Abstract

In this note, we prove a theorem on another presentation for the algebra of the endomorphisms of the permutation representation (Yokonuma-Hecke algebra) of a simple Chevalley group with respect to a maximal unipotent subgroup. This presentation is done using certain non-standard generators.
1. INTRODUCTION

Let $G$ be a simple Chevalley group defined over $\mathbb{F}_q$. In this manuscript, we prove a theorem on a new presentation for the algebra of endomorphisms $\mathcal{Y}_n(q)$ associated to the induced representation of the trivial representation of $U$ up to $G$, where $U$ is a maximal unipotent subgroup of $G$. In [?], this theorem was proved for the case when the Cartan matrix of $G$ is symmetric, that is when $G$ is of type $A_l, D_l, E_6, E_7$ or $E_8$. In this manuscript, we prove the theorem for the other simple Chevalley groups also. More precisely, we prove the nonstandard presentation theorem for the simple Chevalley groups of type $B_l, C_l, F_4$ and $G_2$.

In [?], T. Yokonuma has given a description (presentation) of this algebra $\mathcal{Y}_n(q)$ in terms of the standard generators, that is, in terms of generators given by the double cosets (see 11.30[?]). So, we call the algebra $\mathcal{Y}_n(q)$, the Yokonuma-Hecke algebra. The presentation of Yokonuma is analogous to the classical presentation of the Iwahori-Hecke algebra (see [?]).

In Theorem 2.18[?], the first author of this article has proved that this algebra $\mathcal{Y}_n(q)$ has a presentation with non standard generators for the simple Chevalley groups $G$ whose Cartan matrix is symmetric. This presentation uses non-standard generators defined by a pre-fixed non-trivial additive character of $\mathbb{F}_q$, and a certain non-trivial linear combination involving the standard basis of $\mathcal{Y}_n(q)$ (see Definition ??). Originally, these generators were defined in a geometrical way for the group $GL_n(\mathbb{F}_q)$, that is, like Fourier Transforms on the space of functions of flags vectors on $\mathbb{F}_q^n$. As an application of our main theorem, we recall that abstracting the presentation in the case when $G$ is of type $A_l$, it is possible to define a certain finite dimensional algebra, involving braids and ties, which give new matrix representation for the Artin group of type $A$, see [?]. It is a natural question to study the representation for the Artin groups of types $B_l, C_l, F_4$ and $G_2$.

The aim of this note is to prove that the above mentioned non standard generators give a presentation for the algebra $\mathcal{Y}_n(q)$ for the simple Chevalley groups of type $B_l, C_l, F_4$ and $G_2$.

For more precise statement, see Theorem 2.

The layout of this manuscript is as follows:

Section 2 consists of preliminaries and statement of the main Theorem (for a more precise statement, see Theorem 2.) Section 3 consists of the proof for the case when $G$ is of type $B_l, C_l$ or $F_4$. Section 4 consists of the proof for the case when $G$ is of type $G_2$.

2. PRELIMINARES AND STATEMENT OF THE MAIN RESULT

2.1. Let $k$ denote a finite field with $q$ elements. Let $G$ be a simple simply connected Chevalley group defined over $k$. Let $T$ be a “maximally split” torus of $G$. Let $B$ be a Borel subgroup of $G$ containing $T$. Let $U$ be the unipotent radical of $B$. We will denote the rank of $G$ by $l$.

We denote the set of all roots with respect to $T$ by $\Phi$.

Let $\Delta$ be the set of all simple roots with respect to $T$ and $B$. Let $N$ be the normaliser of $T$ in $G$ and let $W = N/T$ be the Weyl group of $G$ with $S = \{s_\alpha : \alpha \in \Delta\}$ being the set of simple reflections. The pair $(W, S)$ is a Coxeter system and we have the presentation:

$$W = \langle s_\alpha : (s_\alpha s_\beta)^{m_{\alpha \beta}} = 1, \alpha, \beta \in \Delta \rangle,$$

where $m_{\alpha \beta}$ denote the order of $s_\alpha s_\beta$.

Let $\pi$ be the canonical homomorphism from $N$ onto $W$. Using $\pi$, we have an action of the Weyl group $W$ on $T: (w, t) \mapsto w(t) := \omega t \omega^{-1}$, where $\omega \in N$ is such that $\pi(\omega) = w$.

We recall that for any root $\alpha \in \Phi$, there is an $\omega_\alpha \in N$ such that $\pi(\omega_\alpha) = s_\alpha$ and there is a homomorphism $\phi_\alpha : SL_2 \longrightarrow G$ such that

$$\omega_\alpha = \phi_\alpha \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \quad h_\alpha(r) = \phi_\alpha \begin{pmatrix} r & 0 \\ 0 & r^{-1} \end{pmatrix}, \quad (r \in k^\times).$$

2.2. Let $\mathcal{Y}_n(q)$ be the algebra of endomorphism of the induced (permutation) representation $\text{Ind}_{U}^{G}1$, over the field of complex numbers. We call the algebra $\mathcal{Y}_n(q)$ as the Yokonuma-Hecke algebra.
From the Bruhat decomposition, $G = \prod_{n \in \mathbb{N}} UnU$, we have that the standard basis of the Yokonuma-Hecke algebra is parametrised by $N$. Let $\{R_n | n \in \mathbb{N}\}$ be the standard basis.

If $n = \omega \alpha$, we denote $R_n$ by $R_\alpha$.

If $n = t \in T$, we call the elements $R_t$ in $Y_n(q)$ operators of homothety corresponding to $t$. In the case $t = h_\alpha(r)$, we denote $R_t$ by $H_\alpha(r)$. With these notations, we define $E_\alpha$ by

$$E_\alpha := \sum_{r \in k^X} H_\alpha(r) \quad (\alpha \in \Phi).$$

It is clear that the $E_\alpha$'s commute among themselves, and a direct computation shows that

$$E_\alpha^2 = (q-1)E_\alpha.$$

Now, we recall a Theorem due to T. Yokonuma.

**Theorem 1.** (See [?]) The Yokonuma-Hecke algebra $Y_n(q)$ is generated, as an algebra, by $R_\alpha (\alpha \in \Phi)$, and the homotheties $R_t (t \in T)$. Moreover, these generators with the relations below define a presentation for $Y_n(q)$.

(??.1) $R_\alpha^2 = qH_\alpha(-1) + R_\alpha E_\alpha$ (quadratic relation)

(??.2) $\frac{R_\alpha R_\beta R_\alpha R_\beta \cdots}{m_{\alpha\beta}} = \frac{R_\beta R_\alpha R_\beta R_\alpha \cdots}{m_{\alpha\beta}}$ (braid relation)

(??.3) $R_t R_\alpha = R_\alpha R_{t'}$, where $t' = \omega_\alpha t \omega_\alpha^{-1}$ ($t \in T$)

(??.4) $R_u R_v = R_{uv}$ ($u, v \in T$).

2.3. In the following, we fix a non-trivial additive character $\psi$ of $(k, +)$. For any $\alpha$ in $\Phi$, we define $\Psi_\alpha$ as the following linear combination of elements in $Y_n(q)$,

$$\Psi_\alpha := \sum_{r \in k^X} \psi(r)H_\alpha(r).$$

From a direct computation, we have that $\Psi_\alpha$ commutes with $E_\alpha$, and

$$\Psi_\alpha E_\alpha = -E_\alpha.$$

**Definition 1.** Let $\alpha \in \Psi$. We define the element $L_\alpha$, as

$$L_\alpha := q^{-1}(E_\alpha + R_\alpha \Psi_\alpha).$$

Our main goal is to prove the following Theorem.

**Theorem 2.** The Yokonuma-Hecke algebra $Y_n(q)$ is generated (as an algebra), by $L_\alpha (\alpha \in \Phi)$, and the homotheties $R_t (t \in T)$. Moreover, these generators with the relations below define a presentation for $Y_n(q)$.

(??.1) $L_\alpha^2 = 1 - q^{-1}(E_\alpha - L_\alpha E_\alpha)$ (quadratic relation)

(??.2) $\frac{L_\alpha L_\beta L_\alpha L_\beta \cdots}{m_{\alpha\beta}} = \frac{L_\beta L_\alpha L_\beta L_\alpha \cdots}{m_{\alpha\beta}}$ (braid relation)

(??.3) $R_t L_\alpha = L_\alpha R_{t'}$, where $t' = \omega_\alpha t \omega_\alpha^{-1}$ ($t \in T$)

(??.4) $R_u R_v = R_{uv}$ ($u, v \in T$).

To prove this Theorem, we introduce some notations and one useful Proposition. We denote by $E_\alpha^w$ the effect of $w$ on $E_\alpha$ arising from the action of the Weyl group $W$ on $T$. That is,

$$E_\alpha^w = \sum_{r \in k^X} H_\gamma(r) \quad (\alpha \in \Phi, w \in W),$$

where $\gamma$ is the root defined by $w(\alpha) = \gamma$.

In the similar way, we denote by $\Psi_\alpha^w$ the effect of $w$ on $\Psi_\alpha$. 
Proposition 3. Let $s$ be the reflection corresponding to $\alpha$, and let $\beta \in \Phi$. We have

\begin{align*}
(\text{??}.1) \quad & E_\beta R_\alpha = R_\alpha E_\beta^s, \quad \Psi_\beta R_\alpha = R_\alpha \Psi_\beta^s \\
(\text{??}.2) \quad & E_\alpha^s = E_\alpha \\
(\text{??}.3) \quad & E_\alpha \Psi_\alpha^s = -E_\alpha = E_\alpha \Psi_\alpha \\
(\text{??}.4) \quad & \Psi_\alpha \Psi_\alpha^s = \Psi_\alpha^s \Psi_\alpha = qH(1) - E_\alpha.
\end{align*}

**Proof.** The proof of the assertions in **?.1** is an immediate consequence of Yokonuma’s Theorem, part 1.3 and the proofs of **?.2**, **?.3** and **?.4** are straightforward computations. □

2.4. We are now going to sketch the proof of Theorem ?? for the simple Chevalley groups of type $B_l, C_l, F_4$ and $G_2$. The only statement of Theorem ?? that involves the Dynkin diagram of the group is the statement about the braid relation, that is **?.2**. Since Theorem ?? was proved for the cases of type $A_l, D_l, E_6, E_7$ and $E_8$ in [??], to prove the Theorem, we need to prove only **?.2** for the cases when $G$ is of type $B_l, C_l, F_4$ and $G_2$. In Section 2, we prove **?.2** for the case when $G$ is of type $B_l, C_l$ and $F_4$. In Section 3, we prove **?.2** for the case when $G$ is of type $G_2$. The method of proof involves the one parameter subgroups $H(1), t \in k^\times, \alpha \in \Phi$, and some automorphisms of the two dimensional torus $k^\times \times k^\times$.

3. **Cases $B_l, C_l$ AND $F_4$**

3.1. Let $A = \{\alpha_1, \ldots, \alpha_{l-1}, \alpha_l\}$ denote the set of all simple roots of type $B_l$. So, the Dynkin diagram is as follows:

\begin{center}
$B_l$: $\alpha_1 - \underbrace{\quad \cdots} \quad \alpha \quad - \underbrace{\cdots} \quad \beta$
\end{center}

where $\alpha = \alpha_{l-1}$ and $\beta = \alpha_l$. Let $s$ (respectively $s'$) be the reflection corresponding to the root $\alpha$ (respectively $\beta$).

Notice that the simple roots $\alpha_1, \ldots, \alpha_{l-1}$ of $B_l$ turn to the set of simple roots of $A_{l-1}$ and so from Theorem 2.12[??], we deduce:

\begin{align*}
L_{\alpha_i}L_{\alpha_j} &= L_{\alpha_j}L_{\alpha_i} \quad \text{if } |i - j| > 1 \\
L_{\alpha_i}L_{\alpha_j}L_{\alpha_i} &= L_{\alpha_j}L_{\alpha_i}L_{\alpha_j} \quad \text{if } |i - j| = 1.
\end{align*}

Therefore, to prove Theorem ??, we need to prove only the relation $L_{\alpha}L_{\beta}L_{\alpha}L_{\beta} = L_{\beta}L_{\alpha}L_{\beta}L_{\alpha}$. In the proof of this braid relation, we will use the following lemma. The same proof holds for the cases: $C_l$ and $F_4$. (The only difference is $\alpha = \alpha_1, \beta = \alpha_{l-1}$ in the case of $C_l$ and $\alpha = \alpha_2, \beta = \alpha_3$ in the case of $F_4$).

**Lemma 4.** We have

\begin{align*}
(\text{??}.1) \quad & E_\alpha^s E_\beta = E_\beta E_\alpha = E_\alpha E_\beta \\
(\text{??}.2) \quad & (E_\alpha^s)^s = E_\alpha^s, \quad (E_\beta^s)^s = E_\beta^s \\
(\text{??}.3) \quad & E_\beta^s E_\alpha = E_\alpha E_\beta \\
(\text{??}.4) \quad & ((E_\alpha^s)^s)^s = E_\alpha, \quad ((E_\beta^s)^s)^s = E_\beta \\
(\text{??}.5) \quad & (\Psi_\alpha^s)^s = \Psi_\alpha^s, \quad (\Psi_\beta^s)^s = \Psi_\beta^s \\
(\text{??}.6) \quad & (H_{\alpha}(-1))^{s'} = H_{\alpha}(-1), \quad (H_{\beta}(-1))^{s'} = H_{\alpha}(-1)H_{\beta}(-1).
\end{align*}
Proof. We now prove ?? 1. We have $s'(\alpha) = \alpha + 2\beta$ and so we have $E^E \alpha \beta = E^E \alpha + 2\beta$. Hence, we have

$$E^E \alpha \beta \beta = \sum_{t \in k^\times} H^E_{\alpha + 2\beta}(t) \sum_{r \in k^\times} H_{\beta}(r)$$

$$= \sum_{(t,r) \in k^\times \times k^\times} H^E_{\alpha}(t)H_{\beta}(t^2 \cdot r)$$

$$= \sum_{(t,r) \in k^\times \times k^\times} H^E_{\alpha}(t)H_{\beta}(r) = E^E \alpha \beta,$$

since the map $(t, r) \mapsto (t, t^2 \cdot r)$ is an automorphism of $k^\times \times k^\times$. This proves that $E^E \alpha \beta = E^E \alpha \beta$.

