ANOMALY CANCELLATION CONDITION
IN LATTICE GAUGE THEORY

Hiroshi Suzuki*
Department of Mathematical Sciences, Ibaraki University,
Mito 310-8512, Japan†
and
The Abdus Salam International Centre for Theoretical Physics, Trieste, Italy.

Abstract

We show that, to all orders of powers of the gauge potential, a gauge anomaly $\mathcal{A}$ defined on 4-dimensional infinite lattice can always be removed by a local counterterm, provided that $\mathcal{A}$ depends smoothly and locally on the gauge potential and that $\mathcal{A}$ reproduces the gauge anomaly in the continuum theory in the classical continuum limit: The unique exception is proportional to the anomaly in the continuum theory. This follows from an analysis of nontrivial local solutions to the Wess-Zumino consistency condition in lattice gauge theory. Our result is applicable to the lattice chiral gauge theory based on the Ginsparg-Wilson Dirac operator, when the gauge field is sufficiently weak $\|U(n,\mu) - 1\| < \epsilon'$, where $U(n,\mu)$ is the link variable and $\epsilon'$ a certain small positive constant.

MIRAMARE – TRIESTE

February 2000

* E-mail: hsuzuki@ictp.trieste.it
† On leave of absence from.
1. Introduction

If one puts Weyl fermions on lattice while respecting desired physical properties, one has to sacrifice the $\gamma_\sigma$-symmetry [1,2]. This implies that the gauge symmetry is inevitably broken on lattice when Weyl fermions are coupled to the gauge field. This is rather expected, because we know in the continuum theory there exists the gauge anomaly [3–8]. However, even if the anomaly in the continuum theory cancels $\text{tr}_{R-L} T^a \{T^b, T^c\} = 0$ [5–8], the fermionic determinant is not gauge invariant in general when the lattice spacing is finite $a \neq 0$. Then the gauge degrees of freedom do not decouple and it becomes quite unclear whether properties of the continuum theory (such as the unitarity) are reproduced in the continuum limit, after taking effect of dynamical gauge fields into account. Basically this is the origin of difficulties of chiral gauge theories on lattice [9]. It is thus quite important to understand the structure of breakings of the gauge symmetry on lattice, which will be denoted by $\mathcal{A}$, while keeping the lattice spacing finite.

What is the possible structure of $\mathcal{A}$ for $a \neq 0$? This question appears meaningless unless one imposes certain conditions on $\mathcal{A}$. After all, uniqueness of the gauge anomaly in the continuum theory [5–8,10–17] is lost for a finite ultraviolet cutoff and the explicit form of the breaking $\mathcal{A}$ is expected to quite depend on details of the lattice formulation. But what kind of conditions can strongly constrain the structure of $\mathcal{A}$? And, under such conditions, is it possible to relate $\mathcal{A}$ and the anomaly in the continuum theory? It seemed almost impossible to answer these questions. (This statement is not completely true: If one restricts operators with the mass dimension $\leq 5$ (we assign one mass dimension to the ghost field), the complete classification of possible breakings has been known in the context of the Rome approach [18].)

The atmosphere has been changed after Lüscher’s theorem on the $\gamma_\sigma$-anomaly in the abelian lattice gauge theory $G = U(1)$ appeared [19]. Assuming smoothness, locality‡ and the topological nature of the anomaly, he showed the theorem for 4-dimensional infinite lattice, that is equivalent to $\S$

$$\mathcal{A} = \delta_B \ln \text{Det} M'[A]$$
$$= \sum_n c(n) \left[ \alpha + \beta_{\mu\nu} F_{\mu\nu}(n) + \gamma \epsilon_{\mu\nu\rho\sigma} F_{\mu\nu}(n) F_{\rho\sigma}(n + \hat{\mu} + \hat{\nu}) + \Delta^*_\mu k_\mu(n) \right], \quad (1.1)$$

where $\text{Det} M'$ is a fermionic determinant and $\delta_B$ is the BRS transformation [10] corresponding to the gauge transformation in the abelian lattice gauge theory, $\delta_B A_\mu(n) = \Delta_\mu c(n)$ and $\delta_B c(n) = 0$; $c(n)$ stands for the abelian Faddeev-Popov ghost field defined on lattice. In eq. (1.1), $\alpha$, $\beta_{\mu\nu}$ and $\gamma$ are unknown constants and $k_\mu(n)$ in the last term is a local and gauge invariant current. Note that eq. (1.1) holds for finite lattice spacing and the structure is quite independent of details of the formulation. In this sense, this theorem provides a

‡ The meaning of the locality is of course different from that of the continuum theory. We will explain this terminology in detail in the next section.

§ For our notation, see appendix A.
universal characterization of the gauge anomaly in the abelian lattice gauge theory. Moreover, the theorem asserts that the anomaly cancellation in the abelian lattice gauge theory is (almost) equivalent to that of the continuum theory: The last term of the breaking (1.1) can be removed by adding the local counterterm $\mathcal{B} = \sum_n A_\mu(n)k_\mu(n)$ to the effective action $\ln \text{Det} M' \rightarrow \ln \text{Det} M' + \mathcal{B}$, because $\delta_B \mathcal{B} = \sum_n \Delta_\mu c(n)k_\mu(n) = -\sum_n c(n)\Delta^*_\mu k_\mu(n)$. If the anomaly is pseudoscalar quantity, the first two constants vanish $\alpha = 0$, the term proportional to $\gamma$ vanishes if $\sum_R \epsilon^3_R - \sum_L \epsilon^3_L = 0$, here $\epsilon_H$ stands for the $U(1)$ charge, because we have absorbed the $U(1)$ charge in $c$ and in $F_{\mu\nu}$. This argument shows that the effective action with finite lattice spacing can be made gauge invariant if (and only if) the fermion multiplet is anomaly-free! This remarkable observation was fully utilized in the existence proof of an exactly gauge invariant lattice formulation of anomaly-free chiral abelian gauge theories [20].

In this paper, we attempt to generalize the above theorem (1.1) for general (compact) gauge groups. Our scheme is somewhat different from that of refs. [19,21]. In ref. [21], this problem in nonabelian theories was shown to be equivalent to a classification of gauge invariant topological fields in (4+2)-dimensional space, here 4-dimensions are discrete and 2-dimensions are continuous. In this paper, instead, we analyze general nontrivial local solutions to the Wess-Zumino consistency condition [22] in lattice gauge theory. For a generic gauge group, the BRS transformation is defined by:

$$
\delta_B U(n, \mu) = U(n, \mu)c(n + \hat{\mu}) - c(n)U(n, \mu), \quad \delta_B c(n) = -c(n)^2.
$$

(1.2)

Since this BRS transformation is nilpotent $\delta_B^2 = 0$, the breaking $\mathcal{A} = \delta_B \ln \text{Det} M'$ must satisfy the Wess-Zumino consistency condition

$$
\delta_B \mathcal{A} = 0,
$$

(1.3)

as in the continuum theory [22,10]. In the continuum theory, consistency and uniqueness of anomaly-free chiral gauge theories in perturbative level follow from detailed analyses of the consistency condition [10–17] (for a more complete list of references, see ref. [17]). We will see below that the consistency condition (1.3), combined with the locality in the sense of ref. [19], strongly constrains the possible structure of $\mathcal{A}$, as does in the continuum theory. Our basic strategy is to imitate as much as possible the procedure in the continuum theory, especially that of ref. [16]. Of course, there are many crucial differences between continuum and lattice theories and how to handle these differences becomes key of our “algebraic” approach.

The organization of this paper is as follows. Our main theorems which generalize eq. (1.1) are stated in section 3. Our theorems are applicable only if the gauge anomaly $\mathcal{A}$ depends

---

\* This transformation is obtained by parameterizing the gauge transformation parameter in $U(n, \mu) \rightarrow g(n)^{-1}U(n, \mu)g(n + \hat{\mu})$ by $g = \exp(\lambda c)$ where $\lambda$ stands for an infinitesimal Grassmann parameter.
locally on the gauge field. The only framework known to present which possesses this property is the formulation of refs. [20,21] based on the Ginsparg-Wilson Dirac operator [23–25], or equivalently the overlap formulation [26,27]. Therefore, in section 2, we summarize basic properties of the gauge anomaly in the formulation of ref. [21]. At the same time, we introduce notions of admissibility and of locality. We also introduce the gauge potential and define the “perturbative configuration.” The sections from section 3 to section 6 are entirely devoted to a task to determine general nontrivial local solutions to the consistency condition in the abelian theory $G = U(1)^N$. In section 3, we give some preliminaries. In section 4, we prove several lemmas concerning De Rham and BRS cohomologies on infinite lattice. Here the technique of noncommutative differential calculus [28–31] turns to be a powerful tool [32,33]. Utilizing these lemmas, in section 6, we first determine a complete list of nontrivial local solutions to the consistency condition in the abelian theory. Here the ghost number of the solution is arbitrary. Then we restrict the ghost number of the solution unity. After imposing several conditions, we obtain the content of the theorem for the abelian theory. Section 7 is devoted to the nonabelian extension. In section 7.1, we derive a basic lemma which guarantees the adjoint invariance of nontrivial solutions. In section 7.2, under several assumptions, we show uniqueness of the nontrivial local anomaly to all orders of powers of the gauge potential. This establishes the content of our theorem for nonabelian theories, which will be stated in section 3. In section 7.3, we explicitly write down such a nontrivial local anomaly by utilizing the interpolation technique of lattice fields [34,35]. The last section is devoted to concluding remarks. Our notation is summarized in appendix A. In appendix B, we explain the calculation of the Wilson line which appears in the integrability condition of ref. [21].

2. Gauge anomaly in the Ginsparg-Wilson approach

2.1. Admissibility, locality and the gauge potential

The “admissible” gauge field is defined by [21]

$$\|P(n, \mu, \nu) - 1\| < \epsilon, \quad \text{for all } n, \mu, \nu,$$

(2.1)

where $P(n, \mu, \nu)$ is the plaquette variable in the representation to which the Weyl fermion is belonging and $\epsilon$ a certain small positive constant. In this expression, $\|O\|$ is the operator norm that is defined by [36]

$$\|O\| = \sup_{v \neq 0} \frac{\|Ov\|}{\|v\|},$$

(2.2)

where the norm in the right hand side is defined by the standard norm for vectors. The reason for this restriction of field space is two-fold.

Consider a finite lattice. Let us suppose that the Dirac operator satisfies a kind of index theorem. Namely, a difference of numbers of normalizable zero modes of the Dirac operator
with opposite chirality is equal to the topological charge of the gauge field configuration. The index is an integer and thus inevitably jumps even if the gauge field configuration changes smoothly. This argument suggests that such a Dirac operator cannot be a smooth function of the gauge field. Smoothness of the Dirac operator and in turn that of the gauge anomaly are thus expected to hold only within a restricted field space. In fact, a detailed analysis [37] of Neuberger’s overlap Dirac operator [25], which satisfies the index theorem [38,39], shows that the Dirac operator depends smoothly and locally on the gauge field when $\epsilon \leq 1/30$ in eq. (2.1). Our proof is valid only when the gauge anomaly depends on the gauge field smoothly and locally.

Closely relating to the above point, any configuration of the lattice gauge field can smoothly be deformed into the trivial one \( U(n, \mu) = 1 \) and thus topology of the gauge field space is trivial if no restriction is imposed. On the other hand, it has been known [40] that, under the condition (2.1), one can define a nontrivial principal bundle over periodic lattice and the field space is divided into topological sectors. For example, for the fundamental representation of \( SU(2) \), \( \epsilon \leq 0.015 \) is enough for the construction of ref. [40] to work. Later we utilize the interpolation method of ref. [34] which is based on the section of the principal bundle of ref. [40].

Note that eq. (2.1) is a gauge invariant condition. The gauge equivalences of an admissible configuration are all admissible. However, structure of the space of admissible configurations is quite complicated and no simple parameterization in terms of the gauge potential has been known except for abelian cases [19]. This is the reason that our theorem for nonabelian theories is in practice applicable only for the “perturbative configurations” which will be explained below.

As noted in the introduction, our basic strategy is to imitate the argument in the continuum theory. The first important difference from the continuum theory is the notion of locality. The anomaly is a local quantity when the ultraviolet cutoff is sent to infinity. But of course this is not the case for \( a \neq 0 \) and we need an appropriate notion which works with lattice. Here we follow the definition of ref. [19] (see also ref. [41]). Suppose that \( \phi(n) \) is a field on lattice that depends on link variables \( U \). The field \( \phi(n) \) may depend on the link variable \( U(m, \mu) \) at a distant link from the site \( n \). We say that \( \phi(n) \) locally depends on the link variable, if this dependence on \( U(m, \mu) \) becomes exponentially weak as \( |n - m| \to \infty \). To be more precise, consider the following decomposition:

\[
\phi(n) = \sum_{k=1}^{\infty} \phi_k(n),
\]

where \( \phi_k(n) \) depends only on link variables \( U \) inside a block of the size \( k \) centered at the site \( n \) (such a field \( \phi_k(n) \) is called ultra-local). If all these fields \( \phi_k(n) \) and their derivatives \( \phi_k(n; m_1, \mu_1; \cdots; m_N, \mu_N) \) with respect to the link variables \( U(m_1, \mu_1), \ldots, U(m_N, \mu_N) \) are bounded as

\[
|\phi_k(n; m_1, \mu_1; \cdots; m_N, \mu_N)| \leq C_N k^{p_N} \exp(-\theta k), \tag{2.4}
\]
by the constants $C_N$, $p_N$ and $\theta$ all being independent of link variable configurations, then we say that $\phi(n)$ locally depends on the link variable. In what follows, we introduce also the gauge potential and the ghost field. The same terminology will be used by simply replacing “link variable” by the name of each field. When no confusion arises, we say simply that $\phi(n)$ is local. Also when a functional is given by a sum of such local fields, $\Phi = \sum_n \phi(n)$, we simply say that $\Phi$ is local. If $\phi(n)$ is a local field, the effective range of dependences is a finite number in the lattice spacing. Therefore, physically, this locality can be regarded as equivalent to the ultra-locality. The technical reason for this definition of locality is that the Dirac operator which satisfies the Ginsparg-Wilson relation cannot be ultra-local in general [42,43] and, on the other hand, we can apply the Poincaré lemma of ref. [19], if dependences are exponentially weak.

The basic degrees of freedom in lattice gauge theory are link variables. But we stick to the gauge potential, because its use is essential in arguments in the continuum theory. To keep validity of our argument as wide as possible, we consider the following two cases.

Case I. When the gauge group is abelian $G = U(1)^N$. If we take $0 < \epsilon \leq 1$ in eq. (2.1) or equivalently (the superscript $a$ here labels each $U(1)$ factor in $G$)

$$\| \ln P^a(n, \mu, \nu) \| < \frac{\pi}{3}, \quad \text{for all } a, n, \mu, \nu,$$

there exists the relatively simple prescription [19] which allows a complete parameterization of the space of admissible gauge fields. Under the condition (2.5), one can associate the abelian gauge potential such that

$$U^a(n, \mu) = \exp A^a_{\mu}(n), \quad -\infty < \frac{1}{i} A^a_{\mu}(n) < \infty,$$

and moreover

$$\ln P^a(n, \mu, \nu) = \Delta_{\mu} A^a_{\nu}(n) - \Delta_{\nu} A^a_{\mu}(n).$$

From this relation and eq. (2.5), if a configuration $A^a_{\mu}$ is admissible, the rescaled one $tA^a_{\mu}$ with $0 \leq t \leq 1$ is also admissible. In this prescription [19] (a closely related prescription for 2-dimensional periodic lattice was first given in ref. [35]), the abelian gauge potential $A^a_{\mu}(n)$ corresponding to the given link variables $U^a(n, \mu)$ is not unique. Also this mapping does not preserve the locality. Nevertheless, as far as gauge invariant quantities are concerned, such an ambiguity disappears and also the locality becomes common for both variables. See refs. [19,44] for details.

