Chern-Simons field we put
\[ A = -\frac{i}{2}(A^1 - iA^2) = -i \frac{\Phi}{4\pi z} \]
\[ A = -\frac{i}{2}(A^1 + iA^2) = i \frac{\Phi}{4\pi z}. \]

Yet, in a straightforward way, we obtain the following commutation relations
\[ [z, \partial_z] = 1, \quad [\bar{z}, \partial_{\bar{z}}] = 1, \quad [z, \partial_{\bar{z}}] = 0 = [\bar{z}, \partial_z] \]
\[ [\frac{1}{2}, \partial_z] = -\frac{i}{2}, \quad [\frac{1}{2}, \partial_{\bar{z}}] = -\frac{i}{2}, \quad [\frac{1}{2}, \partial_{\bar{z}}] = [\frac{1}{2}, \partial_z] = \pi \delta^{(2)}(z). \]

In the complex notation again, let us start with the definition of some creation and annihilation operators denoted \( b^\dagger \) and \( b \) respectively. These new operators can be written as follows
\[ b^\dagger = \frac{\Phi}{\sqrt{8\pi}} q_\phi \frac{1}{z} - \sqrt{2} \partial_z + \frac{1}{\sqrt{2}} \bar{z} \]
\[ b = \frac{\Phi}{\sqrt{8\pi}} q_\phi \frac{1}{z} - \sqrt{2} \partial_{\bar{z}}, \]
which satisfy the usual commutation relation
\[ [b, b^\dagger] = 1. \]

Such as, owing to equation (44)
\[ b^\dagger b = \frac{\Phi^2}{8\pi^2} q_\phi \frac{1}{z^2} - 2\partial \bar{\partial} + \bar{z} = \bar{z} \partial + \frac{\Phi}{4\pi} \]
\[ b b^\dagger = \frac{\Phi^2}{8\pi^2} q_\phi \frac{1}{z^2} - 2\partial \partial + = \partial \bar{z} + \frac{\Phi}{4\pi}. \]

Thus, \( q_\phi \) is the charge of bosonic and \( \Phi \) is its associated flux.

From these operators, we can define our operators \( B^\dagger \) and \( B \) which appear in the Hamiltonian \( H_b \) as creation and annihilation bosonic anyon operators. Then we assume the following definition
\[ B^\dagger B + \frac{1}{2} = \frac{1}{2} [b^\dagger b - \bar{z} \partial - \bar{\partial} z - \frac{\Phi}{4\pi}] = \frac{1}{2} [\frac{\Phi^2}{8\pi^2} q_\phi \frac{1}{z^2} - 2\partial \bar{\partial}] \]
\[ B B^\dagger + \frac{1}{2} = \frac{1}{2} [b b^\dagger - \partial z - \bar{\partial} \bar{z} - \frac{\Phi}{4\pi}] = \frac{1}{2} [\frac{\Phi^2}{8\pi^2} q_\phi \frac{1}{z^2} - 2\partial \bar{\partial}], \]
which leads us to the usual commutation relation
\[ [B, B^\dagger] = 1. \]

Consequently, the Hamiltonian \( H_b \) can be written as
\[ H_b = 2[B^\dagger B + \frac{1}{2}] \]
\[ = \Phi q_\phi \frac{1}{8\pi^2} \frac{1}{z^2} - 2\partial \bar{\partial}. \]
Also it is very easy to verify that, in cartesian coordinates, the Hamiltonian $H_b$ takes the following expression

$$H_b = -\frac{1}{2}(\nabla - iqA)^2,$$  \hspace{1cm} (51)

taking into account equations (40), (41), (42), (43), (46), (47) and (48).

Otherwise, for the fermionic anyon system the same proposition and the definition given by Wilczek can be associated to it. So, the fermionic anyon is a composite of the charge $q_f$ and the flux $\Phi$.

According to the relations (36) and (37), the associated Hamiltonian $H_f$ reads

$$H_f = 2[F^\dagger F - \frac{1}{2}],$$  \hspace{1cm} (52)

In terms of the Pauli matrices

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix},$$  \hspace{1cm} (53)

we define the fermionic anyon operators as follows

$$F^\dagger = \frac{1}{\sqrt{2}}(\sigma_1 + i \frac{q_f}{2} \delta^{(2)}(z)\sigma_2)$$ \hspace{1cm} (54)

$$F = \frac{1}{\sqrt{2}}(\sigma_1 - i \frac{q_f}{2} \delta^{(2)}(z)\sigma_2)$$

which satisfy

$$\{F, F^\dagger\} = 1 + \frac{q_f^2 \Phi^2}{4} \delta^{(2)}(z).$$  \hspace{1cm} (55)

Then the relation (52) becomes

$$H_f = q_f \Phi \delta^{(2)}(z)\sigma_3 + \frac{q_f^2 \Phi^2}{4} \delta^{(2)}(z).$$ \hspace{1cm} (56)

Let us assume, in this context, that $\Phi$ is a weak flux; i.e. $\Phi \ll 1$. Then we can stop at the first order of $\Phi$ in our results as

$$\{F, F^\dagger\} = 1,$$ \hspace{1cm} (57)

Then the FSUSY Hamiltonian, will be explicitly

$$H = \frac{1}{2}(\nabla - iqA)^2 + q_f \Phi \delta^{(2)}(z)\sigma_3.$$  \hspace{1cm} (58)

From this equation, it is very clear that the FSUSY Hamiltonian $H$ describes a system consisting of one bosonic anyon and one fermionic anyon. Such as the two kinds of particles associated to a Chern-Simons gauge field $A$. Then, in this SUSY there are non-trivial properties for the corresponding statistics. This system is characterized by two statistical parameters which can be defined for the bosonic anyonic system and the fermionic ones respectively as

$$\nu_b = \frac{q_b \Phi}{2}, \quad \nu_f = \frac{q_f \Phi}{2}. \hspace{1cm} (59)$$

Consequently, the FSUSY Hamiltonian, in this context, is well defined for a system characterized by fractional statistics.
5 Conclusion

To summarize, we can say that, in this paper, we have discussed the construction of anyonic operators and its algebras, on the two-dimensional lattice, from bosonic, fermionic operators and quonic ones as a general case. We have used for this subject the Jordan-Wigner transformation\[?] basing on references \[?, ?, ?, ?, ?\].

By using the quonic anyons, we have thus recalled the FSUSY constructed in the work of \[?] on the two-dimensional lattice. Where the supercharges have defined by associating two different kinds of quonic anyons with different deformation parameters. As a particular case, we have tended the deformation parameters to ±1. Then we have obtained one FSUSY Hamiltonian for a system consists of one bosonic anyon and one fermionic anyon. In connection with the Chern-Simons gauge theory, we rewrote the obtained FSUSY Hamiltonian.

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