The equality $E^E \alpha \beta = E^E \alpha \beta$ follows from the fact that $s(\beta) = \alpha + \beta$ and the map $(t, r) \mapsto (tr, r)$ is an automorphism of $k^\times \times k^\times$.

We now prove ?? 2. We have $s(s'(\alpha)) = s(\alpha + 2\beta) = -\alpha + 2(\beta + \alpha) = \alpha + 2\beta = s'(\alpha)$. This proves that $(E^E \alpha \beta)^s = E^E \alpha \beta$.

Proof of $(E^E \alpha \beta)^s = E^E \alpha \beta$ follows from the fact:

$$s'(s'(\alpha)) = s'(\alpha + 2\beta) = (\alpha + 2\beta) - \beta = \alpha + \beta = s(\beta).$$

Proof of ?? 3 follows from the facts that $s(\beta) = \alpha + \beta, s'(\alpha) = \alpha + 2\beta$ and the map $(t, r) \mapsto (tr, tr^2)$ is an automorphism of $k^\times \times k^\times$.

Proof of ?? 4 follows from the facts that

$$s'(s'(\alpha)) = s'(\alpha + 2\beta) = (\alpha + 2\beta) - 2\beta = \alpha$$

and

$$s(s'(s(\beta))) = s(s'(\alpha + \beta)) = s(\alpha + \beta) = -\alpha + \alpha + \beta = \beta.$$

Proof of ?? 5 is similar to the proof of ?? 2. We note here that $\Psi$ does not play an important role in this situation.

We now prove ?? 6. We have $s'(\alpha) = \alpha + 2\beta$ and hence, we have

$$(H^E_{\alpha}(1))^s = H^E_{\alpha}(1)H^E_{\beta}(1)^2 = H^E_{\alpha}(-1)H^E_{\beta}((-1)^2) = H^E_{\alpha}(-1).$$

Since $s(\beta) = \alpha + \beta$, we have $(H^E_{\beta}(-1)^s = H^E_{\alpha}(1)H^E_{\beta}(-1)).$

We now prove the following Lemma which will complete the proof of Theorem ?? for the cases when $G$ is of type $B_2, C_1$ and $F_4$.

Lemma 5. $L^E_{\alpha}L^E_{\beta}L^E_{\alpha}L^E_{\beta} = L^E_{\alpha}L^E_{\beta}L^E_{\alpha}L^E_{\beta}$.

Proof. First, we compute the products:

$p_{\alpha \beta} := q^2L^E_{\alpha}L^E_{\beta}$, and $p_{\beta \alpha} := q^2L^E_{\beta}L^E_{\alpha}$.

From the definition of $L^E_{\alpha}$ and $L^E_{\beta}$, we have

$$p_{\alpha \beta} = (E^E_{\alpha} + R^E_{\alpha} \Psi_{\alpha})(E^E_{\beta} + R^E_{\beta} \Psi_{\beta})$$

$$= E^E_{\alpha}E^E_{\beta} + E^E_{\alpha}R^E_{\beta} \Psi_{\beta} + R^E_{\alpha} \Psi_{\alpha}E^E_{\beta} + R^E_{\alpha} \Psi_{\alpha}R^E_{\beta} \Psi_{\beta}$$

$$= E^E_{\alpha}E^E_{\beta} + \underbrace{R^E_{\alpha} \Psi_{\beta} + R^E_{\alpha} \Psi_{\alpha}E^E_{\beta}}_{a} + \underbrace{R^E_{\alpha} \Psi_{\alpha}R^E_{\beta} \Psi_{\beta}}_{b}.\overbrace{\Psi_{\alpha}}^{c} \overbrace{R^E_{\alpha}}^{d} \overbrace{R^E_{\beta}}^{E^E_{\beta}}.$$

Notice that $b$ (respectively $d$) is obtained from $E^E_{\alpha}R^E_{\beta} \Psi_{\beta}$ (respectively $R^E_{\alpha} \Psi_{\alpha}R^E_{\beta} \Psi_{\beta}$) using proposition 3.

Now, we compute $p_{\alpha \beta}^2$:

$$p_{\alpha \beta}^2 = a^2 + b^2 + c^2 + d^2 + ab + ac + ad + ba + bc + bd + ca + cb + cd + da + db + dc.$$

In the same way, we obtain an analogous expression for $p_{\beta \alpha}^2$, but in the symbols $a', b', c'$ and $d'$.

The proof of this Lemma is as follows. In the expression of $p_{\alpha \beta}^2$ and $p_{\beta \alpha}^2$, we first bring the monomials 1, $R^E_{\alpha}, R^E_{\beta}, R^E_{\alpha}R^E_{\beta}, R^E_{\alpha}R^E_{\beta}R^E_{\alpha}, R^E_{\alpha}R^E_{\beta}R^E_{\alpha}R^E_{\beta},$ and $R^E_{\alpha}R^E_{\beta}R^E_{\alpha}R^E_{\beta}R^E_{\alpha}$ to the
left. After this procedure, we will check that the coefficients of these monomials in $p^2_{\alpha \beta}$ with the corresponding coefficients in $p^2_{\beta \alpha}$ are the same. Then, the lemma follows.

Let $X_0, X_\alpha, X_\beta, X_{\alpha \beta}, X_{\alpha 3 \alpha}, X_{\alpha 3 \beta}, X_{\alpha \beta 3 \alpha}$ and $X_{\alpha 3 \beta \alpha}$ be the coefficient of $1$, $R_\alpha$, $R_\beta$, $R_\alpha R_\beta$, $R_\alpha R_\beta R_\alpha$, $R_\alpha R_\beta R_\alpha R_\alpha$ and $R_\alpha R_\beta R_\alpha R_\beta R_\alpha$ respectively in $p^2_{\alpha \beta}$. Let $Y_0, Y_\alpha, Y_\beta, Y_{\alpha \beta}, Y_{\alpha 3 \alpha}, Y_{\alpha 3 \beta}, Y_{\alpha \beta 3 \alpha}$ and $Y_{\alpha 3 \beta \alpha}$ be the coefficient of $1$, $R_\alpha$, $R_\beta$, $R_\alpha R_\beta$, $R_\alpha R_\beta R_\alpha$, $R_\alpha R_\beta R_\alpha R_\beta R_\alpha$ and $R_\alpha R_\beta R_\alpha R_\beta$ respectively in $p^2_{\beta \alpha}$.

We need to prove that $X_\gamma = Y_\gamma$ for all $\gamma$ (words in $\alpha$ and $\beta$) as above. To do this, we will compute $X_\gamma$ and $Y_\gamma$ using essentially the Lemma ?? Now, as the computations are all very similar, we are going to compute only $X_0, X_{\alpha \beta}, Y_{\alpha \beta}, X_{\alpha \beta \alpha}, Y_{\alpha \beta \alpha}, X_{\alpha \beta \alpha \beta}$ and $Y_{\alpha \beta \alpha \beta}$.

**Computation of $X_0$ and $Y_0$.** It is easy to see that the terms contributing to the constant coefficient in the expression of $p^2_{\alpha \beta}$ (resp. in $p^2_{\beta \alpha}$) are only $a^2, b^2$ and $c^2$ (resp. $(a')^2, (b')^2$ and $(c')^2$).

We have $a^2 = (E_\alpha E_\beta)^2 = (E_\beta E_\alpha)^2 = (a')^2$.

We now compute $b^2$.

We have

$$b^2 = R_\beta E_\alpha^s \Psi_\beta R_\alpha E_\alpha^s \Psi_\beta = R^2_\beta ((E_\alpha^s \Psi_\beta)^s)^s E_\alpha^s \Psi_\beta$$

From the observation ?? of Theorem ??, we have $R^2_\beta = qH_\beta(-1) + R_\beta E_\beta$.

Thus, the constant coefficient yielded by $b^2$ is $qH_\beta(-1)(E_\alpha^s \Psi_\beta)^s E_\alpha^s \Psi_\beta$.

By a similar computation, it is easy to see that $(c')^2$ yields the same constant coefficient as that yielded by $b^2$.

A similar proof shows that the constant coefficient yielded by $c^2$ and that yielded by $(b')^2$ are the same and both are equal to $qH_\gamma(-1)(E_\alpha^s \Psi_\beta)^s E_\alpha^s \Psi_\beta$.

Thus, we have $X_0 = Y_0$.

**Computation of $X_{\alpha 3 \alpha}$ and $Y_{\alpha 3 \alpha}$.** It is clear that the terms having $R_\alpha R_\beta R_\alpha$ in $p_{\alpha \beta}$ (respectively $p_{\beta \alpha}$) is only $dc$ (respectively $b'd'$). We have $dc = b'd'$. Namely,

$$b'd' = (R_\alpha E_\beta^s \Psi_\alpha)(R_\beta R_\alpha \Psi_\beta^s \Psi_\alpha)$$

$$= R_\alpha R_\beta R_\alpha (E_\beta^s)^s (\Psi_\alpha^s)^s \Psi_\beta^s \Psi_\alpha$$

$$= R_\alpha R_\beta R_\alpha E_\beta (\Psi_\alpha^s)^s \Psi_\beta^s \Psi_\alpha$$

$$= R_\alpha R_\beta R_\alpha (\Psi_\alpha^s)^s \Psi_\beta^s \Psi_\alpha E_\beta$$

$$= (R_\alpha R_\beta \Psi_\alpha^s \Psi_\beta)(R_\alpha \Psi_\alpha E_\beta)$$

$$= dc$$

**Computation of $X_\alpha$ and $Y_\alpha$.** The terms having $R_\alpha$ in $p^2_{\alpha \beta}$ are: $c^2, ac, ca$, and $db$. We now compute $c^2$. We have

$$c^2 = (R_\alpha E_\beta^s \Psi_\alpha)(R_\beta E_\alpha^s \Psi_\beta)$$

$$= R^2_\alpha E_\alpha^s \Psi_\alpha^s E_\beta (\text{from proposition ??})$$

$$= (qH_\alpha(-1) + R_\alpha E_\alpha^s \Psi_\alpha^s E_\beta) (\text{from ??1})$$

Hence, $c^2$ yields the coefficient $E_\alpha \Psi_\alpha^s E_\beta \Psi_\alpha E_\beta$. Now, using proposition ?? and lemma ??, we get

$$E_\alpha \Psi_\alpha^s E_\beta^s \Psi E_\beta = -E_\alpha E_\beta^s \Psi_\alpha E_\beta$$

$$= -E_\alpha E_\beta \Psi_\alpha E_\beta$$

$$= -E_\beta E_\alpha \Psi_\alpha E_\beta$$

$$= E_\beta E_\alpha E_\beta$$

$$= (q - 1) E_\alpha E_\beta$$

(from 1).
On the other hand, using Proposition ??, we get
\[ ac = (E_\alpha E_\beta)(R_\alpha \Psi_\alpha E_\beta) = R_\alpha E_\alpha^s E_\beta^s \Psi_\alpha E_\beta, \]
and
\[ ca = R_\alpha \Psi_\alpha E_\beta E_\alpha E_\beta. \]

Using Lemma ??, we deduce:
\[ E_\alpha^s E_\beta^s \Psi_\alpha E_\beta = \Psi_\alpha E_\beta E_\alpha E_\beta = -(q - 1)E_\alpha E_\beta. \]

Let us compute \( db \),
\[ db = (R_\alpha R_\beta \Psi_\alpha^s \Psi_\beta)(R_\beta E_\alpha^s \Psi_\beta) = R_\alpha R_\beta^s(\Psi_\alpha^s)^s \Psi_\beta^s E_\alpha E_\beta \] (from proposition ??).

Therefore, from ??, we have
\[ db = R_\alpha(qH_\beta(-1) + R_\beta E_\beta)(\Psi_\alpha^s)^s \Psi_\beta^s E_\alpha E_\beta. \]

Thus, \( db \) yields the coefficient \( qH_\beta(-1)(\Psi_\alpha^s)^s \Psi_\beta^s E_\alpha E_\beta = qH_\beta(-1)\Psi_\alpha \Psi_\beta^s E_\alpha E_\beta. \)
Now,
\[ qH_\beta(-1)\Psi_\alpha \Psi_\beta^s E_\alpha E_\beta = qH_\beta(-1)\Psi_\alpha E_\beta^s(qH_\beta(-1) - E_\beta) \] (from ??).
\[ = q^2 \Psi_\alpha E_\beta^s - qH_\beta(-1)\Psi_\alpha E_\beta^s E_\beta \]
\[ = q^2 \Psi_\alpha E_\beta^s - qH_\beta(-1)\Psi_\alpha E_\alpha E_\beta \] (from ??).
\[ = q^2 \Psi_\alpha E_\beta^s - qH_\beta(-1)E_\alpha E_\beta \] (from ??).

Thus, \( db \) yields
\[ q^2 \Psi_\alpha E_\beta^s + qE_\alpha E_\beta. \]

Therefore, we have
\[ X_\alpha = q^2 \Psi_\alpha E_\beta^s + E_\alpha E_\beta. \]

It is easy to see that the terms having \( R_\alpha \) in \( p_{\beta \alpha} \) are precisely \( a'b', b'a', c'd', \) and \((b')^2\).