Case II. For a general (compact) gauge group $G$, we define

$$U(n, \mu) = \exp A_{\mu}(n), \quad \| A_{\mu}(n) \| \leq \pi,$$

and we further impose

$$\| A_{\mu}(n) \| < \frac{1}{4} \ln(1 + \epsilon) \leq \pi, \quad \text{for all } \mu \text{ and } n.$$

By noting $\| OO' \| \leq \| O \|\|O' \|$ [36], one can see that configurations which satisfy eq. (2.9) is

6
in fact admissible, i.e., they satisfy eq. (2.1). But note that eq. (2.9) is a very restrictive condition and it contains only a portion of admissible configurations; in fact, the condition (2.9) is not gauge invariant. We call configurations which satisfy eq. (2.9) “perturbative.” For perturbative configurations, all link variables are close to unity \[ \|U(n, \mu) - 1\| < \epsilon' = (1+\epsilon)^{1/4} - 1. \] Unfortunately, our theorem for nonabelian theories is applicable only for this restricted space, when the admissibility (2.1) is required.

2.2. Fermionic determinant and the gauge anomaly

In this subsection, we study basic properties of the gauge anomaly appearing in the formulation based on the Ginsparg-Wilson Dirac operator [20,21], with a particular choice of the integration measure. As noted sometimes [45,46], this formulation can be reinterpreted in terms of the overlap formulation [26,27]. Therefore it must be possible to repeat a similar argument also in the context of the overlap formulation.

Following refs. [20,21], we define the fermionic determinant as

\[
\text{Det} M' = \int d[\psi] d[\bar{\psi}] \exp \left( - \sum_n \bar{\psi}(n) D \psi(n) \right), \quad \hat{P}_H \psi(n) = \psi(n), \quad (2.10)
\]

where the Dirac operator \( D \) satisfies the Ginsparg-Wilson relation \( \gamma_5 D + D \gamma_5 = D \gamma_5 D \) [23]. We assume that the Dirac operator \( D \) is gauge covariant and local in the sense of ref. [20] and it depends smoothly on the gauge field. Thus we assume the admissibility (2.1) for gauge field configurations. The chirality of the fermion is defined with respect to the Ginsparg-Wilson chiral matrix \( \gamma_5 = \gamma_5 (1 - D) \) [47,41,48]. Namely, the projection operator has been defined by \( \hat{P}_H = (1 + \epsilon H \gamma_5)/2 \). The chirality of the anti-fermion is on the other hand defined by the conventional \( \gamma_5 \) matrix.

The integration measure for the fermion \( d[\psi] \) in eq. (2.10) thus depends on the gauge field nontrivially due to the condition \( \hat{P}_H \psi = \psi \). But this condition alone does not specify the integration measure uniquely. For definiteness, we make the following choice which starts with the particular “measure term” [49]

\[
\mathcal{L}_\eta' = -i \epsilon_H \int_0^1 ds \text{ Tr } \hat{P}_H \left[ \partial_s \hat{P}_H, \delta_eta \hat{P}_H \right], \quad (2.11)
\]

where Tr stands for the summation over lattice points \( \sum_n \) of the diagonal \( (n,n) \) components as well as traces over the gauge and the spinor indices. In this expression, \( \eta \) stands for the

* \( \mathcal{L}_\eta' \) identically vanishes when the representation of the Weyl fermion is (pseudo-)real [49].
infinitesimal variation of link variables

\[ \delta_\eta U(n, \mu) = \eta_\mu(n)U(n, \mu). \]  

(2.12)

We have to specify also the \( s \)-dependence in eq. (2.11). As a simple choice, we set

\[ U(n, s, \mu) = \exp[sA_\mu(n)], \quad 0 \leq s \leq 1, \]

(2.13)

for both cases I (2.6) and II (2.8) above. Note that the line in the configuration space \( U(n, s, \mu) \) which connects 1 and \( U(n, \mu) \) is contained in the admissible space (2.1) and, for the case II, in the perturbative region (2.9). The functional (2.11) smoothly and locally depends on the gauge potential due to the assumed properties of the Dirac operator (\( \mathcal{L}'_\eta \) does not contain the inverse of the Dirac operator). Since the functional \( \mathcal{L}'_\eta \) is linear in \( \eta_\mu \), it may be written as

\[ \mathcal{L}'_\eta = \sum_n \eta_\mu(n) j_\mu^a(n), \quad \eta_\mu(n) = \eta_\mu(n)T^a. \]  

(2.14)

This current \( j_\mu^a \) depends smoothly and locally on the gauge potential.

Now, using the Ginsparg-Wilson relation, one can show [49] that \( \mathcal{L}'_\eta \) satisfies the differential form of the integrability condition [20,21]:\(^\dagger\)

\[ \delta_\eta \mathcal{L}'_\zeta - \delta_\zeta \mathcal{L}'_\eta + \mathcal{L}'_{[\eta, \zeta]} = -i\epsilon H \text{Tr} \hat{P}_H [\delta_\eta \hat{P}_H, \delta_\zeta \hat{P}_H]. \]  

(2.15)

Moreover, considering a one parameter family of gauge fields, \( U_t(n, \mu) \) \((0 \leq t \leq 1)\) and introducing the transporting operator \( Q_t \) by [21]

\[ \partial_t Q_t = [\partial_t P_t, P_t]Q_t, \quad P_t = \hat{P}_H|_{U \to U_t}, \quad Q_0 = 1, \]

(2.16)

one can show (appendix B) that \( \mathcal{L}'_\eta \) satisfies the integrability in the integrated form [21] for an arbitrary closed loop \( U_0(n, \mu) = U_1(n, \mu) \) (here \( \eta_\mu(n) = \partial_t U_t(n, \mu)/U_t(n, \mu)^{-1} \))

\[ W' = \exp\left(i \int_0^1 dt \mathcal{L}'_\eta\right) = \text{Det}(1 - P_0 + P_0Q_1)^{-\epsilon H}, \]

(2.17)

as far as the loop \( U_t(n, \mu) \) is contained within the perturbative region (2.9) for the case II. Since both the space of admissible fields (2.5) for the case I and the space of perturbative configurations (2.9) for the case II are contractable, there is no global obstruction [50] which is a lattice counterpart of the Witten’s anomaly [51]. Eq. (2.17) guarantees that there

\(^\dagger\) Here we assume that \( \eta \) and \( \zeta \) are independent of the gauge field.
exists the integration measure $d[\psi]d[\bar{\psi}]$ which corresponds to the measure term $\mathcal{L}^{\prime}_\eta$ [21]. In particular, the infinitesimal variation of the fermion determinant (2.10) is given by

$$\delta_\eta \ln \det M' = \text{Tr} \delta_\eta D \tilde{P}_H D^{-1} + i \epsilon_H \mathcal{L}^{\prime}_\eta.$$  \hspace{1cm} (2.18)

We have completely specified the fermionic determinant (2.10) up to a physically irrelevant proportionality constant. This fermionic determinant is however not gauge invariant in general. The resulting gauge anomaly $A = \delta_B \ln \det M'$ is obtained simply by setting

$$\eta_\mu(n) = U(n, \mu) c(n + \hat{\mu}) U(n, \mu)^{-1} - c(n),$$  \hspace{1cm} (2.19)

in eq. (2.12). Then from eqs. (2.18) and (2.14), we have

$$A = \epsilon_H \text{Tr} c_5 \left(1 - \frac{1}{2} D\right) - i \epsilon_H \sum_n c^a(n) \left[j^a_\mu(n) - U(n - \tilde{\mu}, \mu)^{-1} j^a_\mu(n - \tilde{\mu}) U(n - \tilde{\mu}, \mu)\right]_G,$$  \hspace{1cm} (2.20)

where use of the gauge covariance $\delta_B D = [D, c]$ for $s = 1$ and the Ginsparg-Wilson relation has been made. Manifestly, this anomaly $A$ depends smoothly and locally on the gauge potential (and on the ghost field) from the assumed properties of the Dirac operator and from the properties of the current $j^a_\mu$.

We need to know the classical continuum limit of $A$. The gauge potential in the classical continuum limit $A_\mu(x)$ is introduced by the conventional manner

$$U(n, \mu) = \mathcal{P} \exp \left[a \int_0^1 du A_\mu(n + (1 - u)\tilde{\mu}a)\right],$$  \hspace{1cm} (2.21)

where $\mathcal{P}$ stands for the path ordered product. Then the first term of eq. (2.20) produces the covariant gauge anomaly which can be deduced from the general arguments [52,21] or from explicit calculations using Neuberger’s overlap Dirac operator [53,54] as $\dagger$

$$\epsilon_H \text{Tr} c_5 \left(1 - \frac{1}{2} D\right) \xrightarrow{a \to 0} - \frac{\epsilon_H}{8\pi^2} \int d^4 x \varepsilon_{\mu\rho\sigma} \text{tr} c \partial_\mu \left(A_\nu \partial_\rho A_\sigma + \frac{2}{3} A_\nu A_\rho A_\sigma\right),$$  \hspace{1cm} (2.22)

for a single Weyl fermion. For the second term of eq. (2.20), which corresponds to a divergence of the so-called Bardeen-Zumino current [55] in the continuum theory, it is easier to

$\dagger$ Of course, we assume that parameters in the Dirac operator has been chosen such that there is only one massless degree of freedom.
consider $\mathcal{L}'_\eta$ (2.11) instead of the divergence of the current $j^\mu$. With the choice (2.13), we see in the classical continuum limit:

$$\delta_B \hat{P}_H = s[\hat{P}_H, c] + O(a). \quad (2.23)$$

Then by using the Ginsparg-Wilson relation, we have

$$i\epsilon_H \mathcal{L}'_\eta = -\frac{\epsilon_H}{2} \int_0^1 ds \, s \, \partial_s \text{Tr} \, c \gamma_5 (1 + D) + O(a). \quad (2.24)$$

It is possible to argue that the $O(a)$-term in eq. (2.23) contributes only to $O(a)$-term in eq. (2.24) from the mass dimension and the pseudoscalar nature of $i\epsilon_H \mathcal{L}'_\eta$ (assuming the Lorentz invariance restores in the classical continuum limit). Then, from eq. (2.22) with the substitution $A_\mu \rightarrow sA_\mu$, we have

$$i\epsilon_H \mathcal{L}'_\eta \xrightarrow{a \to 0} \frac{\epsilon_H}{8\pi^2} \int d^4 x \, \varepsilon_{\mu\nu\rho\sigma} \text{Tr} \, c \, \partial_\mu \left( \frac{2}{3} A_\nu \partial_\rho A_\sigma + \frac{1}{2} A_\nu A_\rho A_\sigma \right). \quad (2.25)$$

Combining eqs. (2.22) and (2.25), we have the correct consistent anomaly in the continuum theory:

$$A \xrightarrow{a \to 0} -\frac{\epsilon_H}{24\pi^2} \int d^4 x \, \varepsilon_{\mu\nu\rho\sigma} \text{Tr} \, c \, \partial_\mu \left( A_\nu \partial_\rho A_\sigma + \frac{1}{2} A_\nu A_\rho A_\sigma \right). \quad (2.26)$$

This expression is for a simple gauge group. The gauge anomaly for a generic gauge group $G = \prod G_\alpha$ can be obtained by simply substituting $c \rightarrow \sum_\alpha c^{G_\alpha}$ and $A_\mu \rightarrow \sum_\alpha A_\mu^{G_\alpha}$ in eq. (2.26). To have the standard form of the anomaly for $G = \prod G_\alpha$, we add the local counterterm to the effective action $\ln \text{Det} M'' = \ln \text{Det} M' + S$, for the measure term $i\epsilon_H \mathcal{L}'' = i\epsilon_H \mathcal{L}' + \delta_\eta S$, where

$$S = \frac{\epsilon_H}{144\pi^2} \sum_n \varepsilon_{\mu\nu\rho\sigma} \left[ U(n, \mu) U^{(1)\beta} - 1 \right] \text{Tr} \left[ U(n, \nu)^{(a)} - 1 \right] \left[ U(n, \rho)^{(a)} - 1 \right] \left[ U(n, \sigma)^{(a)} - 1 \right]. \quad (2.27)$$

The superscript $a$ runs over simple groups in $G$ and $\beta$ denotes each $U(1)$ factor in $G$. $S$ depends smoothly and locally on the link variable and the modification does not affect the

---

\(\dagger\) For abelian cases, the relation $\delta_B \hat{P}_H = s[\hat{P}_H, c]$ holds for arbitrary $a$.

\(\ddagger\) Strictly speaking, an explicit calculation of eq. (2.11) or of eq. (2.24) in the classical continuum limit, using say the Neuberger's overlap Dirac operator, has not been carried out in the literature. A corresponding calculation in the linearized level in the overlap formulation was given in the last reference of ref. [27]. See also ref. [21].
integrability, eqs. (2.15) and (2.17). The counterterm $S$ was chosen such that its classical continuum limit becomes $\epsilon \int d^4 x \varepsilon_{\mu \nu \rho \sigma} A^{(1)\beta}_\mu A^{(\alpha)\rho}_\nu A^{(\alpha)\sigma}_\rho / (144 \pi^2)$. Then the gauge anomaly of the modified effective action becomes

$$
\mathcal{A} \rightarrow 0 - \frac{\epsilon H}{24 \pi^2} \int d^4 x \left\{ \varepsilon_{\mu \nu \rho \sigma} \partial_\mu c^{(\alpha)} \partial_\sigma A^{(\alpha)} + \frac{1}{2} A^{(\alpha)} A^{(\alpha)} A^{(\alpha)} \right\}
$$

$$
+ \varepsilon_{\mu \nu \rho \sigma} U^{(1)\beta}_\rho A^{(\alpha)} \partial_\mu A^{(\alpha)} \frac{1}{(144 \pi^2)}
$$

$$
+ \varepsilon_{\mu \nu \rho \sigma} U^{(1)\beta}_\rho \partial_\mu \left[ A^{(\alpha)} A^{(\alpha)} + 2 A^{(\alpha)} A^{(\alpha)} A^{(\alpha)} \right]
$$

in the classical continuum limit.

We have thus observed that the anomaly on lattice $\mathcal{A} = \delta_B \ln \text{Det} \ M''$ smoothly and locally depends on the gauge potential and on the ghost field and that $\mathcal{A}$ reproduces the gauge anomaly in the continuum theory in the classical continuum limit, eq. (2.28). In the following sections, we show that such an anomaly $\mathcal{A} = \delta_B \ln \text{Det} \ M''$ on infinite lattice can always be written as $\mathcal{A} = \delta_B \mathcal{B}$, where $\mathcal{B}$ smoothly and locally depends on the gauge potential, if (and only if) the anomaly in the continuum theory is canceled $\text{tr} \{ T^a T^b, T^c \} = 0$ etc. This statement holds to all orders of powers of the gauge potential for nonabelian cases. This implies that, for an anomaly-free fermion multiplet, one can improve the fermionic determinant as

$$
\text{Det} \ M''[A] \rightarrow \text{Det} \ M[A] = \text{Det} \ M''[A] \exp(-\mathcal{B}[A]),
$$

so that the improved fermionic determinant $\text{Det} \ M[A]$ has the exact gauge invariance.** Therefore, to all orders of the gauge potential, there exists a gauge invariant lattice formulation of anomaly-free nonabelian chiral gauge theories, as far as the perturbative configurations (2.9) on infinite lattice are concerned.

In the context of the overlap formulation, our choice of the measure term (2.11) corresponds to a particular choice of the phase of the vacuum state. The formula corresponds to eq. (2.20) was given in the last reference of ref. [27]. The gauge anomaly and the Witten’s anomaly as local and global obstructions in the overlap formulation were studied in detail in ref. [56]. See also ref. [46].

---

* Note that (when $\eta$ and $\zeta$ are independent of the gauge field) $(\delta_\eta \delta_\zeta - \delta_\zeta \delta_\eta + \delta_{\eta \zeta}) S = 0$ holds for any functional $S$ of the link variable.

** Since the fermionic determinant $\text{Det} \ M[A]$ is gauge invariant, one can then regard it as a functional of the link variable $\text{Det} \ M[U]$ for the case I (this is trivially the case for the case II).
3. Results

In this section, we present our main results in a summarized form. When the gauge group is abelian $G = U(1)^N$, we will show the following theorem.

**Theorem (Abelian theory).** Let $\mathcal{A}[c, A]$ be the gauge anomaly $\mathcal{A} = \delta_B \ln \text{Det} M'[A]$ on 4-dimensional infinite hypercubic lattice, defined from a certain fermionic determinant $\text{Det} M'$. Suppose that $\mathcal{A}$ depends smoothly and locally on the abelian gauge potential $A_\mu^a$ and on the abelian ghost field $c^a$ and that it reproduces for smooth field configurations the gauge anomaly in the continuum theory (2.28) in the classical continuum limit. Then $\mathcal{A}$ is always of the form (for a single Weyl fermion)

$$
\mathcal{A}[c, A] = -\frac{\epsilon H}{96\pi^2} \sum_n \sum_{abc} \varepsilon_{\mu\nu\rho\sigma} c^a(n) F^b_{\mu\nu}(n) F^c_{\rho\sigma}(n + \hat{\mu} + \hat{\nu}) + \delta_B \mathcal{B}[A],
$$

(3.1)

where the functional $\mathcal{B}$ depends smoothly and locally on the gauge potential $A_\mu^a$.