Let us compute \((b')^2\),
\[ (b')^2 = (R_\alpha E_\beta^s \Psi_\alpha)(R_\alpha E_\beta^s \Psi_\alpha) \]
\[ = R_\alpha^2(\Psi_\alpha^s)^s E_\beta^s \Psi_\alpha \]
\[ = R_\alpha^2 E_\beta^s \Psi_\alpha \]
\[ = (qH_\alpha(-1) + R_\alpha E_\alpha E_\beta)(E_\alpha E_\beta)(qH_\alpha(-1) - E_\alpha). \]

Hence \((b')^2\) yield the coefficient \( E_\alpha E_\beta E_\alpha^s qH_\alpha(-1) - E_\alpha = qE_\alpha E_\beta E_\alpha^s - E_\alpha^s E_\beta E_\beta^s. \) Then \((b')^2\) yield precisely
\[ (q - 1)E_\alpha E_\beta. \]

Now, we have \( a'b' = (E_\beta E_\alpha)(R_\alpha E_\beta^s \Psi_\alpha) = R_\alpha E_\beta^s E_\alpha E_\beta^s \Psi_\alpha. \) Therefore, using Lemma ??, we deduce that \( a'b' \) yield the coefficient
\[ E_\beta^s E_\alpha^s \Psi_\alpha = -(q - 1)E_\alpha E_\beta. \]

It is easy to see that \( b'a' \) also yield the same coefficient of \( a'b' \).

Let us now compute \( c'd' \),
\[ c'd' = (R_\beta \Psi_\beta E_\alpha)(R_\beta R_\alpha \Psi_\beta^s \Psi_\alpha) \]
\[ = R_\beta^2 R_\alpha(\Psi_\beta^s)^s(\Psi_\alpha^s)^s \Psi_\alpha \]
\[ = (qH_\beta(-1) + R_\beta E_\beta R_\alpha(\Psi_\beta^s)^s(\Psi_\alpha^s)^s \Psi_\alpha \]
\[ = qH_\beta(-1)R_\alpha(\Psi_\beta^s)^s(\Psi_\alpha^s)^s \Psi_\alpha + R_\beta E_\beta R_\alpha(\Psi_\beta^s)^s(\Psi_\alpha^s)^s \Psi_\alpha. \]
That is,
\[ c'd' = qR_\alpha(H_\beta(-1))^s{(\Psi_\beta^s)}^s(E_\alpha^s)^s\Psi_\alpha + R_\beta R_\alpha E_\beta^s(\Psi_\beta^s)^s(E_\alpha^s)^s\Psi_\alpha. \]

Thus, \( c'd' \) yields the coefficient
\[ q(H_\beta(-1))^s(\Psi_\beta^s)^s(E_\alpha^s)^s\Psi_\alpha = q(H_\beta(-1))^s(\Psi_\beta^s)^s(E_\alpha^s)^s\Psi_\alpha. \]

Now, from the observation ??4 of Proposition ??, and the observation ??2 of Lemma ??, we have
\[ q(H_\beta(-1))^s(\Psi_\beta^s)^s(E_\alpha^s)^s\Psi_\alpha = q(H_\beta(-1))^s(qH_\beta(-1) - E_\beta)^s E_\alpha^s \Psi_\alpha = q^2 E_\alpha^s \Psi_\alpha - qE_\beta^s E_\alpha^s \Psi_\alpha = q^2 E_\alpha^s \Psi_\alpha + qE_\alpha E_\beta. \]

Thus, we have proved \( Y_\alpha = q^2 E_\alpha^s \Psi_\alpha + E_\alpha E_\beta = X_\alpha. \)

**Computation of** \( X_{\alpha\beta} \) **and** \( Y_{\alpha\beta}. **First**, notice that there is only one term having \( R_\alpha R_\beta \) in \( p_{\alpha\beta}^2 \), which is \( b'c' \). Now,
\[ b'c' = (R_\alpha E_\alpha^s \Psi_\alpha)(R_\beta \Psi_\beta E_\alpha) = R_\alpha R_\beta E_\alpha^s \psi_\alpha^s \Psi_\beta \]
\[ = R_\alpha R_\beta E_\alpha^s \psi_\alpha^s \Psi_\beta E_\alpha \]
\[ = R_\alpha R_\beta \psi_\alpha^s \Psi_\alpha E_\alpha \]
\[ = R_\alpha R_\beta \psi_\alpha^s \Psi_\alpha E_\alpha \] (from ??2)
\[ = R_\alpha R_\beta \psi_\alpha^s \Psi_\alpha E_\alpha \] (from ??1)
\[ = -R_\alpha R_\beta \psi_\alpha^s \Psi_\alpha E_\alpha \] (from ??3).

Therefore, we have \( Y_{\alpha\beta} = -\Psi_\alpha^s E_\alpha E_\beta. \)

**On the other side**, the only terms having the monomial \( R_\alpha R_\beta \) in \( p_{\alpha\beta}^2 \) are: \( ad, cb, cd, da \) and \( db \). We have:
\[ ad = (E_\alpha E_\beta)(R_\alpha R_\beta \psi_\alpha^s \Psi_\beta) \]
\[ = R_\alpha R_\beta (E_\alpha^s)^s (E_\beta^s)^s \Psi_\alpha^s \Psi_\beta \]
\[ = R_\alpha R_\beta E_\alpha^s \psi_\alpha^s \psi_\beta \]
\[ = R_\alpha R_\beta (E_\alpha E_\beta) \psi_\alpha^s \psi_\beta \]
\[ = R_\alpha R_\beta (E_\alpha E_\beta) \psi_\alpha^s \psi_\beta \]
\[ = -R_\alpha R_\beta E_\alpha E_\beta \psi_\beta \] (from ??3)
\[ = R_\alpha R_\beta E_\alpha E_\beta. \]
\[ da = (R_\alpha R_\beta \psi_\alpha^s \Psi_\beta)(E_\alpha E_\beta) \]
\[ = R_\alpha R_\beta \psi_\alpha^s \Psi_\beta E_\alpha E_\beta \]
\[ = -R_\alpha R_\beta \psi_\alpha^s E_\alpha E_\beta. \]
\[ cb = (R_\alpha \psi_\alpha E_\beta R_\beta)(E_\alpha^s \Psi_\beta) \]
\[ = R_\alpha R_\beta \psi_\alpha^s E_\beta E_\alpha^s \psi_\beta \]
\[ = R_\alpha R_\beta (\psi_\alpha E_\alpha)^s E_\alpha \psi_\beta \]
\[ = R_\alpha R_\beta E_\alpha^s E_\beta \]
\[ = R_\alpha R_\beta E_\alpha E_\beta \] (from ??1).
From the observation ??1 of Theorem ??, it is easy to see that \( db \) yields the coefficient
\[
E_\beta (\Psi_\alpha^s)^s \Psi_\alpha^s E_\alpha \Psi_\beta = (\Psi_\alpha^s)^s (E_\beta \Psi_\beta)^s E_\alpha \Psi_\beta
\]
\[
  = -\Psi_\alpha E_\beta E_\alpha \Psi_\beta
\]
\[
  = -\Psi_\alpha E_\beta E_\alpha \Psi_\beta
\]
\[
  = -E_\alpha E_\beta.
\]
Again from the observation ??1 of Theorem ??, we deduce that \( cd \) yield the coefficient
\[
E_\alpha^s (\Psi_\beta^s)^s (E_\beta^s)^s \Psi_\alpha^s E_\beta = (E_\alpha \Psi_\alpha^s)^s E_\beta \Psi_\beta
\]
\[
  = -E_\alpha E_\beta \Psi_\beta
\]
\[
  = -(E_\alpha \Psi_\alpha^s)(E_\beta \Psi_\beta)
\]
\[
  = -E_\alpha E_\beta.
\]
Thus, we have \( X_{\alpha \beta} = Y_{\alpha \beta} = -\Psi_\alpha^s E_\alpha E_\beta = E_\alpha E_\beta. \)

**Computation of** \( X_{\alpha 3 \alpha \beta} \) **and** \( Y_{\alpha 3 \alpha \beta} \). It is easy to see that \( d^2 \) is the only term that yields \( X_{\alpha 3 \alpha \beta} \) and this coefficient is \( \Psi_\alpha^s \Psi_\beta ((\Psi_\alpha^s \Psi_\beta)^s)^s \).

Also, it is easy to see that \( Y_{\alpha 3 \alpha \beta} \) is equal to \( \Psi_\alpha^s \Psi_\beta \Psi_\alpha \Psi_\beta^s \). By Lemma ??, it is clear that \((\Psi_\alpha^s)^s = \Psi_\alpha \) and \((\Psi_\beta)^s = \Psi_\beta \). Hence, we have \( X_{\alpha 3 \alpha \beta} = Y_{\alpha 3 \alpha \beta}. \)

4. **Case** \( G_2 \)

Let \( \Pi = \{\alpha, \beta\} \) be a system of positive simple root of \( \Phi \). Let us put
\[
G_2: \quad \alpha \quad \beta
\]
the Dynkin diagram. Let \( W \) denote the Weyl group of \( G_2 \). Let \( s \) (respectively \( s' \)) denote the reflection corresponding to the root \( \alpha \) (respectively \( \beta \)). We have, \( \langle \alpha, \beta \rangle = -1 \) and \( \langle \beta, \alpha \rangle = -3 \). Hence, we have
\[
(3) \quad s(\beta) = 3\alpha + \beta, \quad s'(\alpha) = \alpha + \beta.
\]
In the proof of the braid relation of type \( G_2 \) (Lemma ??), we will use the following Lemma.

**Lemma 6.** We have,
\[
(??1) \quad ((E_\alpha^s)^s)^s = (E_\alpha^s)^s, \quad ((E_\beta^s)^s)^s = (E_\beta^s)^s
\]
\[
(??2) \quad ((\Psi_\alpha^s)^s)^s = (\Psi_\alpha^s)^s, \quad ((\Psi_\beta^s)^s)^s = (\Psi_\beta^s)^s
\]
\[
(??3) \quad E_\alpha E_\alpha = E_\alpha E_\beta, \quad E_\alpha E_\beta E_\beta = (q - 1) E_\alpha E_\beta
\]
\[
(??4) \quad H_\alpha(-1)E_{3\alpha + 2\beta} = E_{3\alpha + 2\beta}, \quad H_\alpha + \beta(-1)E_\beta E_{3\alpha + 2\beta} = E_\beta E_{3\alpha + 2\beta}
\]
\[
(??5) \quad E_{\alpha + \beta} E_{3\alpha + 2\beta} = E_\alpha E_\beta, \quad E_{\alpha + \beta} E_{3\alpha + 2\beta} = E_\beta E_{3\alpha + 2\beta}
\]
\[
(??6) \quad ((\Psi_\alpha^s \Psi_\beta) (\Psi_\alpha^s \Psi_\beta)^s (E_\beta^s)^s)^s = (\Psi_\alpha^s \Psi_\beta) (\Psi_\alpha^s \Psi_\beta)^s (E_\beta^s)^s
\]
\[
(??7) \quad (E_\alpha E_\beta)^w = E_\alpha E_\beta, \quad \Psi_\alpha^w E_\alpha E_\beta = \Psi_\alpha^w E_\alpha E_\beta = -E_\alpha E_\beta, \quad w \in W
\]
\[
(??8) \quad H_\alpha(-1)E_\alpha^s = H_\beta(-1)E_\alpha^s
\]
\[
(??9) \quad \Psi_\alpha^s E_\beta + \Psi_\alpha \Psi_\beta^s E_\beta = 0.
\]
Proof. We have \( s(s'(\alpha)) = s(\alpha + \beta) = -\alpha + (3\alpha + \beta) = 2\alpha + \beta \). Therefore, we have \( s'(s(s'(\alpha))) = s'(2\alpha + \beta) = 2(\alpha + \beta) - \beta = 2\alpha + \beta = s(s'(\alpha)) \). Using a similar argument, it is easy to see that \( s(s'(s'(\beta))) = s'(s'(\beta)) \). These observations prove ???.1 and ???.2.

Now, we will prove ???.3. We have \( s'(\alpha) = \alpha + \beta \), and so we have \( E_{\alpha}' = \sum_{r \in k^\times} H_\alpha(r)H_\beta(r) \).

Then

\[
E_{\alpha}' E_{\alpha}' = \sum_{t \in k^\times} H_\alpha(t) \sum_{r \in k^\times} H_\alpha(r)H_\beta(r) = \sum_{t,r \in k^\times} H_\alpha(tr)H_\beta(r) = \sum t, r \in k^\times H_\alpha(t)H_\beta(r) = E_{\alpha}E_{\beta}
\]

In the above assertion, we use the fact that \((t, r) \mapsto (rt, r)\) is an automorphism of \( k^\times \times k^\times \).

Proof of the other assertion of ???.3 is similar to this proof.

We now prove ???.4.

We have

\[
H_\alpha(-1)E_{3\alpha+2\beta} = \sum_{t \in k^\times} H_\alpha(-t^3)H_\beta(t^2) = \sum_{t \in k^\times} H_\alpha((-t)^3)H_\beta((-t)^2) = E_{3\alpha+2\beta}
\]

We note that here, we use the fact that \( t \mapsto -t \) is a bijection of \( k^\times \) onto itself.

Once again using this fact, we have \( H_\beta E_\beta = E_\beta \). Therefore, we have

\[
H_{\alpha+\beta}(-1)E_{\beta}E_{3\alpha+2\beta} = (H_{\beta}(-1)E_{\beta})(H_{\alpha}(-1)E_{3\alpha+2\beta}) = E_{\beta}E_{3\alpha+2\beta}
\]

This proves ???.4.

We now prove ???.5. We have

\[
E_{\alpha+\beta}E_{3\alpha+2\beta} = \sum_{(t,s) \in k^\times \times k^\times} H_\alpha(ts^3)H_\beta(ts^2) = E_{\alpha}E_{\beta}
\]

Here, we use the fact that \((t, s) \mapsto (ts^3, ts^2)\) is an automorphism of the group \( k^\times \times k^\times \).

Similarly, the other assertion \( E_{3\alpha+2\beta}E_{3\alpha+2\beta} = E_{3\beta}E_{3\alpha+2\beta} \) of ???.5 follows from the fact that \((t, s) \mapsto (ts, s^{-1})\) is an automorphism of \( k^\times \times k^\times \).

We now prove the assertion ???.6.

We first compute \((\Psi_{\alpha}\Psi_{\alpha})'(\Psi_{\beta}\Psi_{\beta}')E_{3\alpha+2\beta}\).

We have \( \Psi_{\alpha}\Psi_{\alpha} = qH_\alpha(-1) - E_\alpha \) and \( s'(\alpha) = \alpha + \beta \). Therefore, we have \( (\Psi_{\alpha}\Psi_{\alpha})' = qH_{\alpha+\beta}(-1) - E_{\alpha+\beta} \).

We also have \( \Psi_{\beta}\Psi_{\beta}' = qH_\beta(-1) - E_\beta \).

Thus, we get

\[
((\Psi_{\alpha}\Psi_{\alpha})'(\Psi_{\beta}\Psi_{\beta}')((E_{\beta})'(s'))s = (qH_{\alpha+\beta}(-1) - E_{\alpha+\beta})(qH_\beta(-1) - E_\beta)E_{3\alpha+2\beta}
\]

Then using ???.4, we have

\[
((\Psi_{\alpha}\Psi_{\alpha}')'(\Psi_{\beta}\Psi_{\beta}')(E_{\beta})'(s'))s = q^2E_{3\alpha+2\beta} - qE_{\beta}E_{3\alpha+2\beta} - qE_\alpha E_\beta + (q - 1)E_\alpha E_\beta
\]

We now prove that \( q^2E_{3\alpha+2\beta} - qE_\beta E_{3\alpha+2\beta} - E_\alpha E_\beta \) is \( s \)-invariant.

To prove this, we prove each of the summand is \( s \)-invariant.
First, we have \( s(3\alpha + 2\beta) = -3\alpha + 2(3\alpha + \beta) = 3\alpha + 2\beta \) and so we have \( E^{3\alpha+2\beta}_{3\alpha+2\beta} \).

Secondly, we have

\[
(E_{\beta}E_{3\alpha+2\beta})^s = E_{3\alpha+\beta}E_{3\alpha+2\beta} = E_{\beta}E_{3\alpha+2\beta}. \quad \text{(from ??5)}
\]

Thirdly, we have

\[
(E_{\alpha}E_{\beta})^s = E_{-\alpha}E_{3\alpha+\beta} = \sum_{(t,s) \in k^n \times k^n} H_{\alpha}(t^{-1}s^3)H_{\beta}(s)
\]

\[
= E_{\alpha}E_{\beta}.
\]

Here, we use the fact that \((t, s) \mapsto (t^{-1}s^3, s)\) is an automorphism of the group \(k^n \times k^n\).

Thus, we have proved ??6.

We now prove ??7.

First, we prove \((E_{\alpha}E_{\beta})^w = E_{\alpha}E_{\beta}\) for any \(w \in W\).

Since the Weyl group of \(G_2\) is generated by \(s\) and \(s'\), it is sufficient to prove that \((E_{\alpha}E_{\beta})^s = E_{\alpha}E_{\beta} = (E_{\alpha}E_{\beta})^{s'}\).

For a proof of the first equality, we have

\[
(E_{\alpha}E_{\beta})^s = E_{-\alpha}E_{3\alpha+\beta} = \sum_{(t,s) \in k^n \times k^n} H_{\alpha}(t^{-1}s^3)H_{\beta}(s)
\]

\[
= \sum_{(t,s) \in k^n \times k^n} H_{\alpha}(t)H_{\alpha}(s)
\]

\[
= E_{\alpha}E_{\beta}.
\]

We note that in this proof, we use the fact that the map \((t, s) \mapsto (t^{-1}s^3, s)\) is an automorphism of \(k^n \times k^n\).

The proof of the second equality follows from the fact that the map \((t, s) \mapsto (t, ts^{-1})\) is an automorphism of \(k^n \times k^n\).

We now prove that \(\Psi_w E_{\alpha}E_{\beta} = -E_{\alpha}E_{\beta}\) for any \(w \in W\).

Since \((E_{\alpha}E_{\beta})^w = E_{\alpha}E_{\beta}\), we have

\[
\Psi_w E_{\alpha}E_{\beta} = (\Psi_w E_{\alpha}E_{\beta})^w = (-E_{\alpha}E_{\beta})^w
\]

\[
= -E_{\alpha}E_{\beta}.
\]

We now prove ??8.

We have

\[
H_{\alpha}(-1)E_{s'}^\alpha = H_{\alpha}(-1)E_{\alpha+\beta}
\]

\[
= H_{\alpha}(-1)(\sum_{t \in k^n} H_{\alpha}(t)H_{\beta}(t))
\]

\[
= \sum_{t \in k^n} H_{\alpha}(-t)H_{\beta}(t)
\]

\[
= \sum_{t \in k^n} H_{\alpha}(t)H_{\beta}(-t)
\]

\[
= H_{\beta}(-1)(\sum_{t \in k^n} H_{\alpha}(t)H_{\beta}(t))
\]

\[
= H_{\beta}(-1)E_{s'}^\alpha.
\]
We now prove ??9.
We have $s'(\alpha) = \alpha + \beta$, and so we have

$$
\sum_{(t,s) \in k^\times \times k^\times} \Psi(t) H_\alpha(t) H_\beta(ts) = \Psi_\alpha E_\beta.
$$

Here, we use the fact that the map $(t,s) \mapsto (t,ts)$ is an automorphism of $k^\times \times k^\times$.

On the other hand, we have

$$
\Psi_\alpha (\Psi_\beta^s E_\beta) = \Psi_\alpha (-E_\beta) = -\Psi_\alpha E_\beta.
$$

Hence, we have

$$
\Psi_\alpha^s E_\beta + \Psi_\alpha \Psi_\beta^s E_\beta = (\Psi_\alpha - \Psi_\alpha) E_\beta = 0.
$$

Thus, we have proved ??9. \hfill \Box

**Lemma 7.** We have $L_\alpha L_\beta L_\alpha L_\beta L_\alpha L_\beta = L_\beta L_\alpha L_\beta L_\alpha L_\beta L_\alpha$.

**Proof.** Set $p_{\alpha \beta} = q^3 L_\alpha L_\beta L_\alpha$, and $p_{\beta \alpha} = q^3 L_\beta L_\alpha L_\beta$. Then, one can re-write the braid relation of the lemma as

$$
(p_{\alpha \beta})^2 = p_{\beta \alpha} p_{\alpha \beta}.
$$

According to Proposition ??, Lemma ?? and ??, we obtain

$$
p_{\alpha \beta} = (q - 1) E_\alpha E_\beta + q H_\alpha(-1) \Psi_\alpha \Psi_\beta^s E_\beta - R_\alpha E_\alpha E_\beta + R_\beta E_\alpha E_\beta \Psi_\beta
$$

and

$$
p_{\beta \alpha} = (q - 1) E_\alpha E_\beta + q H_\beta(-1) \Psi_\beta \Psi_\alpha^s E_\alpha - R_\beta E_\beta E_\alpha + R_\alpha E_\beta E_\alpha \Psi_\alpha
$$

We are now going to compare the coefficients of the monomials on $R_\alpha$ and $R_\beta$ obtained in both sides of equation ??1. For any word $\gamma$ in $\alpha$ and $\beta$, let $X_\gamma$ (resp. $Y_\gamma$) be the coefficient of $R_\gamma$ in the expression of L.H.S (resp. R.H.S) of ??1.

To prove the Lemma, it is sufficient to prove that $X_\gamma = Y_\gamma$ for all words $\gamma$ in $\alpha$ and $\beta$.

**Computation of $X_{\alpha_3 \alpha_3 \beta_3 \beta_3}$ and $Y_{\alpha_3 \alpha_3 \beta_3 \beta_3}$.** On the left in the product of ??1 the monomial $R_\alpha R_\beta R_\alpha R_\beta R_\alpha R_\beta$ appears only in the multiplication $gg'$, and then the coefficient $X_{\alpha_3 \alpha_3 \beta_3 \beta_3}$ of this monomial is $\Psi_\alpha^s (\Psi_\beta^s \Psi_\beta) \Psi_\alpha$. The coefficient $Y_{\alpha_3 \alpha_3 \beta_3 \beta_3}$ of $R_\alpha R_\beta R_\alpha R_\beta R_\alpha R_\beta$ on the right of ??1. Notice that we have $gg' = g'g$.
Computation of $X_{\alpha_{\beta_{\beta_{\alpha_{\beta}}}}}$ and $Y_{\alpha_{\beta_{\beta_{\alpha_{\beta}}}}}$. It is to check that the monomial $R_{\alpha_{\beta_{\beta}}R_{\alpha_{\beta}}R_{\alpha_{\beta}}}$ occurs only in the product $ge'$ on the left of ??,1, and only in the product $f'g$ on the right of ??,1. Now, the coefficient $X_{\alpha_{\beta_{\beta_{\alpha_{\beta}}}}}$ is

$$X_{\alpha_{\beta_{\beta_{\alpha_{\beta}}}}} = ((E_{\beta_{\alpha}}E_{\beta_{\alpha}})^{s_{\alpha}})^{s_{\beta}}(\Psi_{\alpha_{\beta}}\Psi_{\beta_{\alpha}}E_{\beta_{\alpha}})$$

$$= (((E_{\beta_{\alpha}}E_{\beta_{\alpha}})^{s_{\alpha}})^{s_{\beta}}(\Psi_{\alpha_{\beta}}\Psi_{\beta_{\alpha}}E_{\beta_{\alpha}})$$

$$= Y_{\alpha_{\beta_{\beta_{\alpha_{\beta}}}}}.$$

(notice that from ??,1, $((E_{\beta_{\alpha}})^{s_{\alpha}})^{s_{\beta}} = E_{\beta_{\alpha}}$).

Computation of $X_{\alpha_{\beta_{\beta_{\alpha_{\beta}}}}}$ and $Y_{\alpha_{\beta_{\beta_{\alpha_{\beta}}}}}$: On the right of the equation ??,1 the monomial $R_{\alpha_{\beta_{\beta}}R_{\alpha_{\beta}}R_{\alpha_{\beta}}}$ appears only in the product $f'g$. We now compute this coefficient.

We have $s'(s'(s(\beta))) = s(\beta)$ and so $((E_{\beta_{\alpha}})^{s_{\alpha}})^{s_{\beta}} = E_{\alpha_{\beta}}$. Using this observation, we have:

$$Y_{\alpha_{\beta_{\beta_{\alpha_{\beta}}}}} = ((E_{\beta_{\alpha}}E_{\beta_{\alpha}})^{s_{\alpha}})^{s_{\beta}}(\Psi_{\alpha_{\beta}}\Psi_{\beta_{\alpha}}E_{\beta_{\alpha}})$$

$$= ((E_{\beta_{\alpha}}E_{\beta_{\alpha}})^{s_{\alpha}})^{s_{\beta}}(\Psi_{\alpha_{\beta}}\Psi_{\beta_{\alpha}}E_{\beta_{\alpha}})$$

$$= (E_{\beta_{\alpha}}E_{\beta_{\alpha}})(\Psi_{\alpha_{\beta}}\Psi_{\beta_{\alpha}})(\Psi_{\alpha_{\beta}}\Psi_{\beta_{\alpha}})^{s_{\beta}}$$

$$= (-1)^{A_{\alpha}}E_{\alpha_{\beta}}$$

Hence, we have

(??,2) $Y_{\alpha_{\beta_{\beta_{\alpha_{\beta}}}}} = E_{\alpha_{\beta_{\beta}}}.

On the other side, the terms on the left of the equation ??,1. that contain the monomial $R_{\alpha_{\beta_{\beta}}R_{\alpha_{\beta}}R_{\alpha_{\beta}}}$ are the products: $cg'$, $e'f'$, $ge'$, and $gf$.

We now prove that $cg'$ yields the coefficient $E_{\alpha_{\beta_{\beta}}}$. We have

$$cg' = -R_{\alpha_{\beta_{\beta}}R_{\alpha_{\beta}}R_{\alpha_{\beta}}}(E_{\alpha_{\beta}})^{s_{\beta}}(\Psi_{\alpha_{\beta}})^{s_{\beta}}$$

$$= -R_{\alpha_{\beta_{\beta}}R_{\alpha_{\beta}}R_{\alpha_{\beta}}}((-1)^{A_{\alpha}}E_{\beta_{\beta}}).$$

Therefore, the coefficient of $R_{\alpha_{\beta_{\beta}}R_{\alpha_{\beta}}R_{\alpha_{\beta}}}$ in the expression of $cg'$ is

(??,3) $-E_{\alpha_{\beta}}E_{\beta_{\beta}}.$

We now prove that $e'f'$ yields the coefficient $E_{\alpha_{\beta}}E_{\beta_{\beta}}$. We have

$$e'f' = R_{\alpha_{\beta_{\beta}}R_{\alpha_{\beta}}R_{\alpha_{\beta}}}((\Psi_{\alpha_{\beta}}\Psi_{\beta_{\alpha}})^{s_{\beta}}(E_{\beta_{\alpha}})^{s_{\beta}}(\Psi_{\alpha_{\beta}})^{s_{\beta}}$$

$$= R_{\alpha_{\beta_{\beta}}R_{\alpha_{\beta}}R_{\alpha_{\beta}}}((-1)^{A_{\alpha}}E_{\beta_{\beta}})(\Psi_{\alpha_{\beta}})^{s_{\beta}}(\Psi_{\beta_{\alpha}})^{s_{\beta}}$$

$$= R_{\alpha_{\beta_{\beta}}R_{\alpha_{\beta}}R_{\alpha_{\beta}}}((-1)^{A_{\alpha}}E_{\beta_{\beta}}).$$

Therefore, the coefficient yielded by $e'f'$ is

(??,4) $E_{\alpha_{\beta}}E_{\beta_{\beta}}.$

By using ??,7, it is easy to see that $ge'$ yields the coefficient

(??,5) $(-1)^{A_{\alpha}}E_{\alpha_{\beta}}E_{\beta_{\beta}} = E_{\alpha_{\beta}}E_{\beta_{\beta}}.$