This is a natural generalization of the Lüscher’s result (1.1) to multi-$U(1)$ cases. As a consequence of this theorem, the anomaly cancellation in the corresponding continuum theory $\sum_R e_R^b e_R^c - \sum_L e_L^b e_L^c = 0$ guarantees the gauge invariance of the effective action, after subtracting the local counterterm $\mathcal{B}$.

For nonabelian gauge theories, we will show the following statement.

**Theorem (Nonabelian theory).** Let $\mathcal{A}[c, A]$ be the gauge anomaly $\mathcal{A} = \delta_B \ln \text{Det} M''[A]$ on 4-dimensional infinite hypercubic lattice, defined from a certain fermionic determinant $\text{Det} M''$. Suppose that $\mathcal{A}$ depends smoothly and locally on the gauge potential $A_\mu$ and on the ghost fields $c$ and that it reproduces for smooth field configurations the gauge anomaly in the continuum theory (2.28) in the classical continuum limit. Then if the anomaly in the corresponding continuum theory cancels, $\text{tr}_{R\rightarrow L} T^a \{T^b, T^c\} = 0$ etc., $\mathcal{A}$ is always BRS trivial, i.e., $\mathcal{A} = \delta_B \mathcal{B}[A]$, where the functional $\mathcal{B}$ depends smoothly and locally on the gauge potential $A_\mu$. This statement holds to all orders of powers of the gauge potential $A_\mu$. The explicit form of the nontrivial anomaly $\mathcal{A} \neq \delta_B \mathcal{B}$ is given in eq. (7.50).

Therefore, to all orders of powers of the gauge potential, the anomaly cancellation in the continuum theory guarantees that of the lattice theory. This seems remarkable but is not entirely unexpected. Let us recall the expression in the classical continuum limit (2.28). The expression in fact holds to all orders of powers of the lattice spacing $a$ in the classical continuum limit $a \to 0$ (we assume that the Lorentz covariance restores in this limit). In the classical continuum limit, each coefficient of the expansion with respect to $a$ is a local functional of the gauge potential and the ghost field. Then the uniqueness theorem of

---

*** Several variations of this theorem with weaker assumptions are possible. But this form of theorem seems practically reasonable.
nontrivial anomalies in the continuum theory [16,17] can be appealed and one concludes that the anomaly (2.28) is the unique possibility (up to contributions of local counterterms). Therefore the anomaly cancellation $\text{tr}_{R-L} T^a \{ T^b, T^c \} = 0$ guarantees that $A = \delta_B B$ to all orders of powers of the lattice spacing $a$ in the classical continuum limit $a \to 0$. Of course the expansion with respect to $a$ is (presumably at most) asymptotic and this does not prove the anomaly cancellation for $a \neq 0$. Nevertheless, this argument makes the content of the above theorem quite plausible. Finally, we emphasize that the above theorems themselves do not assume the Ginsparg-Wilson relation and they are applicable to any formulation if the prerequisites of the theorems are fulfilled.

4. Preliminaries in the abelian theory

4.1. Noncommutative differential calculus

To determine general nontrivial local solutions to the consistency condition (1.3), we need De Rham and BRS cohomological information, as in the continuum theory [16,17]. To discuss De Rham cohomology on infinite lattice, the technique of noncommutative differential calculus [28–31] is very useful, because it makes the standard Leibniz rule of the exterior derivative valid even on lattice. In fact, this technique was applied successfully [32,33] to an algebraic proof of the higher dimensional extension of the Lüscher’s theorem of ref. [19] (that is basically equivalent to eq. (1.1)). Here we recapitulate its basic setup.

The bases of the 1-form on $D$-dimensional infinite hypercubic lattice are defined as objects which satisfy the Grassmann algebra

$$dx_1, dx_2, \ldots, dx_D, \quad dx_\mu dx_\nu = -dx_\nu dx_\mu. \quad (4.1)$$

A generic $p$-form is defined by

$$f(n) = \frac{1}{p!} f_{\mu_1 \cdots \mu_p}(n) dx_{\mu_1} \cdots dx_{\mu_p}, \quad (4.2)$$

where the summation of repeated indices is understood. The exterior derivative is then defined by the forward difference operator as

$$df(n) = \frac{1}{p!} \Delta_{\mu_1 \cdots \mu_p} f_{\mu_1 \cdots \mu_p}(n) dx_\mu dx_{\mu_1} \cdots dx_{\mu_p}. \quad (4.3)$$

The nilpotency of the exterior derivative $d^2 = 0$ follows from this definition. The essence of the noncommutative differential calculus on infinite lattice is

$$dx_\mu f(n) = f(n + \hat{\mu}) dx_\mu, \quad (4.4)$$

where $f(n)$ is a 0-form (i.e., function). Namely, a function on lattice and the basis of 1-form do not simply commute. The argument of the function is shifted along $\mu$-direction by
one unit when commuting these two objects. The remarkable fact, which follows from the
noncommutativity (4.4), is that the standard Leibniz rule of the exterior derivative $d$ holds.
With eqs. (4.3) and (4.4), one can easily confirm that
\[ d[f(n)g(n)] = df(n)g(n) + (-1)^p f(n)dg(n), \]
for arbitrary forms $f(n)$ and $g(n)$ (here $f(n)$ is a $p$-form). The validity of this Leibniz rule
is quite helpful for following analyses.

We also introduce the abelian gauge potential 1-form and the abelian field strength
2-form by
\[ A^a(n) = A^a_\mu(n) dx^\mu, \quad F^a(n) = \frac{1}{2} F^a_{\mu\nu}(n) dx^\mu dx^\nu = dA^a(n). \]
Note that the Bianchi identity takes the form $dF^a(n) = 0$. We never use the symbol $F^a$
or $F$ for nonabelian field strength 2-form.

**4.2. Abelian BRS transformation**

From eqs. (1.2) and (2.6), the BRS transformation for the gauge potential and for the
ghost field in the abelian theory is defined by
\[ \delta_B A^a_\mu(n) = \Delta_\mu c^a(n), \quad \delta_B c^a(n) = 0. \]
The BRS transformation is nilpotent $\delta_B^2 = 0$ and the abelian field strength is BRS invariant $\delta_B F^a_{\mu\nu}(n) = 0$. We also introduce the Grassmann coordinate $\theta$ [57–59] and define the
BRS exterior derivative by
\[ s = \delta_B \cdot d\theta. \]
The usual 1-form $dx^\mu$ and the BRS 1-form $d\theta$ anticommute with each other $dx^\mu d\theta = -d\theta dx^\mu$ and the BRS 1-form $d\theta$ commutes with itself $d\theta d\theta \neq 0$. Therefore, for a Grassmann-even (-odd) $p$-form $f(n)$, we have
\[ d\theta f(n) = \pm(-1)^p f(n) d\theta. \]
We also have
\[ s^2 = \{s, d\} = 0, \]
where the first relation follows from $\delta_B^2 = 0$. Finally, we introduce the ghost 1-form by
\[ C^a(n) = c^a(n) d\theta. \]
In terms of these forms, the BRS transformation in the abelian theory (4.7) is expressed as
\[ sA^a(n) = -dC^a(n), \quad sC^a(n) = 0, \quad sF^a(n) = 0. \]
The noncommutative rule (4.4) will be always assumed in expressions written in terms of
forms.
5. Basic lemmas in the abelian theory

With the tools introduced in the preceding section, we establish in this section several lemmas which provide cohomological data. The algebraic Poincaré lemma and the covariant Poincaré lemma concern the De Rham cohomology, including nontrivial information about dependences on the gauge potential and on the ghost field. The Poincaré lemma on infinite lattice [19] is the foundation of these lemmas. We also determine the BRS cohomology in the abelian theory \( G = U(1)^N \).

5.1. ALGEBRAIC POINCARE LEMMA

The algebraic Poincaré lemma \(*\) asserts that any \( d \)-closed form on \( D \)-dimensional infinite lattice is always \( d \)-exact up to a constant form; \( D \)-forms are exceptional because any \( D \)-form is \( d \)-closed. Moreover, the lemma asserts that the locality of dependences is preserved between the original form and its “kernel.”

**Algebraic Poincaré lemma.** Let \( \eta \) be a \( p \)-form on \( D \)-dimensional infinite lattice that depends smoothly and locally on the gauge potential \( A_\mu^a \) and on the ghost field \( c^a \). Then

\[
d\eta(n) = 0 \iff \eta(n) = d\chi(n) + \mathcal{L}(n) d^D x + B, \tag{5.1}
\]

where \( B \) is a constant \( p \)-form and the \( (p-1) \)-form \( \chi \) and the function \( \mathcal{L} \) depend smoothly and locally on the gauge potential and on the ghost field. The function \( \mathcal{L} \) satisfies

\[
\sum_n \delta \mathcal{L}(n) \neq 0, \tag{5.2}
\]

for a certain local variation \( \delta \) of the gauge potential and the ghost field.

*Note.* The term \( \mathcal{L} d^D x \) in eq. (5.1) represents a non-topological part in the \( D \)-form \( \eta \). In other words, a \( D \)-form \( \eta_{\text{top}} \), that is topological, \( \sum_n \delta \eta_{\text{top}}(n) = 0 \) for an arbitrary local variation, is always \( d \)-exact up to a constant form.

**Proof.** We define \( \eta_t \) by rescaling fields as \( A_\mu^a \to tA_\mu^a \) and \( c^a \to tc^a \). Then since \( \eta \) depends smoothly on \( A_\mu^a \) and on \( c^a \),

\[
\eta(n) = \eta(n)_{t=0} + \int_0^1 dt \frac{\partial \eta(n)_t}{\partial t} \\
= \eta(n)_{t=0} + \sum_{n'} \left[ A_\mu^a(n') \theta_\mu^a(n', n) + c^a(n') \kappa^a(n', n) \right], \tag{5.3}
\]

\(*\) The present algebraic Poincaré lemma is somewhat different from that of ref. [32]. Practically, the present form is more convenient.
where
\[ \theta^a_{\mu}(n', n) = \int_0^1 dt \frac{\partial \eta(n)_{\mu}}{\partial t} A^a_{\mu}(n'), \quad \kappa^a(n', n) = \int_0^1 dt \frac{\partial \eta(n)_{\mu}}{\partial t} c^a(n'). \] (5.4)

Eq. (5.4) implies
\[ d\theta^a_{\mu}(n', n) = d\kappa^a(n', n) = 0, \] (5.5)
because \( d\eta = 0 \) for arbitrary configurations. Moreover, since \( \eta \) depends locally on \( A^a_{\mu} \) and on \( c^a, \theta^a_{\mu}(n', n) \) and \( \kappa^a(n', n) \) decay exponentially as \(|n - n'| \rightarrow \infty\). This allows us to apply Lüscher’s Poincaré lemma [19] for \( p < D \) to eq. (5.5) which asserts that there exist forms \( \Theta^a_{\mu}(n, n') \) and \( K^a(n, n') \) such that
\[ \theta^a_{\mu}(n', n) = d\Theta^a_{\mu}(n', n), \quad \kappa^a(n', n) = dK^a(n', n). \] (5.6)

These forms \( \Theta^a_{\mu}(n, n') \) and \( K^a(n, n') \) also decay exponentially as \(|n - n'| \rightarrow \infty \) [19]. Substituting this into eq. (5.3), we have
\[ \eta = d\chi + B \] where \( B = \eta_{t=0} \) and
\[ \chi(n) = \sum_{n'} \left[ A^a_{\mu}(n') \Theta^a_{\mu}(n', n) + c^a(n') K^a(n', n) \right]. \] (5.7)

From the locality property of \( \Theta^a_{\mu}(n, n') \) and of \( K^a(n, n') \) [19], one can easily see [32] that \( \chi(n) \) is a local field. Also the smoothness is preserved in the construction (5.7). In this way, the lemma (5.1) is established for \( p < D \).

For \( p = D \), \( d\eta = 0 \) is a trivial statement and thus we decompose \( \eta \) as
\[ \eta = \eta_{\text{top.}} + \mathcal{L} d^D x, \] (5.8)
where \( \sum_n \delta\eta_{\text{top.}}(n) = 0 \) for an arbitrary local variation. Then \( \theta^a_{\mu} \) and \( \kappa^a \) in eq. (5.4) defined from \( \eta_{\text{top.}} \) satisfy
\[ \sum_n \theta^a_{\mu}(n', n) = \sum_n \kappa^a(n', n) = 0. \] (5.9)

Then Lüscher’s Poincaré lemma for \( p = D \) [19] asserts that there exist \( \Theta^a_{\mu} \) and \( K^a \) which satisfy eq. (5.6). The rest is the same as for \( p < D \) and we have \( \eta_{\text{top.}} = d\chi + B \).
Abelian BRS cohomology. Let $X$ be a form on infinite hypercubic lattice that depends smoothly and locally on the gauge potential and on the ghost field. Then,

$$sX(n) = 0 \iff X(n) = C^{a_1}(n) \cdots C^{a_g}(n)X_0^{[a_1 \cdots a_g]}(\{F_i\}; n) + sY(n),$$

(5.10)

where the form $X_0^{[a_1 \cdots a_g]}(n)$ depends smoothly and locally only on the abelian field strength $F^a_{\mu \nu}$. The form $Y(n)$ depends smoothly and locally on the gauge potential and on the ghost field. In particular, differences of the ghost field can appear only in the BRS trivial part $sY$.

Note. The form $X_0^{[a_1 \cdots a_g]}$ is totally antisymmetric on the upper indices because ghost 1-forms $C^a$ simply anticommute with each other. $X_0^{[a_1 \cdots a_g]}(n)$ depends only on the field strength $F^a_{\mu \nu}(n)$ and its differences, such as $\Delta_\mu F^a_{\nu \rho}(n)$, $\Delta_\mu^* F^a_{\nu \rho}(n)$, $\Delta_\mu \Delta_\nu F^a_{\rho \sigma}(n)$ and so on; obviously $X_0^{[a_1 \cdots a_g]}$ is gauge invariant. In what follows, we denote as $X_0^{[a_1 \cdots a_g]}(\{F_i\})$ to indicate this particular dependence on the field strength, including smoothness and locality of the dependence.

Proof. The proof of the abelian BRS cohomology for a single $U(1)$ case [32] can be repeated by simply supplementing another index $a$ to the gauge potential $A_\mu$ and to the ghost field $c$. Thus we do not reproduce it here to save the space. 

5.3. COVARIANT POINCARE LEMMA

As in the continuum theory [16], the following covariant Poincaré lemma is crucial to determine general nontrivial local solutions to the consistency condition. This lemma for a single $U(1)$ case $G = U(1)$ was given in ref. [32]. It turns out that, however, its extension to multi-$U(1)$ cases is not trivial, due to the reason which will be explained after the proof. In fact, we have at present only the following cumbersome proof that works only for 4- or lower dimensional lattice.

Covariant Poincaré lemma. On 4-dimensional infinite hypercubic lattice, if $p$-form $\alpha_p(\{F_i\})$ is $d$-closed for $p < 4$, or if $\alpha_4(\{F_i\}) = d\chi_3 + B_4$ where $B_4$ is a constant 4-form, then $\alpha_p$ is of the structure

$$\alpha_p(\{F_i\}; n) = d\alpha_{p-1}(\{F_i\}; n) + B_p + F^a(n)B^a_{p-2} + F^a(n)F^b(n)B^{(ab)}_{p-4},$$

(5.11)

where $F^a$ is the field strength 2-form and $B$'s are constant forms.

Note. Here all the expressions are written in terms of the noncommutative differential calculus.
Proof. We prove the lemma step by step from 0-form $p = 0$ until 4-form $p = 4$.

For $p = 0$. The lemma trivially holds by the algebraic Poincaré lemma (5.1). Namely, the $d$-closed 0-form $\alpha_0$ must be a constant $\alpha_0 = B_0$.