Now, we compute the coefficient yielded by $eg'$. We have

$$eg' = (R_{\alpha_{\beta_{\beta}}\Psi_{\alpha_{\beta}}})(R_{\beta_{\beta}}R_{\alpha_{\beta}}R_{\beta_{\beta}}(\Psi_{\beta_{\alpha}})^{s_{\beta}}(\Psi_{\beta_{\alpha}})^{s_{\beta}}(\Psi_{\beta_{\alpha}})^{s_{\beta}}$$

$$= R_{\alpha_{\beta_{\beta}}R_{\alpha_{\beta}}R_{\beta_{\beta}}}(\Psi_{\beta_{\alpha}})^{s_{\beta}}(\Psi_{\beta_{\alpha}})^{s_{\beta}}(\Psi_{\beta_{\alpha}})^{s_{\beta}}.$$
Now, notice that \(((\Psi^s_\alpha \Psi^s_\beta E_\alpha)^s')^r(\Psi^s_\alpha \Psi^s_\beta)^s' E_\alpha \Psi^s_\beta = (\Psi^s_\alpha \Psi^s_\beta \Psi^s_\beta)^s' E_\alpha \Psi^s_\beta\). Then using ??1, we get

\[ R_\alpha R^3_\beta R_\alpha R_\beta = R_\alpha (q H_\beta (-1) + R_\beta E_\beta) R_\alpha R_\beta \]

\[ = R^2_\beta R_\beta ((q H_\beta (-1)) E^s + R_\alpha R_\beta R_\alpha R_\beta ((E_\beta)^s')^s'. \]

But, we are interested in computing only the coefficient of \(R_\alpha R_\beta R_\alpha R_\beta\). From the above computations, we first compute the \(E\)'s (without \(\Psi\)'s) in coefficient of \(R_\alpha R_\beta R_\alpha R_\beta\) in the product \(eg'\)

\[ ((E_\beta)^s')^s' (\Psi^s_\alpha \Psi^s_\beta)^s' (\Psi^s_\beta)^s' \Psi^s_\beta = (-1)^5 E_\alpha E_\beta \quad \text{(from ??7)} \]

\[ = -E_\alpha E_\beta. \]

Therefore, \(eg'\) yields the coefficient

( ??6) \(-E_\alpha E_\beta.\)

By a similar computation, we can see that the coefficient of \(R_\alpha R_\beta R_\alpha R_\beta\) in the expression of \(gf'\) is equal to

( ??7) \((-1)^5 E_\alpha E_\beta = -E_\alpha E_\beta.\)

Summing up these five coefficients (from ??3 to ??7), we have

( ??8) \[ X_{\alpha \beta \alpha \beta} = 3 E_\alpha E_\beta - 2 E_\alpha E_\beta = E_\alpha E_\beta. \]

From the observations ??2 and ??8, we have \(X_{\alpha \beta \alpha \beta} = Y_{\alpha \beta \alpha \beta}.\)

**Computation of \(X_{\alpha \beta \alpha \beta}\) and \(Y_{\alpha \beta \alpha \beta}\).** In the left hand side of ??1, the products that contain the monomials \(R_\alpha R_\beta R_\alpha\) are: \(ce', ed', ee', ga', gb'\) and \(gd'\).

Since \(E_\alpha E_\beta\) is occurring in \(c\), by using ??7, it is easy to see that the coefficient of \(R_\alpha R_\beta R_\alpha\) in the expression of the product \(ce'\) is

( ??9) \[ -(1)^3 (q - 1) E_\alpha E_\beta = -(q - 1) E_\alpha E_\beta. \]

Since \(E_\alpha E_\beta\) is also occurring in \(a'\), one can check that the coefficient of \(R_\alpha R_\beta R_\alpha\) in the expression of \(ga'\) to be equal to

( ??10) \[ -(1)^3 (q - 1) E_\alpha E_\beta = -(q - 1) E_\alpha E_\beta. \]

We now compute \(ed'\). We have

\[ ed' = R_\alpha R_\beta R_\alpha (\Psi^s_\alpha \Psi^s_\beta)^s' (E_\alpha E_\beta)^s' E_\beta \Psi^s_\alpha \]

\[ = R_\alpha R_\beta R_\alpha (-1)^5 E_\alpha E^2_\beta \quad \text{(from ??7)} \]

\[ = -R_\alpha R_\beta R_\alpha (q - 1) E_\alpha E_\beta. \]

Therefore, \(ed'\) yields the coefficient:

( ??11) \[-(q - 1) E_\alpha E_\beta.\]

We now compute the coefficient of \(R_\alpha R_\beta R_\alpha\) in the expression of \(ee'\). We have

\[ ee' = (R_\alpha R_\beta \Psi^s_\alpha \Psi^s_\beta E_\alpha) (R_\beta R_\alpha \Psi^s_\beta \Psi^s_\alpha E_\beta) \]

\[ = R^2_\beta R_\alpha (\Psi^s_\alpha \Psi^s_\beta E_\alpha)^s' \Psi^s_\beta \Psi^s_\alpha E_\beta. \]

We also have

\[ R_\alpha R^2_\beta R_\alpha = q R_\alpha H_\beta (-1) R_\alpha E_\alpha E_\beta + R_\alpha R_\beta R_\alpha (E_\beta)^s' E_\alpha E_\beta. \quad \text{(from ??1)} \]

We are interested in computing only the coefficient of \(R_\alpha R_\beta R_\alpha\) in the expression of \(ee'\). We first compute only the product of \(E\)'s (without \(\Psi\)'s).
This coefficient is equal to
\[(E^\alpha_\alpha)^s(E_\beta^E_\beta^s) = ((E_\alpha E_\beta)^s)^{s'} E_\beta^s,\]
since \(E^s_\beta = E_\beta.

We now compute the coefficient of \(R_\alpha R_\beta R_\alpha\) (together with the product of \(\Psi\)'s) in the expression of \(ee'\).

The coefficient of \(R_\alpha R_\beta R_\alpha\) in the expression of \(ee'\) is
\[((\Psi_\alpha^s \Psi_\beta^{s'})^s \Psi_\alpha ((E_\alpha E_\beta)^s)^s E_\beta\] = \((-1)^4 E_\alpha E_\beta^3\) (from \ref{??7})
\[= (q-1)E_\alpha E_\beta.>\) (from 1)

Hence \(ee'\) yields the coefficient
\ref{??12} \( (q-1)E_\alpha E_\beta.\)

By a similar computation, one can check that the coefficient of \(R_\alpha R_\beta R_\alpha\) in the expression of \(gd'\) is
\ref{??13} \( (q-1)E_\alpha E_\beta.\)

We now compute \(gb'\).

It is easy to see that \(gb'\) yields the coefficient
\ref{??14} \( q((\Psi_\alpha^s \Psi_\beta^{s'})^s \Psi_\alpha \Psi_\beta (H_\beta (-1)E_\beta^s)\).

Summing up all these coefficients (using the observations from \ref{??9} to \ref{??14}): it is easy to see that the sum of the coefficients coming from \(ee'\), \(ed'\) and \(gd'\) is \(-3(q-1)E_\alpha E_\beta\) and that the sum of the coefficients coming from \(ee'\) and \(gd'\) is \(2(q-1)E_\alpha E_\beta.\)

Therefore, we have
\[
X_{\alpha\beta\alpha} = -(3(q-1)E_\alpha E_\beta) + 2(q-1)E_\alpha E_\beta + q(H_\beta(-1)E_\alpha^s)((\Psi_\alpha^s \Psi_\beta^{s'})^s \Psi_\alpha \Psi_\beta (H_\beta E_\beta^s).

Thus, we have
\ref{??15} \( X_{\alpha\beta\alpha} = -(q-1)E_\alpha E_\beta + q(H_\beta(-1)E_\alpha^s)((\Psi_\alpha^s \Psi_\beta^{s'})^s \Psi_\alpha \Psi_\beta (H_\beta E_\beta^s).

Now, we will compute \(Y_{\alpha\beta\alpha}.\) The products on the right that contain the monomials \(R_\alpha R_\beta R_\alpha\) are: \(a'g, b'g, d'f, d'g, f'c,\) and \(f'f.\)

Computations are similar to the computations of \(X_{\alpha\beta\alpha}.\)

Since \(d'f\) contains \(E_\alpha E_\beta\) as a factor, by using \ref{??7}, it is easy to see that the coefficient of \(R_\alpha R_\beta R_\alpha\) in the expression of \(a'g\) is
\ref{??16} \( (-1)^3 (q-1)E_\alpha E_\beta = -(q-1)E_\alpha E_\beta.\)

By the same argument, it is easy to see that the coefficient of \(R_\alpha R_\beta R_\alpha\) in the expression of \(f'c\) is
\ref{??17} \( -(1)^2 (q-1)E_\alpha E_\beta = -(q-1)E_\alpha E_\beta.\)

(Here, we use the fact that \(E_\alpha E_\beta\) is a factor of \(c).\)

We now compute \(d'f.\)

We have
\[
d'f = R_\alpha R_\beta R_\alpha (E_\beta E_\beta^s \Psi_\alpha)^s (E_\alpha^s \Psi_\beta)^s \Psi_\alpha
\[= R_\alpha R_\beta R_\alpha (E_\alpha E_\beta)^s ((E_\beta^s)^s (\Psi_\alpha^s)^s \Psi_\alpha^s \Psi_\beta^s \Psi_\alpha.

Therefore, by using \ref{??7}, it is easy to see that the coefficient of \(R_\alpha R_\beta R_\alpha\) in the expression of \(d'f\) is
\ref{??18} \( (-1)^3 (q-1)E_\alpha E_\beta = -(q-1)E_\alpha E_\beta.\)

By a similar computation in \(ee',\) using \ref{??1}, and \ref{??7}, one can check that the coefficient of \(R_\alpha R_\beta R_\alpha\) in the expression of \(f'f\) is equal to
\ref{??19} \( (-1)^4 (q-1)E_\alpha E_\beta = (q-1)E_\alpha E_\beta.\)
By a similar computation in $gd'$, using ??1 and ??7, one can check that the coefficient of $R_\alpha R_\beta R_\alpha$ in the expression of $d'g$ is equal to

$$(-1)^4(q-1)E_\alpha E_\beta = (q-1)E_\alpha E_\beta. \quad (??20)$$

We now compute the coefficient of $R_\alpha R_\beta R_\alpha$ in the expression of $b'g$.

Using Theorem ??, it is easy to see that the coefficient of $R_\alpha R_\beta R_\alpha$ in the expression of the product $b'g$ is:

$$qH_\alpha(-1)^3H_\beta(-1)^2E_{\alpha}(\Psi_\beta)s'((\Psi_\beta')s')^s(\Psi_\alpha s')(\Psi_\beta s')^s\Psi_\alpha = qH_\alpha(-1)^3E_{\alpha}((\Psi_\beta s')^s)\Psi_{\alpha}.$$

Then, from ??8 we obtain that the coefficient of $R_\alpha R_\beta R_\alpha$ in the expression of the product $b'g$ is

$$qH_\beta(-1)E_{\alpha}((\Psi_\beta s')^s)\Psi_{\alpha}. \quad (??21)$$

Summing up these coefficients (using observations ??16 to ??21), we have

$$Y_{\alpha\beta\alpha} = -(q-1)E_\alpha E_\beta + qH_\alpha(-1)(((\Psi_\beta s')^s)^s(\Psi_\alpha s')(\Psi_{\beta} s')\Psi_{\alpha}). \quad (??22)$$

To prove $X_{\alpha\beta\alpha} = Y_{\alpha\beta\alpha}$, from the observations ??15 and ??22, it is sufficient to prove that

$$(\Psi_\beta s')^s(((\Psi_\beta s')^s)^s)E_{\alpha} = \Psi_{\beta}s(\Psi_{\beta} s')E_{\alpha}. \quad (??23)$$

We now prove this assertion.

By a similar proof of ??1, it is easy to see that

$$s(s'(s'(\beta))) = -(3\alpha + 2\beta) = s(s'(\beta)).$$

Hence, we have

$$((((\Psi_\beta s')^s)^s)^s(\Psi_\beta s')^s = (((\Psi_\beta s')^s)^s)^s = (((\Psi_\beta s')^s)^s)^s."$$

Using this and fact that $H_\alpha(-1)^2 = H_\beta(-1)^2 = 1$, we have

$$(((\Psi_\beta s')^s)^s)^s(\Psi_\beta s')^s E_{\alpha} = (qH_\alpha(-1)E_\alpha s' - (E_\beta s')E_{\alpha})^s' \quad (??8)$$

We now prove this assertion.

**Computation of $X_{\alpha\beta\alpha}$ and $Y_{\alpha\beta\alpha}$.** We first compute $X_{\alpha\beta\alpha}$. On the left of ??1, the terms containing the monomials $R_\alpha$ are: $(a + b)d', c(a' + b'), cd', de', ec', cde', and g$. Now, we deduce that $de'$ yields $q(g-1)E_\alpha E_\beta$. In fact,

$$de' = (R_\beta E_\alpha E_{\alpha}' \Psi_\beta)(R_\beta R_\alpha \Psi_\beta \Psi_\alpha E_\beta) = R^2_\beta R_\alpha (E_{\alpha}' s')^s (E_{\alpha}' s')^s (E_\alpha s')(\Psi_\beta s')^s \Psi_{\beta} \Psi_\alpha E_\beta \quad (since (s')^2 = 1 and E_{\alpha}^s = E_\alpha).$$
Now, using ??1, it is easy to see that \( de' \) yields the coefficient \( qH_\beta(-1)(E_\alpha^\prime)^sE_\alpha(\Psi_\alpha^s)^s\Psi_\beta^e\Psi_\alpha E_\beta \).