For $p = 1$. By the algebraic Poincaré lemma, the $d$-closed 1-form $\alpha_1$ is $d$-exact up to a constant 1-form. Also $\alpha_1$ is $s$-closed because it is a function of the field strength. Namely,

$$\alpha_1 = d\chi_0^0 + B_1, \quad s\alpha_1 = 0. \quad (5.12)$$

Since these equations imply $s\alpha_1 = sd\chi_0^0 = -ds\chi_0^0 = 0$, the algebraic Poincaré lemma asserts that

$$s\chi_0^0 = 0, \quad (5.13)$$

where we have used the fact that the right hand side cannot be a constant. The solution to this equation is given by the abelian BRS cohomology (5.10) for $g = 0$ case:

$$\chi_0^0 = \omega_0[\{F_i\}], \quad (5.14)$$

and thus eq. (5.12) shows that the lemma holds for $p = 1$:

$$\alpha_1 = d\omega_0[\{F_i\}] + B_1. \quad (5.15)$$

For $p = 2$. In this case, from the algebraic Poincaré lemma, we have

$$\alpha_2 = d\chi_1^0 + B_1, \quad s\alpha_2 = 0, \quad (5.16)$$

and, in a similar way as the $p = 1$ case, these lead to the following descent equations

$$s\chi_1^0 = d\chi_0^1, \quad s\chi_0^1 = 0. \quad (5.17)$$

The general solution to the last equation is given by the abelian BRS cohomology

$$\chi_0^1 = C^a\omega_0^a[\{F_i\}] + s\beta_0. \quad (5.18)$$

We may however absorb $\beta_0$ in the redefinition of $\chi_0^1$ and $\chi_1^0$,

$$\chi_0^1 \rightarrow \chi_0^1 + s\beta_0, \quad \chi_1^0 \rightarrow \chi_1^0 + d\beta_0, \quad (5.19)$$

without changing $\alpha_2$. We can therefore take $\chi_0^1 = C^a\omega_0^a$. Then the first equation in eq. (5.17)
reads

\[
s\chi_1^0 = dC^a \omega_0^a - C^a d\omega_0^a \\
= -s(A^a \omega_0^a) - C^a d\omega_0^a.
\] (5.20)

Now consider a special configuration of the ghost field \( c^a(n) \to c^a = \text{const.} \). Then the consistency of eq. (5.20) requires

\[
d\omega_0^a = 0, \quad s(\chi_1^0 + A^a \omega_0^a) = 0,
\] (5.21)

because \( s(\text{something}) \) is proportional to differences of the ghost fields such as \( c^a(n + \mu) - c^a(n) \). Note that \( \omega_0^a \) does not depend on the ghost field. The solution to the first equation of eq. (5.21) is given by the present lemma for \( p = 0 \), which we have shown above:

\[
\omega_0^a = B_0^a \quad (\text{const.}),
\] (5.22)

and then the second relation of eq. (5.21) implies

\[
\chi_1^0 = -A^a B_0^a + \omega_1 \{ F_i \},
\] (5.23)

by the BRS cohomology. Going back to the original relation (5.16), we have

\[
\alpha_2 = -F^a B_0^a + d\omega_1 \{ F_i \} + B_2,
\] (5.24)

because \( dA^a = F^a \). This shows the lemma for \( p = 2 \).

For \( p = 3 \). In this case, the counterparts of eqs. (5.16) and (5.17) are

\[
\alpha_3 = d\chi_2^0 + B_3, \quad s\alpha_3 = 0,
\] (5.25)

and

\[
s\chi_2^0 = d\chi_1^1, \quad s\chi_1^1 = d\chi_0^2, \quad s\chi_0^2 = 0.
\] (5.26)

The solution to the last equation is (we have absorbed the BRS trivial part as eq. (5.19))

\[
\chi_0^2 = C^a C^b \omega_0^{[ab]} \{ F_i \},
\] (5.27)

where \( \omega_0^{[ab]} \) is antisymmetric on \( a \leftrightarrow b \). For following arguments, it is quite helpful to introduce the symmetrization symbol, that is defined by

\[
sym(X_1 X_2 \cdots X_N) = \sum_{\sigma} \frac{1}{N!} \epsilon_\sigma X_{\sigma(1)} X_{\sigma(2)} \cdots X_{\sigma(N)},
\] (5.28)

where the summation is taken over all the permutations \( \sigma \). The sign factor \( \epsilon_\sigma \) is defined as the signature arising when the product \( X_1 \cdots X_N \) is made to the order \( X_{\sigma(1)} X_{\sigma(2)} \cdots X_{\sigma(N)} \).
by regarding all $X_i$'s are ordinary (i.e., the form basis $dx_\mu$ simply commutes with functions) forms. With use of the symmetrization symbol, eq. (5.27) is trivially written as

$$\chi_0^2 = \text{sym}(C^a C^b) \omega_0^{[ab]},$$

and then the second relation of eq. (5.26) reads

$$s\chi_1^1 = 2\text{sym}(dC^a C^b) \omega_0^{[ab]} + \text{sym}(C^a C^b) d\omega_0^{[ab]}$$
$$= s \left[ -2\text{sym}(A^a C^b) \omega_0^{[ab]} \right] + \text{sym}(C^a C^b) d\omega_0^{[ab]}.$$

(5.30)

We consider the special configuration $c^a(n) \rightarrow \text{const.}$. As eq. (5.20), the consistency of eq. (5.30) requires

$$d\omega_0^{[ab]} = 0, \quad s \left[ \chi_1^1 + 2\text{sym}(A^a C^b) \omega_0^{[ab]} \right] = 0.$$

(5.31)

The general solution to the first equation is $\omega_0^{[ab]} = B_0^{[ab]}$ and then the second equation implies (by the BRS cohomology) $\chi_1^1 = -2\text{sym}(A^a C^b) B_0^{[ab]} + C^a \omega_1^0 \{F_i\}$. Substituting these into the first relation of eq. (5.26), we have

$$s\chi_2^0 = s \left[ \text{sym}(A^a A^b) B_0^{[ab]} - A^a \omega_1^a \right] - 2\text{sym}(F^a C^b) B_0^{[ab]} - C^a d\omega_1^a.$$

(5.32)

We again consider the configuration $c^a(n) \rightarrow \text{const.}$. Then eq. (5.32) requires

$$2F^a B_0^{[ab]} + d\omega_1^b = 0, \quad s \left[ \chi_2^0 - \text{sym}(A^a A^b) B_0^{[ab]} + A^a \omega_1^a \right] = 0.$$

(5.33)

In deriving the first relation, we have noted the fact that the constant ghost form $C^b$ and the 2-form $F^a$ simply commute and thus $C^b$ can be factored out from the equation. We next consider a configuration $F^a_{\mu}(n) \rightarrow \text{const.}$ Since $\omega_1^a$ depends only on the field strength, we see that the constant $B_0^{[ab]}$ must vanish for the consistency of eq. (5.33) and thus

$$d\omega_1^a = 0 \Rightarrow \omega_1^a = d\omega_1^a \{F_i\} + B_1^a.$$

(5.34)

by the present lemma for $p = 1$. Substituting this into the second relation of eq. (5.33) and then by using the first equation (5.25), we have

$$\alpha_3 = -F^a d\omega_0^a - F^a B_1^a + d\omega_2 + B_3$$
$$= d(-F^a \omega_0^a + \omega_2) - F^a B_1^a + B_3,$$

(5.35)

where we have used the Bianchi identity $dF^a = 0$. This shows the lemma for $p = 3$.

* Later, we apply the covariant Poincaré lemma to the case II above, by regarding components of the nonabelian gauge potential $A^a_{\mu}(n) = \sum_a A^a_{\mu}(n) T^a$ as if the abelian gauge potential. In this case, it is impossible to make $F^a_{\mu}(n) = \text{const.}$ while keeping the range of $A^a_{\mu}(n)$ as eq. (2.9). However, it is possible to make $F^a_{\mu}(n) = \text{const.} = O(1/R)$ inside of a block of the size $R$. The term $d\omega_1^a$ then behaves as $\sim \exp(-aR)$ because the dependence of $\omega_1^a$ is local. Since the first relation of eq. (5.33) holds for an arbitrary $R$, this implies that each terms have to vanish separately.
For $p = 4$. Similarly as the above cases, we have
\[ \alpha_4 = d\chi_3^0 + B_4, \quad s\alpha_4 = 0, \] (5.36)
and
\[ s\chi_3^0 = d\chi_2^1, \quad s\chi_2^1 = d\chi_1^2, \quad s\chi_1^2 = d\chi_0^3, \quad s\chi_0^3 = 0. \] (5.37)

The solution to the last equation is given by $\chi_0^3 = \text{sym}(C^a C^b C^c)\omega_0^{[abc]} \{ F_i \}$ and then the third relation of eq. (5.37) reads
\[ s\chi_1^2 = s \left[ -3\text{sym}(A^a C^b C^c)\omega_0^{[abc]} \right] - \text{sym}(A^a C^b C^c)d\omega_0^{[abc]}, \] (5.38)
The consistency for $c^a(n) \rightarrow \text{const.}$ requires $\omega_0^{[abc]} = B_0^{[abc]} \text{(const.)}$ and thus
\[ \chi_1^2 = -3\text{sym}(A^a C^b C^c)B_0^{[abc]} + \text{sym}(C^a C^b)\omega_1^{[ab]} \{ F_i \}, \] (5.39)
and the second equation of eq. (5.37) becomes
\[ s\chi_2^1 = s \left[ -3\text{sym}(A^a A^b C^c)B_0^{[abc]} - 2\text{sym}(A^a C^b)\omega_1^{[ab]} \right] \\
- 3\text{sym}(F^a C^b C^c)B_0^{[abc]} + \text{sym}(C^a C^b)d\omega_1^{[ab]} \] (5.40)
Setting $c^a(n) \rightarrow \text{const.}$ in this equation and then setting $F_\mu^a(n) \rightarrow \text{const.}$, we see that $B_0^{[abc]} = 0$ and $d\omega_1^{[ab]} = 0$. The present lemma for $p = 1$ then asserts that $\omega_1^{[ab]} = d\omega_0^{[ab]} \{ F_i \} + B_1^{[ab]}$. The general structure of $\chi_2^1$ is therefore given by
\[ \chi_2^1 = -2\text{sym}(A^a C^b)(d\omega_0^{[ab]} + B_1^{[ab]}) + C^a \omega_2^{[a]} \{ F_i \}. \] (5.41)
Substituting this into the first relation of eq. (5.37), we have
\[ s\chi_3^0 = s \left[ \text{sym}(A^a A^b)(d\omega_0^{[ab]} + B_1^{[ab]}) - A^a \omega_2^a \right] \\
- 2\text{sym}(F^a C^b)(d\omega_0^{[ab]} + B_1^{[ab]}) - C^a d\omega_2^a. \] (5.42)
The consistency for $c^a(n) \rightarrow \text{const.}$ requires
\[ 2F^a(d\omega_0^{[ab]} + B_1^{[ab]}) + d\omega_2^b = 0, \] (5.43)
and the consistency for $F_\mu^a(n) \rightarrow \text{const.}$,
\[ B_1^{[ab]} = 0, \quad 2F^a d\omega_0^{[ab]} + d\omega_2^b = 0. \] (5.44)
The last equation can be written as $d(\omega_2^a - 2F^b \omega_0^{[ab]}) = 0$ and then the present lemma
for $p = 2$ asserts that

$$
\omega_2^a = 2 F^b \omega_0^{[ab]} + d \omega_1^a \{ [F_i] \} + B_2^a + F^b B_0^{ab}.
$$

(5.45)

Note that $B_0^{ab}$ is not necessarily symmetric under $a \leftrightarrow b$ at this stage. From this, it is not difficult to see that eq. (5.42) yields

$$
s \chi_3^0 = \left[ s \left[ \text{sym}(A^a A^b) \omega_0^{[ab]} \right] + s \left[ -A^a (d \omega_1^a + B_2^a + F^b B_0^{ab}) \right] \right.

\left. + d \left\{ 2 \left[ \text{sym}(F^a C^b) + C^a F^b \right] \omega_0^{[ab]} \right\}\right].
$$

(5.46)

We now arrived at the final stage which requires a special consideration. In eq. (5.46), the last term of the right hand side is not manifestly $s$-exact. So define

$$
\varphi_2^1 = 2 \left[ \text{sym}(F^a C^b) + C^a F^b \right] \omega_0^{[ab]} = (C^a F^b - F^b C^a) \omega_0^{[ab]}.
$$

(5.47)

In the context of ordinary differential calculus, $\varphi_2^1$ identically vanishes because $C^a$ and $F^b$ commute with each other. However we cannot simply throw away $\varphi_2^1$ in the context of noncommutative differential calculus. We first note $s \varphi_2^1 = 0$. Also, when $c^a(n) \to \text{const.}$, $C^a$ and $F^b$ commute and $\varphi_2^1 = 0$ as noted above. Therefore $\varphi_2^1 \propto \Delta \mu c^a$. But these facts combined with the BRS cohomology (5.10) show that $\varphi_2^1$ is $s$-trivial, $\varphi_2^1 = s Y_2$ (actually, otherwise eq. (5.46) becomes inconsistent). In fact, by noting the noncommutative rule (4.4), one finds

$$
\varphi_2^1 = - [\Delta \mu c^a(n) + \Delta \nu c^a(n) + \Delta \mu \Delta \nu c^a(n)] d \theta F^b(n) \omega_0^{[ab]} = s Y_2,
$$

(5.48)

where

$$
Y_2 = - \left[ A_{\mu}^a(n) + A_{\nu}^a(n) + \Delta \nu A_{\mu}^a(n) \right] F^b(n) \omega_0^{[ab]}.
$$

(5.49)

Therefore eq. (5.46) gives

$$
\chi_3^0 = - A^a (d \omega_1^a + B_2^a + F^b B_0^{ab}) + \omega_3 \{ [F_i] \} + d \left[ \text{sym}(A^a A^b) \omega_0^{[ab]} - Y_2 \right],
$$

(5.50)

and from the first equation (5.36), we have

$$
\alpha_4 = d(- F^a \omega_1^a + \omega_3) - F^a B_2^a - F^a F^b B_0^{ab} + B_4.
$$

(5.51)

Finally, we have to show that the term proportional to the antisymmetric part of $B_0^{ab}$ under $a \leftrightarrow b$, which again vanishes in the ordinary differential calculus, can be expressed
as $d\omega_3(\{F_i\})$. We first note that $F^a F^b B_0^{[ab]} = d\varphi_3$ where

$$\varphi_3 = \frac{1}{2} (A^a F^b - F^b A^a) B_0^{[ab]} - \frac{1}{2} dY_2,$$  \hspace{1cm} (5.52)

and $Y_2$ in the second term is defined by $\omega_0^{[ab]} \rightarrow B_0^{[ab]}$ in eq. (5.49). The last term $-dY_2/2$ of course does not contribute to $F^a F^b B_0^{[ab]}$, but it makes $\varphi_3$ gauge invariant. In fact,

$$s \varphi_3 = -\frac{1}{2} (dC^a F^b - F^b dC^a) B_0^{[ab]} + \frac{1}{2} dsY_2 = 0,$$ \hspace{1cm} (5.53)

where use of eqs. (5.48) and (5.47) has been made. More explicitly, after some calculation, we have

$$\varphi_3 = \frac{1}{8} \left[ F^a_{\alpha\beta}(n) F^b_{\beta\gamma}(n) + F^a_{\alpha\beta}(n + \widehat{\gamma}) F^b_{\beta\gamma}(n) ight.$$

$$\left. + F^a_{\alpha\beta}(n) F^b_{\beta\gamma}(n + \widehat{\alpha}) + F^a_{\alpha\beta}(n + \widehat{\gamma}) F^b_{\beta\gamma}(n + \widehat{\alpha}) \right] dx_\alpha dx_\beta dx_\gamma B_0^{[ab]}.$$ \hspace{1cm} (5.54)

This establishes the lemma for $p = 4$.

If one repeats the above argument for $p = 5$ (assuming that the dimension of lattice is greater than 4), the treatment becomes much more involved due to the noncommutativity of forms. Because of this, we could not find an iterative formula for $\alpha_p$ with general $p$, unlike the treatment in the continuum theory [16]. This fact suggests that our noncommutative differential calculus is not powerful enough and there exists another hidden algebraic structure. This is an interesting problem although we do not investigate it here. Of course, the proof in this subsection is sufficient for applications in 4-dimensional lattice.