We also have \( (E_\alpha^\prime)^sE_\alpha E_\beta = (q - 1)E_\alpha E_\beta \). Therefore, using Lemma ?? we deduce that \( de' \) yields the coefficient

\[
(-1)^3q(q - 1)E_\alpha E_\beta = -q(q - 1)E_\alpha E_\beta. 
\]

In the same way, one can check that \( ce' \) yields the same coefficient

\[
(-1)^4q(q - 1)E_\alpha E_\beta = q(q - 1)E_\alpha E_\beta. 
\]

Since \( E_\alpha E_\beta \) is a factor of \( c' \), using ??7 and the fact that \( R_\beta^2 = qH_\beta(-1) + R_\beta E_\beta \), it is easy to see that the coefficient of \( R_\alpha \) in the expression of \( de' \) is

\[
(-1)^3(q(q - 1))E_\alpha E_\beta = -q(q - 1)E_\alpha E_\beta. 
\]

We now compute the coefficient of \( R_\alpha \) in the expression of \( cd' \). We have

\[
\begin{align*}
cd' &= (R_\alpha E_\alpha)(R_\alpha E_\beta E_\alpha^\prime \Psi_\alpha) \\
 &= R_\alpha^2 E_\alpha E_\beta E_\beta^\prime \Psi_\alpha \\
 &= (qH_\alpha(-1) + R_\alpha E_\alpha E_\beta E_\alpha^\prime \Psi_\alpha).
\end{align*}
\]

Therefore, the coefficient of \( R_\alpha \) in the expression of \( cd' \) is \( E_\alpha E_\beta E_\alpha^\prime \Psi_\alpha \), which turns out to be equal to

\[
(-1)^2(q - 1)^3E_\alpha E_\beta = (q - 1)^3E_\alpha E_\beta. 
\]

(by using ??7).

We now compute the coefficient of \( R_\alpha \) in the expression of \( c(a' + b') \).

Since \( E_\alpha E_\beta \) is a factor of \( c \), using ??7, it is easy to see that the coefficient of \( R_\alpha \) in the expression of \( ad' \) yields \( -(q - 1)^3E_\alpha \). Now, again using ??7, it is easy to see that the coefficient of \( R_\alpha \) in the expression of \( cd' \) is \( -(1)^3(q - 1)E_\alpha E_\beta = -q(q - 1)E_\alpha E_\beta \).

Therefore \( c(a' + b') \) yields the coefficient

\[
-(q - 1)^3E_\alpha E_\beta = -(q - 1)(q^2 - q + 1)E_\alpha E_\beta. 
\]

We now compute the coefficient of \( R_\alpha \) in the expression of \((a + b)d' \). Since \( E_\alpha E_\beta \) is a factor of \( a \), using ??7, it is easy to see that \( -(q - 1)^3E_\alpha E_\beta \) is the coefficient of \( R_\alpha \) in the expression of \( ad' \).

On the other hand, we have

\[
\begin{align*}
bd' &= (qH_\alpha(-1)\Psi_\alpha \Psi_\alpha^s E_\beta)(R_\alpha E_\beta E_\alpha^\prime \Psi_\alpha) \\
 &= qR_\alpha H_\alpha(-1)\Psi_\alpha \Psi_\alpha^s E_\beta E_\alpha^\prime \Psi_\alpha \quad \text{(since } s^2 = 1 \text{ and } H_\alpha(-1)^s = H_\alpha(-1)) \\
 &= q(q - 1)R_\alpha H_\alpha(-1)\Psi_\alpha \Psi_\alpha^s E_\beta E_\alpha^\prime \Psi_\alpha.
\end{align*}
\]

Thus, \((a + b)d' \) yields

\[
-(q - 1)^3E_\alpha E_\beta + q(q - 1)H_\alpha(-1)\Psi_\alpha \Psi_\alpha^s E_\beta E_\alpha^\prime \Psi_\alpha. 
\]

We now compute the coefficient of \( R_\alpha \) in the expression of \( gf' \). We have

\[
\begin{align*}
gf' &= (R_\alpha R_\beta R_\alpha(\Psi_\alpha^s \Psi_\beta)^s \Psi_\alpha)(R_\alpha R_\beta(E_\beta^\prime \Psi_\alpha)^s \Psi_\beta) \\
 &= R_\alpha R_\beta(R_\alpha R_\beta \Psi_\alpha^s \Psi_\beta)^s(\Psi_\alpha)^s(E_\beta^\prime \Psi_\alpha)^s \Psi_\beta \\
 &= \frac{q}{R_\alpha R_\beta}qH_\alpha(-1) + R_\alpha E_\alpha \Psi_\alpha R_\beta E_\beta \Psi_\beta. \\

\text{Using } R_\alpha^2 &= qH_\alpha(-1) + R_\alpha E_\alpha, \text{ we get} \\
gf' &= qR_\alpha R_\beta H_\alpha(-1)^s(\Psi_\alpha^s \Psi_\beta)^s(\Psi_\alpha)^s(E_\beta^\prime \Psi_\alpha)^s \Psi_\beta + R_\alpha R_\beta R_\alpha R_\beta E_\alpha^s(\Psi_\alpha^s \Psi_\beta)^s(\Psi_\alpha)^s(E_\beta^\prime \Psi_\alpha)^s \Psi_\beta. \\

\text{Now, using } R_\beta^2 &= qH_\beta(-1) + R_\beta E_\beta, \text{ and using the fact that} \\
H_\beta(-1)(H_\alpha(-1))^s = H_\alpha(-1)H_\beta(-1)^2 = H_\alpha(-1), \\
\text{one can see that the coefficient of } R_\alpha \text{ in the expression of } gf' \text{ is} \\
(??29) &= q^2 H_\alpha(-1)\Psi_\alpha \Psi_\beta^s(\Psi_\alpha)^s(E_\beta^\prime \Psi_\alpha)^s \Psi_\beta 
\]
Therefore, using the observations from ??23 to ??29, we have

\[
X_\alpha = -(q-1)(q^2 + 1)E_\alpha E_\beta + q(q-1)H_\alpha(-1)\Psi_\alpha \Psi_\alpha^s E_\beta E_\beta^s \Psi_\alpha^s \\
+ q^2 H_\beta(-1)H_\alpha(-1)^s \Psi_\alpha \Psi_\beta^s \Psi_\alpha^s (E_\beta^s \Psi_\alpha)^s \Psi_\beta.
\]

We now compute \( Y_\alpha \). On the right of ??1, the terms having the monomials \( R_\alpha \) are: \( (a' + b')c, \ d'(a + b), \ d'c, \ d'f, \ f'd, \ f'f, \text{ and } e'g. \)

Since \( E_\alpha E_\beta \) is a factor of \( c \), using ??7 and the facts that \( E_\alpha^2 = (q-1)E_\alpha \), \( E_\beta^2 = (q-1)E_\beta \), it is easy to see that the coefficient of \( R_\alpha \) in the expression of \( a'c \) is \(-1)^3(q-1)E_\alpha E_\beta.\)

Again, since \( E_\alpha E_\beta \) is a factor of \( c \), using ??7 and the facts that \( E_\alpha^2 = (q-1)E_\alpha \), \( H_\beta(-1)E_\beta = E_\beta \), it is easy to see that the coefficient of \( R_\alpha \) in the expression of \( b'c \) is

\[
-(q-1)^3(q-1)E_\alpha E_\beta = -(q-1)(q^2 - q + 1)E_\alpha E_\beta.
\]

Thus, the coefficient of \( R_\alpha \) in the expression of \((a' + b')c\) is

\[
(??31) \quad -(q-1)^3 + q(q-1)E_\alpha E_\beta = -(q-1)(q^2 - q + 1)E_\alpha E_\beta.
\]

We now compute the coefficient of \( R_\alpha \) in the expression of \( d'(a + b) \). Since \( E_\alpha E_\beta \) is a factor of \( a \) using ??7 and the fact that \( E_\beta^2 = (q-1)E_\beta \), it is easy to check that the coefficient of \( R_\alpha \) in the expression of \( d'a \) is \(-1)^3E_\alpha E_\beta.\)

Also, it is clear that the coefficient of \( R_\alpha \) in the expression of \( d'b \) is

\[
(E_\beta^s \Psi_\alpha)(qH_\alpha(-1)\Psi_\alpha \Psi_\alpha^s E_\beta^s) = q(q-1)H_\alpha(-1)\Psi_\alpha \Psi_\alpha^s E_\beta E_\beta^s.
\]

Thus, the coefficient of \( R_\alpha \) in the expression of \( d'(a + b) \) is

\[
(??32) \quad -(q-1)^3E_\alpha E_\beta + q(q-1)H_\alpha(-1)(\Psi_\alpha)^2 \Psi_\alpha^s E_\beta (E_\beta^s)^2.
\]

We now compute the coefficient of \( R_\alpha \) in the expression of \( d'c \). We have

\[
d'c = (R_\alpha E_\beta E_\beta^s \Psi_\alpha)(R_\alpha E_\alpha E_\beta) \\
= R_\alpha E_\beta^s E_\beta \Psi_\alpha^s E_\alpha E_\beta.
\]

Using \( R_\alpha^2 = qH_\alpha(-1) + R_\alpha E_\alpha \) and ??7, it is easy to see that the coefficient of \( R_\alpha \) in the expression of \( d'c \) is

\[
(??33) \quad E_\alpha E_\beta^s E_\beta \Psi_\alpha^s E_\alpha E_\beta = (q-1)^3E_\alpha E_\beta.
\]

Let us compute the coefficient the yields \( f'd. \) We have

\[
f'd = (R_\alpha R_\beta(E_\beta^s \Psi_\alpha)^s \Psi_\beta)(R_\beta E_\alpha E_\alpha^s \Psi_\beta) \\
= R_\alpha R_\beta^2 E_\beta^s \Psi_\beta^s E_\alpha E_\alpha^s \Psi_\beta.
\]

Therefore from ??1, we get

\[
 f'd = qR_\alpha H_\beta(-1)E_\beta^s \Psi_\alpha \Psi_\beta^s E_\alpha E_\alpha^s \Psi_\beta = R_\alpha R_\beta E_\beta E_\beta^s \Psi_\alpha \Psi_\beta^s E_\alpha E_\alpha^s \Psi_\beta.
\]

We have \( E_\alpha^s = E_\alpha \) and so we have

\[
H_\beta(-1)E_\alpha E_\beta^s = H_\beta(-1)(E_\alpha E_\beta)^s \\
= E_\alpha(H_\beta(-1)E_\beta) \quad \text{(from ??7)} \\
= E_\alpha E_\beta \quad \text{(since } H_\beta(-1)E_\beta = E_\beta \text{)}.
\]

Therefore, \( E_\alpha E_\beta \) is a factor of the coefficient of \( R_\alpha \) in the expression of \( f'd \) and hence using ??7 and the fact that \( E_\beta^2 = (q-1)E_\beta \), it is easy to see that the coefficient of \( R_\alpha \) in the expression of \( f'd \) is

\[
(??34) \quad -(q-1)^3(q(q-1))E_\alpha E_\beta = -(q-1)(q-1)E_\alpha E_\beta.
\]
Since $E_\alpha E_\beta$ is a factor of $c'$, using ??7, 1.1 and the fact that $E_\alpha^2 = (q - 1)E_\alpha$, it is easy to see that the coefficient of $R_\alpha$ in the expression of $cf$ is

( ??35) \[ (-1)^3q(q - 1)E_\alpha E_\beta = -(q(q - 1))E_\alpha E_\beta. \]

We now compute the coefficient of $R_\alpha$ in the expression of the product $f'f$. We have

\[
\begin{align*}
    f'f &= (R_\alpha R_\beta(E_\beta^s E_\alpha) s^s \Psi_\beta')(R_\beta R_\alpha(E_\alpha^s E_\beta) s^s \Psi_\alpha) \\
    &= R_\alpha R_\beta^2 E_\alpha^s E_\beta s^s \Psi_\alpha \\
    &= qH_\alpha^2 H_\beta(-1)^s E_\beta s^s \Psi_\beta'(E_\beta^s E_\alpha) s^s \Psi_\alpha + R_\alpha R_\beta R_\alpha E_\alpha^s E_\beta s^s \Psi_\beta'(E_\beta^s E_\alpha) s^s \Psi_\alpha.
\end{align*}
\]

Thus, from ??1 we deduce that $f'f$ yields the coefficient

\[
qE_\alpha (H_\beta(-1)) E_\beta^s E_\beta s^s \Psi_\alpha = -qE_\alpha E_\beta (H_\beta(-1)) E_\beta s^s \Psi_\alpha.
\]

Using ??7, we deduce that $f'f$ yields

( ??36) \[ (-1)^3q(q - 1)E_\alpha E_\beta = q(q - 1)E_\alpha E_\beta. \]

We now compute the coefficient of $R_\alpha$ in the expression of $e'g$. We have

\[
\begin{align*}
e'g &= (R_\beta R_\alpha E_\beta) s^s \Psi_\beta(R_\alpha R_\beta R_\alpha(E_\alpha^s E_\beta) s^s \Psi_\alpha) \\
    &= R_\alpha R_\beta^2 R_\beta R_\alpha (E_\alpha^s E_\beta) s^s \Psi_\alpha.
\end{align*}
\]

Using twice the relation ??1 we deduce that $e'g$ yields the coefficient

\[
q^2 H_\alpha(-1) (H_\alpha s^s)(((E_\beta^s E_\alpha) s^s)(((E_\beta^s E_\alpha) s^s)) s^s \Psi_\alpha.
\]

which can be written by

( ??37) \[ q^2 H_\alpha(-1) (H_\alpha s^s)(((E_\beta^s E_\alpha) s^s)(((E_\beta^s E_\alpha) s^s)) s^s \Psi_\alpha.
\]

(Here, we use $(H_\beta(-1))(H_\alpha(-1))s^s = (H_\alpha(-1))^2s^s = H_\alpha(-1)s^s = H_\alpha(-1))$.

Therefore, using the observations from ??31 to ??37, we conclude that

\[
Y_\alpha = -q(q - 1)^2 + q(q - 1)H_\alpha(-1)E_\alpha E_\beta E_\beta^s \\
+ q^2 H_\alpha(-1)E_\beta^s ((E_\beta^s E_\alpha) s^s)((E_\beta^s E_\alpha) s^s) s^s \Psi_\alpha \\
= X_\alpha (from ??30).
\]

**Computation of $X_{\alpha \beta}$ and $Y_{\alpha \beta}$.** The products on the left of equation ??1 involving the monomial $R_\alpha R_\beta$ are: $(a + b)f', cc', cf', e(a' + b')', cc', eg', dg', gd'$, and $gf'$.

We now compute the coefficient of $R_\alpha R_\beta$ in the expression of $af'$.