5.4. TOPOLOGICAL FIELDS IN THE ABELIAN THEORY

Once the above three lemmas are established, it is straightforward to show the following theorem which generalizes the theorem of ref. [19] to multi-$U(1)$ cases.

**Theorem.** Let $q(n)$ be a gauge invariant field on 4-dimensional infinite hypercubic lattice that depends smoothly and locally on the abelian gauge potential $A^a_\mu$. Suppose that $q(n)$ is topological, namely

$$\sum_n \delta q(n) = 0,$$ \hspace{1cm} (5.55)

for an arbitrary local variation of the gauge potential. Then $q(n)$ is of the form

$$q(n) = \alpha + \beta^a_{\mu\nu} F^a_{\mu\nu}(n) + \gamma^{(ab)} \varepsilon_{\mu\nu\rho\sigma} F^a_{\mu\nu}(n) F^b_{\rho\sigma}(n + \widehat{\mu} + \widehat{\nu}) + \Delta^*_{\mu} k_\mu(n),$$ \hspace{1cm} (5.56)

where the current $k_\mu(n)$ depends smoothly and locally only on the field strength and thus is gauge invariant.
Proof. We multiply the volume form $d^4x$ to $q(n)$ and define the 4-form $Q_4 = q d^4 x$. $Q_4$ is gauge invariant $sQ_4 = 0$ and thus, from the BRS cohomology (5.10), $Q_4 = Q_4({\{F_i\}})$. From the algebraic Poincaré lemma (5.1), on the other hand, $Q_4 = d\chi_3 + B_4$ because the 4-form $Q_4$ is topological $\sum_n \delta Q_4(n) = 0$ from the assumption (5.55). From these, we can apply the covariant Poincaré lemma (5.11) to $Q_4$ which yields

$$q(n)d^4x = Q_4(n) = B_4 + F^a(n)B^a_2 + F^a(n)F^b(n)B_0^{(ab)} + d\alpha_3(n).$$  \hspace{1cm} (5.57)$$

Finally we factor out the volume form $d^4x$ from the both sides of this equation. Noting the noncommutative rule (4.4), we have eq. (5.56).

Note. The third term of eq. (5.56) is a total difference on lattice and thus in fact satisfies the topological property (5.55):

$$\varepsilon_{\mu
u\rho\sigma}F_{\mu\nu}(n)F_{\rho\sigma}(n + \hat{\mu} + \hat{\nu}) = 4\varepsilon_{\mu
u\rho\sigma}\Delta_{\mu}[A^a_{\mu}(n)\Delta_{\rho}A^b_{\sigma}(n + \hat{\nu})].$$  \hspace{1cm} (5.58)$$

This relation can easily be derived from the relation $F^aF^b = d(A^aF^b)$ valid in the context of noncommutative differential calculus. See also ref. [32].

6. Nontrivial anomalies in the abelian theory

Because of the nilpotency $\delta_B^2 = 0$, any functional of the form $A = \delta_B B$ is a solution to eq. (1.3). If the functional $B$ is local, such an anomaly can be removed by the redefinition of the effective action $\ln \det M' \rightarrow \ln \det M' - B$ which does not change the physical content of the theory. Therefore the solution to the consistency condition (1.3) of the form $A = \delta_B B$ with a local functional $B$ will be referred as trivial or BRS trivial.

6.1. Nontrivial local solutions

In this subsection, we study the structure of local solutions to the consistency condition (1.3) in the abelian theory $G = U(1)^N$. The BRS transformation is given by eq. (4.7). The ghost number of the solution is not restricted. We will find a very close analogue to the solutions in the continuum theory [16].

We seek the solution $A$ by regarding $A$ as a smooth and local functional of the gauge potential $A^a_{\mu}$ and the ghost field $c^a$. Since eq. (1.3) must hold for arbitrary configurations of $A^a_{\mu}$ and $c^a$, we have the variational equation

$$\delta\delta_B A = \sum_n \left\{ \delta A^a_{\mu}(n) \frac{\partial A}{\partial A^a_{\mu}(n)} - \delta c^a(n) \left[ \delta B \frac{\partial A}{\partial c^a(n)} + \Delta^* \frac{\partial A}{\partial A^a_{\mu}(n)} \right] \right\} = 0,$$  \hspace{1cm} (6.1)$$

where $\delta\delta_B A^a_{\mu} = \Delta_{\mu}\delta c^a$ and $\delta\delta_B c^a = 0$ have been used. The coefficients of the variations
\[ \delta A^{a}_{\mu}(n) \text{ and } \delta c^{a}(n) \text{ have to vanish separately:} \]

\[ \delta_B \frac{\partial A}{\partial A^{a}_{\mu}(n)} = 0, \quad \delta_B \frac{\partial A}{\partial c^{a}(n)} + \Delta^{*}_{\mu} \frac{\partial A}{\partial A^{a}_{\mu}(n)} = 0. \quad (6.2) \]

Since \( A \) is local, \( \partial A/\partial A^{a}_{\mu}(n) \) is a local field. Then the abelian BRS cohomology \((5.10)\) gives the general solution to the first equation with the ghost number \( g \),

\[ \frac{\partial A}{\partial A^{a}_{\mu}(n)} = c^{a_{1}}(n) \ldots c^{a_{g}}(n) \omega^{a_{1} \ldots a_{g}}(n) + \delta B Y^{a}(n), \quad (6.3) \]

where \( \omega^{a_{1} \ldots a_{g}} \) depends only on the field strength. The second relation of eq. \((6.2)\) then reads,

\[ \delta_B \left[ \frac{\partial A}{\partial c^{a}(n)} + \Delta^{*}_{\mu} Y^{a}(n) \right] = -\Delta^{*}_{\mu} \left[ c^{a_{1}}(n) \ldots c^{a_{g}}(n) \omega^{a_{1} \ldots a_{g}}(n) \right] \]

\[ = -c^{a_{1}}(n) \ldots c^{a_{g}}(n) \omega^{a_{1} \ldots a_{g}}(n) + c^{a_{1}}(n - \tilde{\mu}) \ldots c^{a_{g}}(n - \tilde{\mu}) \omega^{a_{1} \ldots a_{g}}(n - \tilde{\mu}) \]

\[ = \delta_B \left[ -\sum_{i=1}^{g} \binom{g}{i} A^{a_{1}}(n - \tilde{\mu}) \delta_{B} A^{a_{2}}(n - \tilde{\mu}) \ldots \delta_{B} A^{a_{i}}(n - \tilde{\mu}) \right. \]

\[ \times c^{a_{i+1}}(n - \tilde{\mu}) \ldots c^{a_{g}}(n - \tilde{\mu}) \omega^{a_{1} \ldots a_{g}}(n) \bigg] \]

\[ - c^{a_{1}}(n - \tilde{\mu}) \ldots c^{a_{g}}(n - \tilde{\mu}) \Delta^{*}_{\mu} \omega^{a_{1} \ldots a_{g}}(n), \quad (6.4) \]

where from the second line to the third line, we have used \( c^{a}(n) = c^{a}(n - \tilde{\mu}) + \delta_{B} A^{a}_{\mu}(n - \tilde{\mu}). \)

Considering the consistency of the above equation under \( c^{a}(n) \rightarrow \text{const.} \), we have

\[ \Delta^{*}_{\mu} \omega^{a_{1} \ldots a_{g}}(n) = 0, \quad (6.5) \]

and then again from the BRS cohomology,

\[ \frac{\partial A}{\partial c^{a}(n)} = -\Delta^{*}_{\mu} Y^{a}(n) - \sum_{i=1}^{g} \binom{g}{i} A^{a_{1}}(n - \tilde{\mu}) \delta_{B} A^{a_{2}}(n - \tilde{\mu}) \ldots \delta_{B} A^{a_{i}}(n - \tilde{\mu}) \]

\[ \times c^{a_{i+1}}(n - \tilde{\mu}) \ldots c^{a_{g}}(n - \tilde{\mu}) \omega^{a_{1} \ldots a_{g}}(n) \quad (6.6) \]

\[ + c^{a_{1}}(n) \ldots c^{a_{g-1}}(n) X^{a_{1} \ldots a_{g-1}}(n) + \delta B Y^{a}(n), \]

where \( X^{a_{1} \ldots a_{g-1}} \) depends only on the field strength.
Now the functional $\mathcal{A}$ can be reconstructed from its variations (6.3) and (6.6). We introduce $\mathcal{A}_t$ by rescaling variables as $A^a_\mu \rightarrow tA^a_\mu$ and $c^a \rightarrow tc^a$. Noting $\mathcal{A}_{t=0} = 0$ for $g > 0$, we have

$$
\mathcal{A} = \int_0^1 dt \frac{\partial \mathcal{A}_t}{\partial t} = \int_0^1 dt \sum_n \left[ A^a_\mu(n) \frac{\partial \mathcal{A}_t}{\partial t A^a_\mu(n)} + c^a(n) \frac{\partial \mathcal{A}_t}{\partial t c^a(n)} \right].
$$

(6.7)

After substituting eqs. (6.3) and (6.6) and shifting the coordinate $s \rightarrow s + \mu$, this yields

$$
\mathcal{A} = \sum_n \left\{ \left[ A^a_\mu(n) c^{a_1}(n) \cdots c^{a_s}(n) 
- c^{a_0}(n + \mu) \sum_{i=1}^g \left( \frac{g}{i} \right) A^a_\mu(n) \delta_B A^{a_2}_\mu(n) \cdots \delta_B A^{a_{i+1}}_\mu(n) c^{a_{i+1}}(n) \cdots c^{a_s}(n) \right] 
\times \tilde{\omega}^{a_0[a_1 \cdots a_s]}_\mu(n) \right. 
+ c^{a_1}(n) \cdots c^{a_s}(n) \tilde{X}^{[a_1 \cdots a_s]}(n) \left. \right\} 
+ \delta_B \sum_n \left[ A^a_\mu(n) \tilde{Y}^a_\mu(n) - c^a(n) \tilde{Y}^a(n) \right],
$$

(6.8)

where the following abbreviations have been introduced

$$
\tilde{\omega}^{a[a_1 \cdots a_s]}_\mu = \int_0^1 dt \, t^{g-1} \omega^{a[a_1 \cdots a_s]}_\mu, \quad \tilde{X}^{[a_1 \cdots a_s]}(n) = \int_0^1 dt \, t^{g-1} X^{[a_1 \cdots a_s]}(n),
$$

(6.9)

Note that $\Delta^{a_0} \tilde{\omega}^{a[a_1 \cdots a_s]}_\mu = 0$ from eq. (6.5) and all these fields are local from the above construction. In particular, $\tilde{\omega}^{a[a_1 \cdots a_s]}_\mu$ and $\tilde{X}^{[a_1 \cdots a_s]}(n)$ depend only on the field strength.

Eq. (6.8) provides the most general local solutions to the consistency condition. Yet it contains trivial solutions in various ways. First, by noting $c^{a_0}(n + \mu) = c^{a_0}(n) + \delta_B A_{a_0}^a(n)$ and $\delta_B A_{a_2}^a(n) = c^{a_2}(n + \mu) - c^{a_2}(n)$, it is easy to see that the symmetric part of $\tilde{\omega}^{a_0[a_1 \cdots a_s]}_\mu$ on $a_0 \leftrightarrow a_1$ contributes only to a BRS trivial part:

$$
\left[ A^a_\mu(n) c^{a_1}(n) \cdots c^{a_s}(n) - g c^{a_0}(n + \mu) A^a_\mu(n) c^{a_2}(n) \cdots c^{a_s}(n) \right] \tilde{\omega}^{(a_0 a_1)}_{a_2 \cdots a_s}(n) \tilde{Y}^a(n)
= \delta_B \left[ -\frac{g}{2} A^a_\mu(n) A_{a_1}^a(n) c^{a_2}(n) \cdots c^{a_s}(n) \tilde{\omega}^{(a_0 a_1)}_{a_2 \cdots a_s}(n) \tilde{Y}^a(n) \right],
$$

(6.10)

* For $g = 0$, the following expressions hold by simply adding a constant $\mathcal{A}_{t=0}$ to $\mathcal{A}$.  

26
\[ c^{a_0}(n + \hat{\mu}) A^{a_1}_\mu(n) \delta_B A^{a_2}_\mu(n) \cdots \delta_B A^{a_i}_\mu(n) c^{a_{i+1}}(n) \cdots c^{a_s}(n) \tilde{\omega}^{(a_0 a_1) a_2 \cdots a_s}_\mu(n) \]
\[ = \delta_B \left[ -\frac{1}{2} A^{a_0}_\mu (n) A^{a_1}_\mu (n) c^{a_2}(n) \delta_B A^{a_3}_\mu (n) \cdots \delta_B A^{a_i}_\mu (n) c^{a_{i+1}}(n) \cdots c^{a_s}(n) \tilde{\omega}^{(a_0 a_1) a_2 \cdots a_s}_\mu(n) \right]. \]  

(6.11)

Therefore, for nontrivial solutions, we can assume that \( \tilde{\omega}^{a_0[a_1 \cdots a_s]}_\mu \) is antisymmetric under the exchange \( a_0 \leftrightarrow a_1 \), namely, \( \tilde{\omega}^{a_0[a_1 \cdots a_s]}_\mu \) is totally antisymmetric \( \tilde{\omega}^{a_0 \cdots a_s} = \tilde{\omega}^{a_0[a_1 \cdots a_s]}_\mu \) in nontrivial solutions.

Henceforth we use the symbol \( \simeq \) to indicate the equivalence relation modulo BRS trivial parts. The last term of eq. (6.8) is BRS trivial. Also, as noted above, \( \tilde{\omega}^{a_0 \cdots a_s} \) is totally antisymmetric in nontrivial solutions. Then inserting \( \delta_B A^{a_i}_\mu (n) = c^{a_j}(n + \hat{\mu}) - c^{a_j}(n) \) to eq. (6.8), after some rearrangements, we have the following relatively simple expression

\[ \mathcal{A} \simeq \sum_n \left[ \sum_{k=0}^g c^{a_1}(n) \cdots c^{a_k}(n) A^{a_0}_\mu(n) c^{a_{k+1}}(n + \hat{\mu}) \cdots c^{a_s}(n + \hat{\mu}) \tilde{\omega}^{(a_0 \cdots a_s)}_\mu(n) \right. \]
\[ + \left. c^{a_1}(n) \cdots c^{a_s}(n) \tilde{X}^{[a_1 \cdots a_s]}(n) \right]. \]  

(6.12)

This expression takes a particularly simple form in terms of the noncommutative differential calculus. We introduce the dual 3-form of \( \tilde{\omega}^{a_0 \cdots a_s}_\mu \) by

\[ \tilde{\omega}^{[a_0 \cdots a_s]}_\mu(n) = \frac{1}{3!} \varepsilon_{\mu \rho \sigma} \frac{(-1)^g}{g} \Omega^{[a_0 \cdots a_s]}_\nu(n + \hat{\mu}). \]  

(6.13)

Then by using the noncommutative rule (4.4), it is easy to see that

\[ \mathcal{A} d^4 x (d\theta)^g \simeq \sum_n \left[ \text{sym} (A^{a_0} C^{a_1} \cdots C^{a_s}) \Omega^{[a_0 \cdots a_s]} + C^{a_1} \cdots C^{a_s} \tilde{X}^{[a_1 \cdots a_s]} d^4 x \right]. \]  

(6.14)

On the other hand, the divergence-free condition (6.5) becomes

\[ d\Omega^{[a_0 \cdots a_s]} = 0. \]  

(6.15)

We can now apply the covariant Poincaré lemma (5.11) to the 3-form \( \Omega^{[a_0 \cdots a_s]} \) because it depends only on the field strength. This yields

\[ \Omega^{[a_0 \cdots a_s]} = d\alpha^{[a_0 \cdots a_s]}_2 [\{ F_1 \}] + B_3^{[a_0 \cdots a_s]} + F^b B_1^{[a_0 \cdots a_s]} b. \]  

(6.16)

But the contribution of \( \alpha^{[a_0 \cdots a_s]}_2 \) can be absorbed into the second term of eq. (6.14) up to a
trivial part, because

\[
\sum_n \text{sym}(A^a_0 C^{a_1} \cdots C^{a_q}) d\alpha^{[a_0 \cdots a_q]}_2
\]

\[
= \sum_n (-1)^g \text{sym}(F^{a_0} C^{a_1} \cdots C^{a_q}) \alpha^{[a_0 \cdots a_q]}_2
\]

\[
+ s \sum_n \left[ (-1)^{g+1} \frac{1}{2} \text{sym}(A^a_0 A^{a_1} C^{a_2} \cdots C^{a_q}) \alpha^{[a_0 \cdots a_q]}_2 \right].
\]

(6.17)

Similarly, from the covariant Poincaré lemma, we have

\[
\tilde{X}^{|[a_1 \cdots a_q]} d^4 x
\]

\[
= d\alpha^{[a_1 \cdots a_q]}_3 \{F_i\} + L^{|[a_1 \cdots a_q]} d^4 x + B^{|[a_1 \cdots a_q]}_4 + F^b B^{|[a_1 \cdots a_q]}_2 + F^b F^c B^{|[a_1 \cdots a_q]}_0 (bc),
\]

(6.18)

but it is easy to see that \(\alpha^{[a_1 \cdots a_q]}_3\) does not contribute to the nontrivial part.

So, up to this stage, we have obtained

\[
\mathcal{A} d^4 x (d\theta)^g \simeq \sum_n \left[ C^{a_1} \cdots C^{a_q} L^{|[a_1 \cdots a_q]} d^4 x
\right.
\]

\[
+ C^{a_1} \cdots C^{a_q} (B_4^{|[a_1 \cdots a_q]} + F^b B^{|[a_1 \cdots a_q]}_2 + F^b F^c B^{|[a_1 \cdots a_q]}_0 (bc)
\]

\[
+ \text{sym}(A^a_0 C^{a_1} \cdots C^{a_q}) (B_3^{|[a_0 \cdots a_q]} + F^b B^{|[a_0 \cdots a_q]}_1 (bc))
\]

(6.19)

where \(\sum_n \delta L^{|[a_1 \cdots a_q]} \neq 0\) under a certain local variation of the gauge potential. Formally this expression is identical to the list of nontrivial solutions in the continuum theory (see eq. (6.24) of the second reference of ref. [16]). Recall however that eq. (6.19) is an expression in the context of noncommutative differential calculus and it is valid for a finite lattice spacing \(a \neq 0\).

It is easy to see that eq. (6.19) satisfies \(\delta_B \mathcal{A} = 0\). Doesn’t eq. (6.19) contain BRS trivial parts anymore? \(\delta_B\) (something) is always proportional to a difference of the ghost field such as \(\Delta_{\mu} C^a\). But this does not necessarily imply that all terms of eq. (6.19) are BRS nontrivial. In contrast to the BRS cohomology (5.10), this expression contains the summation \(\sum_n\). Therefore, after the “integration by parts”, a difference of the ghost field may be resulted.

The term proportional to \(B_2\) in fact contains BRS trivial parts. Namely, by noting \(dC^a = -sA^a\), we have

\[
\sum_n C^{a_1} \cdots C^{a_q} F^b B_2^{[a_1 \cdots a_q]} b \simeq \sum_n \text{sym}(C^{a_1} \cdots C^{a_q} dA^b) B_2^{[a_1 \cdots a_q]} b
\]

\[
= \sum_n (-1)^g g \text{sym}(sA_1 A^{a_2} \cdots C^{a_q} A^b) B_2^{[a_1 \cdots a_q]} b
\]

\[
\simeq - \sum_n g \text{sym}(A^{a_1} C^{a_2} \cdots C^{a_q} sA^b) B_2^{[a_1 \cdots a_q]} b
\]

\[
= \sum_n (-1)^g g \text{sym}(sA^b C^{a_2} \cdots C^{a_q} A^{a_1}) B_2^{[a_1 \cdots a_q]} b.
\]

(6.20)
Note that the commutator of $C^a$ and the field strength 2-form $F^b$ is proportional to a difference of the ghost field and thus, from the BRS cohomology, it is BRS trivial. Therefore the ordering of $C^a$ and $F^b$ is arbitrary in the first expression of eq. (6.20) up to BRS trivial parts. We have used this fact for the first $\simeq$ equality. By comparing the second line and the fourth line of the above expression, we see that eq. (6.20) is equivalent to

$$
\sum_n C^b C^{a_2} \cdots C^{a_n} F^{a_1} B_2^{[a_1 \cdots a_n]b} = \sum_n C^{a_1} \cdots C^{a_n} F^b B_2^{[ba_2 \cdots a_n]a_1}.
$$

(6.21)

A comparison with the left hand side of eq. (6.20) shows that the antisymmetric part of $B_2^{[a_1 \cdots a_n]b}$ under $a_1 \leftrightarrow b$ is BRS trivial $\simeq 0$. Therefore $B_2^{[a_1 \cdots a_n]b}$ must be symmetric under $a_1 \leftrightarrow b$ to contribute nontrivial solutions.

Similarly, we have (suppressing $\sum_n$)

$$
C^{a_1} \cdots C^{a_n} F^b F^c B_0^{[a_1 \cdots a_n](bc)} \simeq \text{sym}(C^{a_1} \cdots C^{a_n} F^b) F^c B_0^{[a_1 \cdots a_n](bc)}
\simeq C^b C^{a_2} \cdots C^{a_n} F^{a_1} F^c B_0^{[a_1 \cdots a_n](bc)},
$$

(6.22)

and therefore $B_0^{[a_1 \cdots a_n](bc)}$ must be symmetric under $a_1 \leftrightarrow b$.

The term proportional to $B_1$ also might contain BRS trivial parts depending on symmetry of indices. However, the noncommutativity prevented us to imitate the procedure in the continuum theory [16].

Let us summarize the result: The general structure of local solutions to the consistency condition eq. (1.3) is given by eq. (6.19). The constant forms $B_2$ and $B_0$ have the following symmetries:

$$
B_2^{[a_1 \cdots a_n]b} = B_2^{[ba_2 \cdots a_n]a_1}, \quad B_0^{[a_1 \cdots a_n](bc)} = B_0^{[ba_2 \cdots a_n](a_1c)}.
$$

(6.23)

The solution (6.19) is nontrivial, i.e., it cannot be written as $\mathcal{A} = \delta_B \mathcal{B}$ by using a local functional $\mathcal{B}$. The classical continuum limit of eq. (6.19) with eq. (6.23) coincides with the nontrivial solutions in the continuum theory [16] (with a partial exception for $B_1^{[a_0 \cdots a_n]b}$ mentioned above). Then if eq. (6.19) was BRS trivial, the classical continuum limit of the local functional $\mathcal{B}$ would counter the nontrivial solutions in the continuum theory. But this contradicts with the result of ref. [16].

In this subsection, we obtained the general nontrivial local solutions with an arbitrary ghost number. For discussions of the gauge anomaly in the next subsection, knowledge of ghost number one solutions is enough. The solutions with higher ghost number, however, might become relevant in future applications. For example, it might be possible to address the commutator anomaly [60] in the context of lattice gauge theory starting with the above expressions.

* The information about these becomes important [16] when one explicitly computes the higher order sequence $\mathcal{A}_\ell$ with $\ell \geq 4$ for the nonabelian anomaly, although we do not pursue this in this paper. See the discussion in sec. 7.2. 
6.2. Gauge anomaly in Abelian theory

If we restrict solutions with the ghost number unity, eq. (6.19) tells us that

$$\mathcal{A} \simeq \sum_n \left\{ c^a(n) \mathcal{L}^a(n) + c^a(n) \left[ \alpha^a + \beta_{[\mu\nu]}^{(ab)} F_{\mu\nu}(n) + \gamma^{(abc)} \varepsilon_{\mu\nu\rho\sigma} F_{\mu\nu}^b(n) F_{\rho\sigma}^c(n + \hat{\mu} + \hat{\nu}) \right] \right\}$$

where we have used the noncommutative rule (4.4) and the symmetry of indices (6.23). In this expression, the function $\mathcal{L}^a(n)$ satisfies $\sum_n \delta \mathcal{L}^a(n) \neq 0$ for a certain local variation.

Eq. (6.24) provides the general candidate of nontrivial local gauge anomalies in the abelian theory $G = U(1)^N$. However, depending on the situation, we may further restrict the coefficients in various ways.

1. When $G = U(1)$, the last line vanishes due to the anti-symmetrization of indices. Eq. (6.24) then reproduces Lüscher’s result (1.1) except the “non-topological term” $\mathcal{L}^a$.

2. The non-topological term $\mathcal{L}^a$ and the term proportional to $f_{[ab]}^\mu$ never appear in the gauge anomaly of the actual system. In the actual system, the gauge anomaly is defined from a fermionic determinant by $\mathcal{A} = \delta_B \ln \det M'[A]$. Since $\delta_B A^a_\mu(n) \to 0$ for $c^a(n) \to \text{const}$, in the abelian theory, $\mathcal{A}$ vanishes as $c^a(n) \to \text{const}$. This limit might be dangerous† but it is enough to assume that $\delta \mathcal{A}[c, A] = 0$ for $c^a(n) \to \text{const}$., where $\delta$ is an arbitrary local variation of the gauge potential. It is obvious that the non-topological term $\mathcal{L}^a$ and the term proportional to $f_{[ab]}^\mu$ do not satisfy this criterion.

3. If the couplings of the Weyl fermion to gauge fields have the same structure for all $U(1)$ factors except coupling constants (practically this is always the case), then all the coefficients are independent of group indices and we have $\alpha^a \to \alpha$, $\beta_{[\mu\nu]}^{(ab)} \to \beta_{[\mu\nu]}$, $\gamma^{(abc)} \to \gamma$, $f_{\mu}^{[ab]} \to 0$, $g_{[\mu\nu\rho]}^{[abc]} \to 0$.

4. From the dimensional counting, all the terms except $\mathcal{L}^a$ and the term proportional to $\gamma^{(abc)}$ have negative powers of the lattice spacing as the overall coefficient. Therefore, if the classical continuum limit $a \to 0$ of $\mathcal{A}$ is finite (for a smooth background), all the terms except $\mathcal{L}^a$ and $\gamma^{(abc)}$ must be absent. In particular, if $\lim_{a \to 0} \mathcal{A} \text{reproduces the gauge anomaly in the continuum theory, } \gamma^{(abc)} = -e_H/(96\pi^2)$ for a single Weyl fermion.

Physically, we are always interested in cases for which (2) and (4) hold. So let us accept these. Then we have a content of the theorem for the abelian gauge theory which we stated in section 3.

† In fact, when the lattice volume is finite, the index is obtained in this limit as $\mathcal{A} \to c^a \times \text{(index)}$ up to the proportionality constant.
In the section, we study the gauge anomaly for a general (compact) gauge group \( G = \prod_\alpha G_\alpha \), where \( G_\alpha \) is a simple group or a \( U(1) \) factor. The candidate of the anomaly is given by the ghost number one solution to the consistency condition (1.3). The general solution to eq. (1.3) is expressed as

\[
\mathcal{A} = \tilde{\mathcal{A}} + \delta_B \mathcal{B},
\]

(7.1)

where \( \tilde{\mathcal{A}} \neq \delta_B \mathcal{B} \) is the BRS nontrivial part (\( \mathcal{B} \) is a local functional). The BRS transformation is given by eq. (1.2) and it takes the following form in terms of the gauge potential

\[
\delta_B A_\mu(n) = \frac{1}{2} A_\mu(n) \wedge \left[ \coth \frac{1}{2} A_\mu(n) \wedge \Delta_\mu c(n) + c(n) + c(n + \bar{\mu}) \right],
\]

(7.2)

where \( X \wedge Y = [X,Y], \ X^2 \wedge Y = [X,[X,Y]] \) and so on, and \( 1 \wedge Y = Y \) is understood. We shall use both the matrix notation \( A^a(n) \) and \( c(n) \), and the component notation \( A_\mu(n) = \sum_s A^a_\mu(n) T^a \) and \( c(n) = \sum_a c^a(n) T^a \).

To make our problem tractable, we set the following assumptions on the anomaly \( \mathcal{A} \).

(I) \( \mathcal{A} \) is defined from a certain effective action as \( \mathcal{A} = \delta_B \ln \text{Det} M^n \).

(II) \( \mathcal{A} \) is a smooth and local functional of the gauge potential \( A_\mu \) and the ghost field \( c \).

(III) The classical continuum limit of \( \mathcal{A} \) reproduces the anomaly in the continuum theory, eq. (2.28).

Under these assumptions, in section 7.2, we show that the anomaly \( \mathcal{A} \), if it exists, is unique (up to the BRS trivial part) to all orders of the gauge potential. The unique anomaly is proportional to the gauge anomaly in the continuum theory and this establishes the theorem for nonabelian theories, stated in section 3. In section 7.3, we show that such a solution in fact exists. As a preparation for sec. 7.2, we need the following lemma.

7.1. BASIC LEMMA: ADJOINT INVARIANCE

Adjoint invariance. Without loss of generality, one can assume that a nontrivial local solution \( \tilde{\mathcal{A}} \) is invariant under the adjoint transformation

\[
\delta^a \tilde{\mathcal{A}} = 0,
\]

(7.3)

where the adjoint transformation \( \delta^a \) is defined by

\[
\delta^a U(n, \mu) = [T^a, U(n, \mu)], \quad \delta^a A^b_\mu(n) = -i f^{abc} A^c_\mu(n), \quad \delta^a c^b(n) = -i f^{abc} c^c(n).
\]

(7.4)

Note. There is freedom to add a BRS trivial part \( \delta_B \mathcal{B} \) to a nontrivial solution. The above lemma asserts that it is always possible to choose \( \mathcal{B} \) such that \( \mathcal{A} \) is adjoint invariant. The
following relations hold for the adjoint transformation $\delta^a$

$$[\delta_B, \delta^a] = 0, \quad [\Delta^*_\mu, \delta^a] = 0, \quad [\delta^a, \delta^b] = if^{abc}\delta^c. \quad (7.5)$$

**Proof.** The functional $\tilde{A}$ is local, i.e., the field $\tilde{a}(n)$ in $\tilde{A} = \sum_n \tilde{a}(n)$ is a local field. We express $\tilde{a}(n)$ in terms of the following set of variables, which was introduced in the proof of the abelian BRS cohomology in ref. [32]:

$$A_i^a = (\Delta_1)^{p_1} \cdots (\Delta_\mu)^{p_\mu} A_\mu^a(n), \quad F_i^a = (\Delta_1)^{p_1} \cdots (\Delta_D)^{p_D} F_{\mu\nu}^a(n), \quad (7.6)$$

for the gauge potential ($D$ is the dimension of the lattice) and

$$c_i^a = \delta_0 A_i^a = (\Delta_1)^{p_1} \cdots (\Delta_\mu)^{p_\mu} \Delta_\mu c^a(n), \quad \text{and} \quad c^a(n), \quad (7.7)$$

for the ghost field, here $\delta_0$ is the abelian BRS transformation, $\delta_0 A_\mu^a(n) = \Delta_\mu c^a(n)$ and $\delta_0 c^a(n) = 0$. In these expressions, the symbol $(\Delta_\mu)^p$ ($p$ is a nonzero integer) has been defined by

$$(\Delta_\mu)^p = \begin{cases} \Delta_\mu^p, & \text{for } p > 0, \\ \Delta_\mu^{-p}, & \text{for } p < 0. \end{cases} \quad (7.8)$$

It can be shown [32] that these variables, $A_i^a, F_i^a, c_i^a$ and $c^a(n)$ span a (over)complete set, i.e., the field $\tilde{a}(n)$ can be expressed as a function of these variables. A little thought shows that the relation

$$\left[ \frac{\partial}{\partial c^a(n)}, \Delta^*_\mu \right] = 0, \quad (7.9)$$

holds on these variables.