Since $E_\alpha E_\beta$ is a factor of $a$, using ??7, and the fact that $E_\beta^2 = (q - 1)E_\beta$, it is easy to see that the coefficient of $R_\alpha R_\beta$ in the expression of $af'$ is

( ??38) \[ (-1)^2(q - 1)^2 E_\alpha E_\beta = (q - 1)^2 E_\alpha E_\beta. \]

We now compute $bf'$. We have

\[
\begin{align*}
    bf' &= qH_\alpha(-1)E_\alpha s^s E_\beta s^s R_\alpha R_\beta(E_\beta^s E_\alpha) s^s \\
    &= R_\alpha R_\beta ((qH_\alpha(-1)) s^s (E_\beta^s E_\alpha) s^s) s^s \Psi_\beta' \\
    &= (from ??3).
\end{align*}
\]

Therefore, using the fact that $(H_\alpha(-1)) s^s E_\beta^s = -H_\alpha(-1)E_\beta$ and the fact that $\Psi_\beta E_\beta^s = -E_\beta$, it is easy to see that the coefficient of $R_\alpha R_\beta$ in the expression of $bf'$ is

( ??39) \[ -q(H_\alpha(-1))(E_\alpha s^s E_\beta(E_\beta^s E_\alpha) s^s \Psi_\alpha. \]

We now compute $cc'$. Since $E_\alpha E_\beta$ is a factor of $c$, using ??7 and the facts that $E_\alpha^2 = (q - 1)E_\alpha$ and $E_\beta^2 = (q - 1)E_\beta$, it is easy to see that the coefficient of $R_\alpha R_\beta$ in the expression of $cc'$ is

( ??40) \[ (-1)^2(q - 1)^2 E_\alpha E_\beta = (q - 1)^2 E_\alpha E_\beta. \]

We now compute $cf'$. 

We have
\[
    cf' = -R_\alpha E_\alpha E_\beta R_\alpha R_\beta (E_\alpha^s \Psi_\alpha)^s \Psi_\beta
\]
\[
    = -R_\alpha^2 R_\beta E_\alpha E_\beta (E_\alpha^s \Psi_\alpha)^s' \quad \text{(from ??3)}
\]
\[
    = (q H_\alpha (1) + R_\alpha E_\alpha) R_\beta E_\alpha E_\beta (E_\alpha^s \Psi_\alpha)^s \Psi_\beta. \quad \text{(from ??1)}
\]

But, we are interested only in the coefficient of \( R_\alpha R_\beta \), which is equal to
\[
    -E_\alpha^s' E_\alpha E_\beta (E_\alpha^s \Psi_\alpha)^s' \Psi_\beta = (-1)^3 E_\alpha E_\beta (E_\alpha E_\beta)^s' \quad \text{(from ??7)}
\]
\[
    = -((q - 1) E_\alpha E_\beta)^s' \quad \text{(since } E_\alpha^2 = (q - 1) E_\alpha)\)
\[
    = -(q - 1)^2 (E_\alpha E_\beta)^s' \quad \text{(from ??7)}
\]
\[
    = -(q - 1)^2 E_\alpha E_\beta. \quad \text{(from ??7)}
\]

Therefore, the coefficient of \( R_\alpha R_\beta \) in the expression of \( cf' \) is
\[
    ( ??41) \quad -(q - 1)^2 E_\alpha E_\beta.
\]

We now compute \( cc' \).

We have
\[
    ec' = - R_\alpha R_\beta^2 (\Psi_\alpha^s \Psi_\beta E_\alpha)^s E_\alpha E_\beta
\]
\[
    = R_\alpha (q H_\alpha (1) + R_\beta E_\beta) (\Psi_\alpha^s \Psi_\beta E_\alpha)^s E_\alpha E_\beta \quad \text{(from ??1)}
\]

But, we are interested only in the coefficient of \( R_\alpha R_\beta \), which is equal to
\[
    -(\Psi_\alpha \Psi_\beta) E_\alpha E_\beta E_\beta^2 = (-1)^3 E_\alpha E_\beta^2 E_\alpha^2 \quad \text{(from ??7)}
\]
\[
    = -(q - 1) E_\alpha E_\beta E_\alpha^2 \quad \text{(since } E_\alpha^2 = (q - 1) E_\alpha)\)
\[
    = -(q - 1)^2 E_\alpha E_\beta^2 \quad \text{(from ??7)}
\]
\[
    = -(q - 1)^2 E_\alpha E_\beta. \quad \text{(from ??7)}
\]

Therefore, the coefficient of \( R_\alpha R_\beta \) in the expression of \( ec' \) is
\[
    ( ??42) \quad -(q - 1)^2 E_\alpha E_\beta.
\]

We now compute \( eg' \).

We have
\[
    eg' = R_\alpha R_\beta (\Psi_\alpha^s \Psi_\beta E_\alpha R_\beta R_\alpha R_\beta (\Psi_\alpha^s \Psi_\alpha)^s \Psi_\beta
\]
\[
    = R_\alpha R_\beta^2 R_\beta (\Psi_\alpha^s \Psi_\beta E_\alpha)^s \Psi_\beta \Psi_\alpha^s \Psi_\beta \Psi_\beta
\]
\[
    = R_\alpha R_\beta^2 R_\beta (q H_\alpha (1) + R_\beta E_\beta)^s \Psi_\beta (\Psi_\alpha^s \Psi_\alpha E_\alpha)^s \Psi_\beta (\Psi_\beta^s \Psi_\beta)^s \Psi_\alpha \Psi_\beta \Psi_\alpha \Psi_\beta \quad \text{(from ??1 and ??3)}
\]

Using \( R_\alpha^2 = q H_\alpha (1) + R_\alpha E_\alpha \), it is easy to see that the coefficient of \( R_\alpha R_\beta \) in the expression of \( eg' \) is
\[
    E_\alpha^s (q H_\alpha (1))^s \Psi_\beta \Psi_\alpha \Psi_\alpha \Psi_\beta \Psi_\beta. \quad \text{(from ??1 and ??3)}
\]

Here, we first consider the term \( E_\alpha^s (E_\alpha^s)^s \). This can be written as
\[
    (E_\alpha (E_\alpha^s)^s)^s = (E_\alpha E_\alpha^s)^s \quad \text{(since } E_\alpha = E_\alpha^s)\)
\[
    = (E_\alpha E_\alpha^s)^s \quad \text{(since } E_\alpha = E_\alpha)\)
\[
    = E_\alpha E_\beta.
Now, from the above two observations, using ??, it is easy to see that the coefficient of $R_{\alpha}R_{\beta}$ in the expression of $e\gamma'$ is

\[(??.43) \quad (-1)^5 qE_{\alpha}E_{\beta} = -qE_{\alpha}E_{\beta}.\]

We now compute $e\alpha'$.

We have

\[e\alpha' = R_{\alpha}R_{\beta}((q-1)\Psi_{\alpha}'\Psi_{\beta}E_{\alpha}^2E_{\beta}\]
\[= R_{\alpha}R_{\beta}(q-1)(-1)^2 E_{\alpha}E_{\beta} \quad \text{(from ??)}\]
\[= (q-1)^2 E_{\alpha}E_{\beta} \quad \text{(since $E_{\alpha}^2 = (q-1)E_{\alpha}$)}\]

Therefore, the coefficient of $R_{\alpha}R_{\beta}$ in the expression of $e\alpha'$ is

\[(??.44) \quad (q-1)^2E_{\alpha}E_{\beta}.\]

We now compute the coefficient of $R_{\alpha}R_{\beta}$ in the expression of $e\beta'$. Using the fact that $E_{\alpha}E_{\alpha}' = E_{\alpha}E_{\beta}$, we have

\[e\beta' = R_{\alpha}R_{\beta}qH_{\beta}(-1)\Psi_{\beta}'(\Psi_{\beta}'\Psi_{\alpha})'E_{\alpha}E_{\beta}\]
\[= R_{\alpha}R_{\beta}qH_{\beta}(-1)(-1)^4 E_{\alpha}E_{\beta} \quad \text{(from ??)}\]
\[= R_{\alpha}R_{\beta}qE_{\alpha}E_{\beta} \quad \text{(since $H_{\beta}(-1)E_{\beta} = E_{\beta}$)}\]

Therefore, the coefficient of $R_{\alpha}R_{\beta}$ in the expression of $e\beta'$ is

\[(??.45) \quad qE_{\alpha}E_{\beta}.\]

We now compute $d\beta'$. We have

\[d\beta' = R_{\beta}E_{\alpha}E_{\alpha}'\Psi_{\beta}R_{\beta}R_{\alpha}R_{\beta}(\Psi_{\beta}'\Psi_{\alpha})'\Psi_{\beta}\]
\[= R_{\alpha}R_{\beta}E_{\alpha}E_{\alpha}'R_{\beta}R_{\alpha}R_{\beta}(\Psi_{\beta}'\Psi_{\alpha})'\Psi_{\beta}\]
\[= R_{\alpha}R_{\beta}((qH_{\beta}(-1) + R_{\beta}E_{\beta})'(\Psi_{\beta}'\Psi_{\alpha})'\Psi_{\beta} \quad \text{(from ??, ??)}\]
\[= R_{\alpha}R_{\beta}((qH_{\beta}(-1) + R_{\beta}E_{\beta})'(\Psi_{\beta}'\Psi_{\alpha})'\Psi_{\beta} \quad \text{(from ??)}\]

Therefore, the coefficient of $R_{\alpha}R_{\beta}$ in the expression of $d\beta'$ is

\[(??.46) \quad qE_{\alpha}E_{\beta}.\]

Here, we use the fact that $H_{\beta}(-1)E_{\beta} = E_{\beta}$.

We now compute $R_{\alpha}R_{\beta}$ in the expression of $d\gamma'$.

We have

\[d\gamma' = R_{\alpha}R_{\alpha}R_{\alpha}R_{\beta}(\Psi_{\alpha}'\Psi_{\beta})'\Psi_{\alpha}R_{\alpha}E_{\beta}E_{\beta}'\]
\[= R_{\alpha}R_{\alpha}R_{\beta}E_{\alpha}E_{\alpha}'\Psi_{\beta}R_{\alpha}R_{\beta}(\Psi_{\alpha}'\Psi_{\alpha})'\Psi_{\alpha}\]
\[= R_{\alpha}R_{\alpha}R_{\beta}E_{\alpha}E_{\alpha}'\Psi_{\beta}R_{\alpha}R_{\beta}(\Psi_{\alpha}'\Psi_{\alpha})'\Psi_{\alpha}\]
\[= R_{\alpha}R_{\alpha}R_{\beta}E_{\alpha}E_{\alpha}'\Psi_{\beta}R_{\alpha}R_{\beta}(\Psi_{\alpha}'\Psi_{\alpha})'\Psi_{\alpha}\]

Using the quadratic relation $R_{\alpha}^2 = qH_{\alpha}(-1) + R_{\alpha}E_{\alpha}$ and the fact that $s^2 = 1$, it is easy to see that the coefficient of $R_{\alpha}R_{\beta}$ in the expression of $d\gamma'$ is

\[(??.47) \quad qH_{\alpha}(-1)(\Psi_{\alpha}'\Psi_{\alpha}'\Psi_{\alpha}'\Psi_{\beta})E_{\beta}E_{\beta}'.\]

We now compute the coefficient of $R_{\alpha}R_{\beta}$ in the expression of $g\beta'$. We have

\[g\beta' = R_{\alpha}R_{\beta}R_{\alpha}R_{\beta}(\Psi_{\alpha}'\Psi_{\beta})'\Psi_{\alpha}R_{\alpha}E_{\beta}E_{\beta}.'\]

Using $R_{\alpha}^2 = qH_{\alpha}(-1) + R_{\alpha}E_{\alpha}$ and $R_{\beta}^2 = qH_{\beta}(-1) + R_{\beta}E_{\beta}$ and the fact that $H_{\alpha}(-1)'\Psi_{\beta}E_{\beta} = -H_{\alpha}(-1)E_{\beta}$, it is easy to see that the coefficient of $R_{\alpha}R_{\beta}$ in the expression of $g\beta'$ is

\[(??.48) \quad -qH_{\alpha}(-1)(\Psi_{\alpha}'\Psi_{\alpha}'\Psi_{\beta}E_{\beta}E_{\beta}).\]
Using the observations \( ??39, ??48 \), and the observation \( ??9 \) (of Lemma \( ?? \)), it is easy to see that the sum of coefficients yielded by \( bf \) and \( gf \) is equal to

\[-qH_\alpha(-1)(\Psi_\alpha \Psi_\beta')^s E_\beta^s (\Psi_\alpha E_\beta + \Psi_\alpha \Psi_\beta E_\beta) = 0.\]

Therefore, summing up all the other coefficients (using the observations from \( ??38 \) to \( ??48 \)), we have

\[( ??49) \quad X_{\alpha\beta} = (q^2 - q + 1)E_\alpha E_\beta + qH_\alpha(-1)\Psi_\alpha \Psi_\beta^s \Psi_\beta E_\beta^s.\]

We now compute the coefficient of \( R_\alpha R_\beta \) in the expression of \( p_{\alpha\beta}p_{\alpha\beta} \).

In the product \( p_{\beta\alpha}p_{\alpha\beta} \), the terms involving \( R_\alpha R_\beta \) are \( d'e, b'e, d'd, d'e, f'a, f'b \) and \( f'd \).

We now compute \( d'e \).

Since \( E_\alpha E_\beta \) is a factor of \( d' \), using \( ??7 \), and the fact that \( E_\alpha^2 = (q - 1)E_\alpha \), it is easy to see that the coefficient of \( R_\alpha R_\beta \) in the expression of \( d'e \) is

\[( ??50) \quad (q^2 - q + 1)E_\alpha E_\beta.\]

We now compute \( b'e \).