Since the nonabelian BRS transformation (1.2) or (7.2) has the structure,

$$\begin{align*}
\delta_B A_\mu^a(n) &= if^{abc} A_\mu^b(n) c^c(n) + (\text{terms proportional to } \Delta_\mu c^a(n)), \\
\delta_B c^a(n) &= -\frac{1}{2} if^{abc} c^b(n) c^c(n),
\end{align*} \quad (7.10)$$

we have

$$\delta_B \left\{ \begin{array}{c} A_i^a \\ F_i^a \\ c_i^a \end{array} \right\} = -c^b(n) \delta^b \left\{ \begin{array}{c} A_i^a \\ F_i^a \\ c_i^a \end{array} \right\} + (\text{terms proportional to } c_i^a), \quad (7.11)$$

and thus

$$\delta^a = -\left\{ \delta_B, \frac{\partial}{\partial c^a(n)} \right\}, \quad (7.12)$$

on the variables $A_i^a, F_i^a, c_i^a$ and $c^a(n)$ (for $c^a(n)$ this follows from eq. (7.10)).
The remaining argument to show the lemma is almost identical to that of ref. [16]. We introduce the Casimir operator:

\[ \mathcal{O}_K = g^{a_1 \cdots a_m(K)} \delta^{a_1} \cdots \delta^{a_m(K)}, \]

where \( g^{a_1 \cdots a_m(K)} = \text{str} \, T^{a_1} \cdots T^{a_m(K)} \) are totally symmetric constants and \( K \) runs from 1 to the rank of the semisimple part of the group \( G \) [61]. Using the completeness of eigenfunctions of \( \mathcal{O}_K \), we decompose \( \tilde{a}(n) \) according to the representation \( \lambda \),

\[ \mathcal{O}_K \tilde{a}^\lambda(n) = k(K, \lambda) \tilde{a}^\lambda(n), \]

and \( k(K, \lambda) \) is the eigenvalue. Since \( \delta_B \tilde{A} = \sum \delta_B \tilde{a}(n) = 0 \), the dual of the algebraic Poincaré lemma (5.1) \((p = D)\) shows that

\[ \delta_B \tilde{a}(n) = \Delta^*_\mu X_\mu(n), \]

where \( X_\mu(n) \) is a local field. We again apply to this equation the decomposition similar to eq. (7.14):

\[ \delta_B \tilde{a}^\lambda(n) = \Delta^*_\mu X^\lambda_\mu(n), \]

where use of relations (7.5) has been made.

Now suppose that there exist \( K \) and \( \lambda \) such that \( k(K, \lambda) \neq 0 \) in eq. (7.14). Then, by using eqs. (7.13), (7.12), (7.5) and (7.16), we have

\[ \tilde{a}^\lambda(n) = \delta_B \frac{-1}{k(K, \lambda)} g^{a_1 \cdots a_m(K)} \delta^{a_m(K)} \cdots \delta^{a_2} \frac{\partial}{\partial c^{a_1}(n)} \tilde{a}^\lambda(n) \]

\[ + \Delta^*_\mu \frac{-1}{k(K, \lambda)} g^{a_1 \cdots a_m(K)} \delta^{a_m(K)} \cdots \delta^{a_2} \frac{\partial}{\partial c^{a_1}(n)} X^\lambda_\mu(n). \]

Namely, \( \tilde{A} \) contains a BRS trivial part \( \sum \tilde{a}^\lambda(n) \) which we remove by using \( \delta_B \mathcal{B} \). After repeating this procedure, all the eigenvalues \( k(K, \lambda) \) in eq. (7.14) are made to vanish and this implies that \( \lambda \) is the singlet representation. Therefore, we can always assume that a BRS nontrivial solution is adjoint invariant \( \delta^a \tilde{A} = 0 \).

We next derive a constraint for \( \tilde{A} \) following from the assumptions (I) made above and the lemma (7.3). Set \( c^a(n) \to c^a = \text{const} \). Since the ghost number of \( \tilde{A} \) is unity, we can write as

\[ \tilde{A} = c^a k^{a \lambda} X^\lambda[A], \]

where \( k^{a \lambda} \) are constants and \( \lambda \) labels linearly independent functional \( X^\lambda[A] \). We have to then consider the following two cases separately:
(1) When the index \(a\) of the ghost field in eq. (7.18) is belonging to a \(U(1)\) factor group \(U(1)_a\). In this case, we have
\[
\tilde{A} = c^{U(1)_a} X[A],
\] (7.19)
but since \(\delta_B A^{U(1)_a}_\mu(n) = \Delta_\mu c^{U(1)_a} = 0\), we have \(\delta_B B = 0\) in eq. (7.1) and, from the assumption (I), \(\tilde{A} = \mathcal{A} = \delta_B \ln \mathrm{Det} M''[A] = 0\) in the limit \(c^{U(1)_a} (n) \rightarrow \text{const.}\). This limit might be dangerous as noted at the end of section 6, but the condition \(\delta \tilde{A} = 0\), where \(\delta\) is an arbitrary local variation of the gauge potential, is enough for the following argument.

(2) When the index \(a\) of the ghost field in eq. (7.18) is belonging to a simple group. The BRS transformation (1.2) or (7.2) becomes for \(c^a(x) \rightarrow c^a = \text{const.}\),
\[
\delta_B A^a_\mu(n) = -i f^{abc} c^b A^a_\mu(n) = -c^b \delta^b A^a_\mu(n),
\]
\[
\delta_B c^a = -\frac{1}{2} i f^{abc} c^b c^c = -\delta_B c^a - c^b \delta^b c^a,
\] (7.20)
where \(\delta^a\) is the adjoint transformation. But since the lemma (7.3) asserts that \(\delta^a \tilde{A} = 0\), the consistency condition becomes for \(c^a(n) \rightarrow \text{const.}\),
\[
\delta_B \tilde{A} = -\delta_B (c^a k^{a\lambda}) X^\lambda[A] = 0.
\] (7.21)
This requires \(\delta_B (c^a k^{a\lambda}) = 0\). Then the Lie algebra cohomology in ref. [16] asserts that \(c^a k^{a\lambda} = \text{tr} c = 0\) for a simple group. This shows that \(\tilde{A} = 0\) for \(c^a(n) \rightarrow \text{const.}\). This limit might again be dangerous but it is sufficient to assume that \(\delta \tilde{A} = 0\), where \(\delta\) is an arbitrary local variation of the gauge potential.

Combining (1) and (2), we see that the assumption (I) implies
\[
\delta \tilde{A} = 0, \quad \text{for} \quad c^a(n) \rightarrow \text{const.},
\] (7.22)
where \(\delta\) is an arbitrary local variation of the gauge potential. This provides a strong constraint for the possible form of \(\tilde{A}\) as will be seen in the next subsection. In what follows, we do not need the assumption (I) itself, but use this constraint (7.22) instead.

7.2. UNIQUENESS OF THE NONTRIVIAL ANOMALY

We now expand the anomaly \(\mathcal{A}\) (7.1) in powers of the gauge potential as
\[
\mathcal{A} = \sum_{\ell=1}^{\infty} \mathcal{A}_\ell, \quad \tilde{A} = \sum_{\ell=1}^{\infty} \tilde{A}_\ell, \quad B = \sum_{\ell=1}^{\infty} B_\ell,
\] (7.23)
where \(\ell\) stands for the number of powers of \(c\) and \(A_\mu\) (recall that the ghost number of \(\mathcal{A}\) is unity). We decompose also the BRS transformation (7.2) according to powers of the fields,
\[ \delta_B = \sum_{\ell=0}^{\infty} \delta_\ell, \]

where

\[ \begin{align*}
\delta_0 A_\mu(n) &= \Delta_\mu c(n), \\
\delta_1 A_\mu(n) &= \frac{1}{2} [A_\mu(n), c(n) + c(n + \hat{\nu})], \\
\delta_1 c(n) &= -c(n)^2, \\
\delta_{2k} A_\mu(n) &= (-1)^{k-1} \frac{B_k}{(2k)!} \left[ A_\mu(n), \underbrace{[A_\mu(n), \ldots, [A_\mu(n), \Delta_\mu c(n)] \ldots]}_{2k} \right], \\
\delta_{2k} c(n) &= 0, \quad \text{for } k \geq 1,
\end{align*} \]

(7.24)

and \( B_k \) is the Bernoulli number and \( \delta_{2k+1} = 0 \) for \( k \geq 1 \). Note that, in terms of components \( A_\mu^a \) and \( c^a \), \( \delta_0 \) has an identical form as the abelian BRS transformation (4.7). The nilpotency \( \delta_B^2 = 0 \) implies

\[ \sum_{k=0}^{\ell} \delta_k \delta_{\ell-k} = 0, \quad \text{for } \ell \geq 0, \]

(7.25)

and the consistency condition (1.3) takes the form

\[ \delta_0 \tilde{A}_\ell = - \sum_{k=1}^{\ell-1} \delta_k \tilde{A}_{\ell-k}, \quad \text{for } \ell \geq 1. \]

(7.26)

Since

\[ \begin{align*}
\tilde{A}_\ell &= \tilde{A}_\ell + \sum_{k=0}^{\ell-1} \delta_k \tilde{B}_{\ell-k}, \\
\end{align*} \]

(7.27)

if \( \tilde{A}_\ell \) contains \( \delta_0 \)-trivial part \( \delta_0 \tilde{B}_\ell \), \( \tilde{B}_\ell \) can always be absorbed into \( \tilde{B}_\ell \). Therefore we can assume that

\[ \tilde{A}_\ell \text{ does not contain } \delta_0 \tilde{B}_\ell. \]

(7.28)

The constraint (7.22) has to hold for each order:

\[ \delta \tilde{A}_\ell = 0, \quad \text{for } c^a(n) \rightarrow \text{const.}, \]

(7.29)

where \( \delta \) is an arbitrary local variation of the gauge potential. Also the correct classical continuum limit (III) requires

\[ \tilde{A}_\ell \xrightarrow{a \rightarrow 0} O(A^{\ell-1}) \text{ term of eq. (2.28)}. \]

(7.30)

We consider the local solution to the consistency condition (7.26) which satisfies the conditions (7.28), (7.29) and (7.30), order by order. The first equation in eq. (7.26) is

\[ \delta_0 \tilde{A}_1 = 0. \]

(7.31)

This equation is completely identical to the consistency condition in the abelian theory. Therefore, from our result in the preceding section, eq. (6.24), the general form of \( \tilde{A}_1 \) which
satisfies eqs. (7.28)–(7.30) is given by

\[ \tilde{A}_1 = 0. \] (7.32)

The next equation in eq. (7.26) is

\[ \delta_0 \tilde{A}_2 = 0. \] (7.33)

Again from the result in the abelian theory (6.24), we have

\[ \tilde{A}_2 = 0, \] (7.34)

where use of eqs. (7.28)–(7.30) has been made to conclude this.

The solution to the next equation

\[ \delta_0 \tilde{A}_3 = 0, \] (7.35)

has a \( \delta_0 \)-nontrivial part. From eq. (6.24), and from the conditions (7.28)–(7.30), we have

\[
\tilde{A}_3 = -\frac{\epsilon H}{24\pi^2} \sum_n \left\{ \varepsilon_{\mu\nu\rho\sigma} \text{tr} c^{(\alpha)}(n) \Delta_\mu \left[ A^{(\alpha)}_\nu(n) \Delta_\rho A^{(\alpha)}_\sigma(n + \hat{\nu}) \right] 
+ \varepsilon_{\mu\nu\rho\sigma} c^{U(1)_\beta}(n) \Delta_\mu \left[ A^{U(1)_\beta}_\nu(n) \Delta_\rho A^{U(1)_\beta}_\sigma(n + \hat{\nu}) \right] 
+ \varepsilon_{\mu\nu\rho\sigma} c^{U(1)_\beta}(n) \text{tr} \Delta_\mu \left[ A^{(\alpha)}_\nu(n) \Delta_\rho A^{(\alpha)}_\sigma(n + \hat{\nu}) \right] 
+ \varepsilon_{\mu\nu\rho\sigma} \text{tr} c^{(\alpha)}(n) \Delta_\mu \left[ A^{(\alpha)}_\nu(n) \Delta_\rho A^{(\alpha)}_\sigma(n + \hat{\nu}) \right] \right\},
\] (7.36)

where we have used the relation (5.58) to make the property (7.29) manifest. Note that the condition (7.29) is crucial to eliminate the possibility that the \( \mathcal{L}_a \) term of eq. (6.24) appears.

The next equation in eq. (7.26) is

\[ \delta_0 \tilde{A}_4 = -\delta_1 \tilde{A}_3. \] (7.37)

The solution to this equation \( \tilde{A}_4 \), if it exists, is unique. If another \( \tilde{A}_4 \) which also satisfies the conditions (7.28)–(7.30) exists,

\[ \delta_0 (\tilde{A}_4 - \tilde{A}_4) = 0, \] (7.38)

and the quantity inside the brackets again satisfies eqs. (7.28) and (7.29). But then eq. (6.24) shows that \( \tilde{A}_4' - \tilde{A}_4 = 0 \).
The above argument can be repeated for higher $\tilde{A}_\ell$'s. Suppose that a sequence for the nontrivial part, $\tilde{A}_3, \tilde{A}_4, \ldots, \tilde{A}_{\ell-1}$, has been obtained. Then the next term $\tilde{A}_\ell$ has to satisfy eq. (7.26) and the conditions (7.28)-(7.30). Then the same argument as above shows that the solution $\tilde{A}_\ell$, if it exists, is unique.

We have seen that the sequence $\tilde{A}_\ell$ for the nontrivial anomaly $\tilde{A}$, which satisfies eq. (7.22) and the assumptions (II) and (III), if it exists, is unique under the condition (7.28). There is no free parameter which can appear in a higher $\tilde{A}_\ell$. Moreover, this uniqueness shows that the anomaly cancellation in the continuum theory implies that of the lattice theory: If the first nontrivial term $\tilde{A}_3$ (7.36), which is proportional to the anomaly in the continuum theory $\text{tr}_{R-L} T^a \{T^b, T^c\}$ etc., is canceled among the fermion multiplet, then the subsequent sequence of $\tilde{A}_\ell$ for $\ell \geq 4$ are completely canceled. In other words, the possible nontrivial local anomaly on lattice $\tilde{A}$, under the assumptions (I)-(III), is always proportional to the anomaly in the continuum theory, to all orders of powers of the gauge potential. This establishes our theorem for nonabelian theories, stated in section 3.

The existence of the nontrivial sequence $\tilde{A}_\ell$ for $\ell \geq 4$ might be examined by repeatedly solving eq. (7.26). However, eq. (7.24) suggests that the explicit form of higher $\tilde{A}_\ell$'s will become quite complicated as $\ell$ increases. In the next subsection, instead of this analysis, we will give a “compact” form of a nontrivial solution which manifestly satisfies eq. (7.22) and the assumptions (II) (at least for the perturbative region (2.9)) and (III). This explicitly shows the existence of the nontrivial sequence, $\tilde{A}_\ell$ with $\ell \geq 4$. To write down the compact solution, however, we need the interpolation technique of lattice fields with which the BRS transformation takes a quite simple form. Therefore, we give a quick summary of the method of ref. [34] in the first part of the next subsection.

7.3. COMPACT FORM OF THE NONTRIVIAL ANOMALY

First we recapitulate essence of the interpolation method of lattice fields in ref. [34] (simply extended for infinite lattice).* For our argument, the interpolation method has to possess several properties which we will verify. To distinguish from the fields defined on lattice sites $n$, we use the continuous coordinate $x$ to indicate use of interpolated fields.

The method of ref. [34] consists of the following two steps.

Step 1. One first constructs the interpolated gauge potential $A^{(m)}(x)$ within each hypercube $h(m)$, here $m$ stands for the origin of the hypercube, such that the gauge potentials in neighboring hypercubes $h(m - \tilde{\mu})$ and $h(m)$ are related by the transition function $v_{m,\mu}(x)$ of the Lüscher’s principal fiber bundle [40],

$$A^{(m-\tilde{\mu})}(x) = v_{m,\mu}(x) [\partial_\lambda + A^{(m)}(x)] v_{m,\mu}(x)^{-1}, \quad (7.39)$$

on the intersection of the two hypercubes $x \in h(m - \tilde{\mu}) \cap h(m)$ (that is a 3-dimensional cube). For the transition function $v_{m,\mu}(x)$ to be well-defined, the gauge field configuration

* Under the same conditions we assume, the method of ref. [35] might be adopted as well.
must be “non-exceptional” \[40\]. As already noted, it can be shown that if \( \epsilon \) in eq. (2.1) is sufficiently small, the gauge field configuration is non-exceptional. So we assume that \( \epsilon \) has been chosen as this holds. The gauge potential which satisfies eq. (7.39) can be constructed, starting with a special gauge \( A^{(m)}_{\lambda}(x) = 0 \) at \( x \sim m \). The explicit expression of \( A^{(m)}_{\lambda}(x) \) in terms of \( v_{m,\mu}(x) \), which is eventually expressed by link variables \( U \) \[40\], is given in ref. [34]. We do not reproduce it here because it is rather involved and we do not need the explicit form in what follows. The interesting property of \( A^{(m)}_{\mu}(x) \) is [34]

\[
\mathcal{P} \exp \left[ \int_0^1 dt \, A^{(m)}_{\mu}(n + (1 - t)\hat{\mu}) \right] = u^m_{n,n+\hat{\mu}}. \tag{7.40}
\]

Namely, the Wilson line constructed from the interpolated gauge potential \( A^{(m)}_{\mu}(x) \) coincides with the link variable in the complete axial gauge of ref. [40].