We have

\[
b'e = qH_\beta(-1)\Psi_\beta \Psi_\beta' E_\alpha R_\alpha R_\beta \Psi_\alpha' \Psi_\beta E_\alpha,
\]

\[
= R_\alpha R_\beta((H_\beta(-1)\Psi_\beta \Psi_\beta')^s\Psi_\alpha' \Psi_\beta E_\alpha), \quad \text{from} \quad ??3
\]

\[
= R_\alpha R_\beta((H_\beta(-1)\Psi_\beta \Psi_\beta')^s)\Psi_\alpha' \Psi_\beta E_\alpha, \quad \text{from} \quad ??1, \text{and} \quad ??3
\]

\[
= R_\alpha R_\beta(-1)^4 q((E_\alpha E_\beta H_\beta(-1))^s)\Psi_\beta, \quad \text{from} \quad ??7
\]

\[
= R_\alpha R_\beta(E_\alpha E_\beta)^s, \quad \text{since} \quad H_\beta(-1)E_\beta = E_\beta
\]

\[
= R_\alpha R_\beta q E_\alpha E_\beta, \quad \text{from} \quad ??7)
\]

Here, we use the fact that

\[
E_\alpha((E_\alpha')^s) = E_\alpha ((E_\alpha)^s) \quad \text{from} \quad ??1
\]

\[
= (E_\alpha E_\alpha^s) \quad \text{since} \quad E_\alpha^s = E_\alpha
\]

\[
= E_\alpha E_\beta \quad \text{from} \quad ??3, \text{and} \quad ??7.
\]

Therefore, the coefficient of \( R_\alpha R_\beta \) in the expression of \( b'e \) is

\[
( ??51) \quad q E_\alpha E_\beta.
\]

We now compute \( d'd \).

We have

\[
d'd = R_\alpha E_\beta E_\beta^s \Psi_\alpha R_\beta E_\alpha E_\alpha' \Psi_\beta
\]

\[
= R_\alpha R_\beta(E_\beta E_\beta^s \Psi_\alpha)^s E_\alpha E_\alpha' \Psi_\beta, \quad \text{from} \quad ??3
\]

\[
= R_\alpha R_\beta(E_\beta E_\beta^s \Psi_\alpha)^s E_\alpha E_\beta \Psi_\beta, \quad \text{from} \quad ??7
\]

\[
= R_\alpha R_\beta(-1)^2 (q - 1)^2 E_\alpha E_\beta, \quad \text{from} \quad ??7, \text{and} \quad E_\beta^3 = (q - 1)^2 E_\alpha E_\beta
\]

Therefore, the coefficient of \( R_\alpha R_\beta \) in the expression of \( d'd \) is

\[
( ??52) \quad (q - 1)^2 E_\alpha E_\beta.
\]

We now compute the coefficient of \( R_\alpha R_\beta \) in the expression of \( d'e \).

We have

\[
d'e = R_\alpha E_\beta E_\beta^s \Psi_\alpha R_\alpha R_\beta \Psi_\alpha' \Psi_\beta E_\alpha
\]

\[
= R_\alpha^2 R_\beta((E_\beta E_\beta^s \Psi_\alpha)^s)^s \Psi_\alpha' \Psi_\beta E_\alpha, \quad \text{from} \quad ??3
\]

\[
= (qH_\alpha(-1) + R_\alpha E_\alpha) R_\beta((E_\beta E_\beta^s \Psi_\alpha)^s)^s \Psi_\alpha' \Psi_\beta E_\alpha, \quad \text{from} \quad ??1
\]
But, we are interested in computing only the coefficient of \( R_\alpha R_\beta \), which is equal to

\[
(E_\alpha^s E_\beta)(E_\beta E_\alpha^s \Psi_\alpha)^s' \Psi_\beta = (-1)^3(q - 1)^2 E_\alpha E_\beta \quad \text{(from ??3 and ??7)}.
\]

Therefore, the coefficient of \( R_\alpha R_\beta \) in the expression of \( d' c \)

is

\[
(??53) \quad -(q - 1)^2 E_\alpha E_\beta.
\]

We now compute \( f' a \).

We have

\[
f' a = R_\alpha R_\beta (E_\beta^s \Psi_\alpha)^s' \Psi_\beta(q - 1)(E_\alpha E_\beta)
\]

\[
= R_\alpha R_\beta(-1)^2(q - 1)^2((E_\alpha E_\beta^s)^s') \quad \text{(from ??7)}
\]

\[
= R_\alpha R_\beta(q - 1)^2 E_\alpha E_\beta \quad \text{(since } E_\beta^2 = (q - 1)E_\beta) \]

Therefore, the coefficient of \( R_\alpha R_\beta \) in the expression of \( f' a \)

is

\[
(??54) \quad -(q - 1)^2 E_\alpha E_\beta.
\]

We now compute \( f' b \).

A straightforward computation shows that the coefficient of \( R_\alpha R_\beta \) in the expression of \( f' b \)

is

\[
(??55) \quad q(E_\alpha^s \Psi_\alpha)^s' \Psi_\beta H_\alpha(-1)\Psi_\alpha \Psi_\alpha^s E_\beta.
\]

We now compute the coefficient of \( R_\alpha R_\beta \) in the expression of \( f' d \).

We have

\[
f' d = R_\alpha R_\beta (E_\beta^s \Psi_\alpha)^s' \Psi_\beta R_\beta(E_\alpha E_\alpha^s') \Psi_\beta
\]

\[
= R_\alpha R_\beta (E_\beta^s \Psi_\alpha)^s' (E_\alpha E_\alpha^s') \Psi_\beta \quad \text{(from ??3)}
\]

\[
= R_\alpha(qH_\alpha(-1) + R_\beta E_\beta)((E_\beta^s \Psi_\alpha)^s' \Psi_\beta)^s'(E_\alpha E_\alpha^s') \Psi_\beta.
\]

But, we are interested in computing only the coefficient of \( R_\alpha R_\beta \), which is equal to

\[
E_\beta((E_\beta^s \Psi_\alpha)^s' \Psi_\beta)^s'(E_\alpha E_\alpha^s') \Psi_\beta = (-1)^3((E_\alpha E_\beta^s)^s') E_\alpha^s' \quad \text{(from ??7 and } (s')^2 = 1)
\]

\[
= -(q - 1)(E_\alpha E_\beta^s)^s' E_\alpha^s' \quad \text{(since } E_\beta^2 = (q - 1)E_\beta)
\]

\[
= -(q - 1)(E_\alpha E_\beta^s)^s' \quad \text{(from ??7)}
\]

\[
= -(q - 1)^2 E_\alpha E_\beta \quad \text{(since } E_\alpha^2 = (q - 1)E_\alpha)
\]

Therefore, the coefficient of \( R_\alpha R_\beta \) in the expression of \( f' d \) is

\[
(??56) \quad -(q - 1)^2 E_\alpha E_\beta.
\]

Summing up these coefficients (using the observations from ??50 to ??56), we have

\[
(??57) \quad Y_{\alpha \beta} = (q^2 - q + 1)E_\alpha E_\beta + qH_\alpha(-1)\Psi_\alpha \Psi_\alpha^s \Psi_\alpha^s' \Psi_\beta E_\beta^s (E_\alpha^s').
\]

We have \( E_\alpha^s = E_{3\alpha + 1} \beta \) and \( (E_\beta^s)^s' = E_{3\alpha + 2} \beta \).

Using a similar proof ??5, it is easy to see that

\[
(??58) \quad E_\beta(E_\beta)^s' E_\beta E_{3\alpha + 1} \beta = E_{3\alpha + 1} \beta E_{3\alpha + 2} \beta = E_\beta^s(E_\beta)^s'.
\]

Using the observations, ??49, ??57 and ??58, it is easy to see that \( X_{\alpha \beta} = Y_{\alpha \beta} \).

We now prove that

\[
X_0 = Y_0.
\]

The terms yielding the constant coefficients in the expression of \( p_{\alpha \beta} p_{3\alpha} \) are \( aa', ab', ba', bb', cd', dc', ec' \) and \( ff' \).

The terms yielding the constant coefficients in the expression of \( p_{3\alpha} p_{\alpha \beta} \) are \( a'd', a'b', b'a, b' b, c'd', d' c', e'e \) and \( ff' \).
It is easy to see that these terms yield the following coefficients.

\[
\begin{align*}
    aa' &= a'a = (q - 1)^4 E_\alpha E_\beta \\
    ab' &= b'a = q(q - 1) E_\alpha E_\beta \\
    ba' &= a'b = q(q - 1) E_\alpha E_\beta \\
    bb' &= b'b = q^2 H_\beta (-1)^E_\alpha E_\beta \\
    cd &= a'b = q(q - 1)^2 E_\alpha E_\beta \\
    dd &= c'd = q(q - 1)^2 E_\alpha E_\beta \\
    ee' &= f' f = q^2 H_\beta (-1)^E_\alpha E_\beta (E_\alpha E_\beta) \\
    ff' &= e' e = q(q - 1)^2 E_\alpha E_\beta
\end{align*}
\]

Hence, we have

\[X_0 = Y_0.\]

Since the computations of \(X_\beta, X_\alpha, X_{\alpha_\beta}, X_{\alpha_\beta_\alpha}, X_{\beta_\alpha_\beta_\alpha}\) are similar to the computations of \(X_\alpha, X_{\alpha_\beta}, X_{\alpha_\beta_\alpha}, X_{\alpha_\beta_\alpha_\beta}, X_{\beta_\alpha_\beta_\alpha}\) respectively and the same is true for the \(Y_\alpha\)'s, we will only quote the coefficients.

We first quote the terms involving \(R_\beta\) in the expression of \(p_{\alpha_\beta} p_{\beta_\alpha}\) and the terms involving \(R_\beta\) in the expression of \(p_{\beta_\alpha} p_{\alpha_\beta}\).

The terms involving \(R_\beta\) in the expression of \(p_{\alpha_\beta} p_{\beta_\alpha}\) are \(ac', bc', cf, da', db', dc', eg' fd'\). On the other hand, the terms involving \(R_\beta\) in the expression of \(p_{\beta_\alpha} p_{\alpha_\beta}\) are \(a'd, b'd, c'a, db, c'd, d'e, e'c, e'e, g'f\).

The coefficients yielded by these are as follows:

\[
\begin{align*}
    ac' &= c'a = -(q - 1)^2 E_\alpha E_\beta \\
    bc' &= c'b = -(q(q - 1)) E_\alpha E_\beta \\
    cd &= a'b = -(q - 1)^2 E_\alpha E_\beta \\
    dd &= b'd = -(q(q - 1)) E_\alpha E_\beta \\
    ed' &= c'd = -(q - 1)^2 E_\alpha E_\beta \\
    eg' &= g' f = q^2 H_\beta (-1)^E_\alpha E_\beta (E_\alpha E_\beta) \\
    fd' &= d'e = -(q - 1)^2 E_\alpha E_\beta \\
    ff' &= e' e = q(q - 1)^2 E_\alpha E_\beta
\end{align*}
\]

We note that here, we write the only coefficients (not with the monomials). Therefore, we have

\[X_\beta = Y_\beta.\]

We now do the same for \(R_\beta R_\alpha\).

The terms involving \(R_\beta R_\alpha\) in the expression of \(p_{\alpha_\beta} p_{\beta_\alpha}\) are \((a + b)e', dd', de', f(a' + b')\) and \(fd'\). On the other hand, the terms involving \(R_\beta R_\alpha\) in the expression of \(p_{\beta_\alpha} p_{\alpha_\beta}\) are \(a' f, b' f, c'e, c' f, d' g, e' a, e' b, e' c, e' g, g' d\) and \(g' f\).
The coefficient yielded by these are as follows:

\[ ae' = e'a = (q-1)^2E_\alpha E_\beta \]

\[ be' = e'b = q(E_\beta E_\beta^* - (q-1)E_\alpha E_\beta) \]

\[ de' = e'd = (q-1)^2E_\alpha E_\beta \]

\[ de' = e'f = -(q(q-1))E_\alpha E_\beta \]

\[ fa' = a'f = (q-1)^2E_\alpha E_\beta \]

\[ fb' = b'f = qE_\alpha E_\beta \]

\[ df = e'c = -(q-1)^2E_\alpha E_\beta \]

\[ d'g = -(e'g) = q(H_\alpha(-1)^s)^s((\Psi_\alpha^s)^s)^s((\Psi_\alpha^s)^s)^s E_\alpha E_\beta \]

\[ g'd = -(g'f) = E_\alpha E_\beta . \]

(We note that we write only the coefficients)

Hence, we have

\[ X_{\beta \alpha} = Y_{\beta \alpha} . \]

We now do the same for \( R_\beta R_\alpha R_\beta \).

The terms involving \( R_\beta R_\alpha R_\beta \) in the expression of \( a^q, b^q, d^q, f^q \) and \( f^q'. \)

On the other hand, terms involving \( R_\beta R_\alpha R_\beta \) in the expression of \( p_{\beta \alpha}p_{\alpha \beta} \) are \( e^q, e^q', g^q, g^q' \).

The coefficients yielded by these are as follows:

\[ ag' = g'a = -(q-1)E_\alpha E_\beta \]

\[ bg' = g'b = qH_\alpha(-1)^s\Psi_\beta^s E_\beta \]

\[ dg' = g'd = (q-1)^2E_\alpha E_\beta \]

\[ df = e'c = -(q-1)^2E_\alpha E_\beta \]

\[ f'e' = e'f = E_\alpha E_\beta \]

\[ e'g = -e'g = E_\alpha E_\beta \]

\[ g'c = -g'f = E_\alpha E_\beta . \]

Hence, we have

\[ X_{\beta \alpha \beta} = Y_{\beta \alpha \beta} . \]

We now do the same for \( R_\beta R_\alpha R_\beta R_\alpha \).

The only term involving \( R_\beta R_\alpha R_\beta R_\alpha \) in the expression of \( p_{\alpha \beta}p_{\beta \alpha} \) is \( f^q' \).

On the other hand, the only term involving \( R_\beta R_\alpha R_\beta R_\alpha \) is \( g^q c \).

The coefficients yielded by these are as follows:

\[ fg' = g'e = \Psi_\beta^s((\Psi_\beta^s)^s)^s((\Psi_\alpha^s)^s)^s E_\alpha . \]
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