Step 2.1. The section of the principal fiber bundle \[40\]

\[
w^m(n) = U(m,1)^{z_1} U(m+z_1^1,2)^{z_2} U(m+z_1^1+z_2^2,3)^{z_3} U(m+z_1^1+z_2^2+z_3^3,4)^{z_4} \in G, \tag{7.41}
\]
is defined for each lattice site \( n \) belonging to the hypercube \( h(m) \) where \( n = m + \sum \mu z^\mu \). This section is then smoothly interpolated, first on the links, next on the plaquettes, on the cubes, and finally inside the hypercube \( h(m) \). At this stage, if the homotopy group \( \pi_{M-1}(G) \) is nontrivial, the smooth interpolation of the section \( w^m(x) \) into a \( M \)-dimensional (sub)lattice may fail, depending on the configuration of the section \( w^m(x) \) on a boundary of the \( M \)-dimensional (sub)lattice. For example, for \( G = U(1) \), \( \pi_1(U(1)) = Z \), and if the local winding of \( w^m(x) \) around a boundary of the plaquette \( p(m,\mu,\nu) \),

\[
Q(m,\mu,\nu) = \frac{i}{2\pi} \int_{\partial p(m,\mu,\nu)} dx^{\mu} \varepsilon^{\mu\nu\rho\sigma} w^m(x)^{-1} \partial_{\nu} w^m(x), \tag{7.42}
\]
does not vanish, then the interpolation of the section \( w^m(x) \) into the plaquette \( p(m,\mu,\nu) \) develops a singularity. Similarly, for \( G = SU(2) \), \( \pi_3(SU(2)) = Z \), and the local winding is given by†

\[
Q(m) = \frac{1}{24 \pi^2} \int_{\partial h(m)} d^3 x^{\mu} \varepsilon^{\mu\nu\rho\sigma} \text{tr} \, w^m(x)^{-1} \partial_{\nu} w^m(x) w^m(x)^{-1} \partial_{\rho} w^m(x) w^m(x)^{-1} \partial_{\sigma} w^m(x). \tag{7.43}
\]

If \( Q(m) \) does not vanish, then the interpolation of \( w^m(x) \) into the hypercube \( h(m) \) develops the singularity. If these singularities arise, the description in term of the interpolated fields

---

† The total winding \( Q = \sum_m Q(m) \) on finite periodic lattice is nothing but the Lüscher’s topological charge \[40\].

38
becomes inadequate. Fortunately, all local windings are vanishing within the perturbative region (2.9), for a sufficiently small \( \epsilon \). If \( \epsilon \) in (2.9) is sufficiently small, the norm of exponent of a product of several link variables is also small and the expression of the interpolated section [34] cannot have the “jump” on a boundary of the \( M \)-dimensional (sub)lattice. This implies that there is no local winding.

Step 2.2. With the smooth interpolated section \( w^m(x) \), we define the “global” interpolated gauge potential by

\[
A_\lambda(x) = w^m(x)^{-1} \left[ \partial_\lambda + A^{(m)}_\lambda(x) \right] w^m(x),
\]

for \( x \in h(m) \). The resulting interpolated gauge potential \( A_\lambda(x) \) is Lie algebra valued.

Now, we need the following properties of the interpolation method to express the non-trivial local solution.

(i) The gauge covariance. This is the most important property for our purpose. Namely, there exists a smooth interpolation of the gauge transformation (in our present context this becomes a smooth interpolation of the ghost field) and the lattice gauge (BRS) transformation on the link variables takes an identical form as that of the continuum theory. This property was shown in ref. [34]. Therefore, the BRS transformation (1.2) induces

\[
\delta_B A^a_\mu(x) = \partial_\mu c^a(x) + i f^{abc} A^b_\mu(x) c^c(x), \quad \delta_B c^a(x) = -\frac{1}{2} i f^{abc} c^b(x) c^c(x),
\]

on the interpolated fields.

(ii) The transverse continuity. This means the following. The gauge potential \( A_\lambda(x) \) is continuous inside each hypercube and, on the intersection of two neighboring hypercubes \( x \in h(m - \mu) \cap h(m) \), the component transverse to this intersection (namely, \( \lambda \neq \mu \)) is continuous across this intersection. We need this property because otherwise integration by parts is not allowed in the following expression. It is easy to see this property, if one notes that the Lüscher’s transition function [40] and the interpolated section [34] are related by

\[
v_{m,\mu}(x) = w^{m-\mu}(x) w^m(x)^{-1}, \quad \text{for} \quad x \in h(m - \mu) \cap h(m).
\]

Then from eqs. (7.39) and (7.44),

\[
w^{m-\mu}(x)^{-1} \left[ \partial_\lambda + A^{(m-\mu)}_\lambda(x) \right] w^{m-\mu}(x) = w^m(x)^{-1} \left[ \partial_\lambda + A^{(m)}_\lambda(x) \right] w^m(x)
\]

for \( x \in h(m - \mu) \cap h(m) \). Namely, the interpolated gauge potentials defined from a side of the hypercube \( h(m - \mu) \) and defined from a side of \( h(m) \) coincide on the intersection when \( \lambda \neq \mu \). For \( \lambda \neq \mu \), the component may jump across the intersection [34], but this

\[\text{‡} \quad \text{The procedure of ref. [35] can avoid this difficulty.}\]
causes no problem for our purpose. The interpolation for the ghost field is obtained by setting $g(n) = \exp[\lambda c(n)]$ in the interpolation formula for the gauge transformation $g(x)$ in ref. [34]. This gives the smooth ghost field (which is also Lie algebra valued) throughout the whole lattice.

(iii) The smoothness and locality. The interpolated gauge potential $A_\lambda(x)$ and the ghost field $c(x)$ are smooth functions of link variables (and of the gauge transformation function) residing nearby the point $x$. The smoothness (for the perturbative configurations) and the locality are manifest from the explicit expressions of $A_\lambda^{(m)}(x)$ and of $g(x)$ in ref. [34]. In fact, in this case, the relation is ultra-local.

(iv) The correct continuum limit. In the classical continuum limit, $a \to 0$, the interpolated gauge potential $A_\mu(x)$ and the ghost field $c(x)$ reduce to (for smooth configurations) to the gauge potential and the ghost field in the continuum theory. From eq. (7.40), we have

$$
\mathcal{P} \exp \left[ \int_0^1 du A_\mu(n + (1 - u)\nu) \right] = w^n(n)^{-1} u_{n,n+\nu} w^{(n + \nu)}
= U(n, \nu),
$$

(7.48)
where we have used the definition of the link variable in the complete axial gauge $u_{n,n+\nu} [40]$.

This is nothing but the conventional expression that one assumes in the classical continuum limit (2.21). For the interpolated ghost field, the formula in ref. [34] shows that $c(x = n) = c(n)$.

(v) The constant ghost field. From the formula in ref. [34], it is easy to see that the constant ghost field on the sites induces the constant interpolated ghost field,

$$
c(n) = c = \text{const.} \Rightarrow c(x) = c = \text{const.}
$$

(7.49)

Now we can write down the nontrivial local solution to the consistency condition (1.3) which satisfies eq. (7.22) and the assumptions (II) and (III) in terms of the interpolated fields. It is

$$
\mathcal{A} = -\frac{\epsilon H}{24\pi^2} \sum_{h(n)} \int d^4 x \left\{ \varepsilon_{\mu\nu\rho\sigma} \text{tr} \left[ c^{(a)}(x) \partial_\mu \left[ A_\nu^{(a)}(x) \partial_\sigma A_\sigma^{(a)}(x) + \frac{1}{2} A_\nu^{(a)}(x) A_\rho^{(a)}(x) A_\sigma^{(a)}(x) \right] \right]
+ \varepsilon_{\mu\nu\rho\sigma} \text{tr} \left[ c^{(a)}(x) \partial_\mu \left[ A_\nu^{(a)}(x) \partial_\sigma A_\sigma^{(a)}(x) + \frac{2}{3} A_\nu^{(a)}(x) A_\rho^{(a)}(x) A_\sigma^{(a)}(x) \right] \right]
+ 2\varepsilon_{\mu\nu\rho\sigma} \text{tr} \left[ A_\nu^{(a)}(x) \partial_\mu A_\sigma^{(a)}(x) \partial_\rho U^{(1)a}(x) \right] \right\},
$$

(7.50)

In this expression, $h(n)$ is the hypercube whose origin is the site $n$. Note that this is a

* In fact, from the formulas of ref. [34], it can be shown that $A_\mu(x)$ is constant along the link, $A_\mu(x) = A_\mu(n)$ for $x \in [n, n + \rho]$ where $U(n, \rho) = \exp A_\mu(n)$.
functional of the link variable $U(n, \mu)$ and the ghost field $c(n)$, through the interpolation formulas of ref. [34].

It is easy to see that eq. (7.50) satisfies the consistency condition (1.3), because the BRS transformation of the interpolated fields (7.45) has an identical form as that of the continuum theory and because eq. (7.50) has formally an identical form as the gauge anomaly in the continuum theory (2.28). More precisely, we need to perform an integration by parts within each hypercube to show eq. (1.3). Then the transverse continuity (ii) guarantees that contributions from a boundary of hypercubes cancel with each other. This solution eq. (7.50) is moreover $\delta_B$-nontrivial: From the property (iv) of the interpolation, in the classical continuum limit (assuming background fields are smooth), eq. (7.50) reproduces the gauge anomaly in the continuum theory (2.28) that is BRS nontrivial. In other words, if eq. (7.50) was $\delta_B$-trivial, there exists a local functional $B$ on lattice such that $A = \delta_B B$. Then the classical continuum limit of $B$, which is a local functional in the continuum theory, would cancel the gauge anomaly in the continuum theory.

The nontrivial solution (7.50) manifestly fulfills the condition (7.22) from the properties (v) and (ii) of the interpolation. From the above arguments, it is also clear that eq. (7.50) satisfies the assumptions (II) (within the perturbative region (2.9)) and (III).

The existence of the nontrivial solution $A$ (7.50), which satisfies eq. (7.22) and the assumptions (II) and (III), shows the existence of the unique nontrivial sequence $\hat{A}_\ell$ (7.23) that is characterized by eq. (7.28). We first expand $A$ (7.50) in powers of the gauge potential (2.8) as eq. (7.23). Then using eq. (7.27), we can determine $B_\ell$ order by order such that the sequence $\hat{A}_\ell$ satisfies eq. (7.28). This procedure gives the unique sequence $\hat{A}_\ell$. Note that when the anomaly in the continuum is canceled, the anomaly $A$ (7.50) vanishes. Therefore the above procedure gives $\hat{A}_\ell = 0$ for all $\ell$. This is consistent with the conclusion in the preceding subsection.

8. Conclusion

In this paper, we have shown that any gauge anomaly $A$ on 4-dimensional infinite lattice can always be removed by local counterterms order by order in powers of the gauge potential, provided that $A$ depends smoothly and locally on the gauge potential and that $A$ reproduces the gauge anomaly in the continuum theory. The unique exception is proportional to the anomaly in the continuum theory. This implies that the anomaly cancellation condition in lattice gauge theory is identical to that of the continuum theory.

The most important prerequisite for our result is the locality of anomaly $A$. As we have shown, the gauge anomaly in the formulation based on the Ginsparg-Wilson Dirac operator is local (for a particular choice of the integration measure) and thus our theorems are applicable (at least in the perturbative region in which a parameterization of the admissible space in terms of the gauge potential is possible). Unfortunately, the gauge anomaly $A$ appearing in formulations based on the more familiar Wilson Dirac operator or on the Kogut-Susskind Dirac operator is not local, although these Dirac operators themselves are ultra-
local. For these operators, the chiral gauge symmetry is broken in the tree level and as a result the anomaly is given as $\mathcal{A} = \text{Tr} \text{(explicit breaking term)} \times \text{(propagator)}$. The (massless) propagator in this expression breaks the locality. (In the classical continuum limit, the locality restores and $\mathcal{A}$ reproduces the gauge anomaly in the continuum theory [62,63].)

Let us discuss possible extensions of results in this paper. The most severe limitation of our result for nonabelian theories is that it holds only in an expansion in powers of the gauge potential. An interesting observation relating this is that the expansion of the anomaly density $a(n) (A(n) = \sum_{n} a(n))$ $a(n) = \sum_{\ell=1}^{\infty} a_{\ell}(n)$ in powers of the gauge potential has a finite radius of convergence. This follows from the smoothness and the locality of the anomaly $\mathcal{A}$ which we have assumed. If these hold for the admissible configurations (2.1), the radius of convergence of this series is given by the right hand side of eq. (2.9). Another interesting point is that the compact solution (7.50) is smooth and local at least in the perturbative region (2.9). These observations suggest that our result is valid beyond the expansion with respect to the gauge potential, at least within the perturbative region. What is not clear at present is a convergence of the individual series $\tilde{A} = \sum_{\ell=1}^{\infty} \tilde{A}_{\ell}$ and $B = \sum_{\ell=1}^{\infty} B_{\ell}$ in eq. (7.23), after imposing eq. (7.28).

By using similar arguments as above, it seems straightforward to classify general topological fields on 4-dimensional infinite lattice (that is a nonabelian analogue of the theorem (5.56)) at least to all orders of powers of the gauge potential. According to the result of ref. [21], this analysis is relevant to see existence of an exactly gauge invariant formulation of anomaly-free two-dimensional chiral gauge theories. For four-dimensional chiral gauge theories, we have to generalize the covariant Poincaré lemma (5.11) to 6-dimensional lattice. This generalization would be straightforward, although the proof may become quite cumbersome.

The restriction to the perturbative region (2.9) for nonabelian theories is due to a complicated structure of the admissible space (2.1). If it is possible to parameterize the admissible space in terms of the gauge potential, as in the abelian case, the restriction may be relaxed. It is highly plausible that the “rewinding” technique of ref. [35] is useful in this context.

Our results are not yet “realistic” because these are for infinite lattice. For the abelian case $G = U(1)$, it has been shown [20] that the anomaly cancellation works even for finite periodic lattice. To generalize the argument in ref. [20] for nonabelian theories, we have to first understand the structure of the admissible space on finite lattice (see the above remark). Another possible approach to this problem is to clarify De Rham cohomology on finite (periodic) lattice. This approach corresponds to the argument of ref. [17] in the continuum theory.

In this paper, we adopted a “classical” algebraic viewpoint based on the Wess-Zumino consistency condition. In the continuum theory, the algebraic approach to the anomaly has a close relationship to a higher dimensional theory [64]. It seems very important to investigate

* However the existence of the global obstruction [50] shows that this must be in general impossible.
such a relationship in the context of lattice gauge theory. In fact, there are some indications for such relation [21,45,65].

Finally, let us mention on the physical implication of these analyses. After all, even a local counterterm which makes the effective action gauge invariant exists (for anomaly-free cases), the implementation of gauge invariance requires a fine tuning of parameters in the counterterm, that is highly unnatural. One might thus be tempting to appeal to the mechanism of ref. [66] which dynamically restores the gauge invariance. However, for the mechanism of ref. [66] to work, the gauge breaking $A$ (with the ghost field is replaced by a logarithm of the gauge transformation field) has to be “small.” In particular, if $A \neq \delta B$ for a local functional $B$, the effective lagrangian for the gauge transformation field would be given by a lattice analogue of the Wess-Zumino lagrangian which modifies the physical content (thus it cannot be regarded “small”). Therefore, study of the gauge anomaly on lattice is important also to examine the necessary condition for the mechanism of ref. [66]. In this respect, it seems interesting to study the locality (in a four dimensional sense) of the gauge anomaly appearing in the overlap formulation with the Brillouin-Wigner phase convention, in connection with the result of ref. [67].

Acknowledgments

The author has greatly benefited from correspondence with T. Fujiwara, Y. Kikukawa and K. Wu and from discussions with P. Hernandez. The author is particularly grateful to M. Lüscher for various helpful remarks, without which this work would not have been completed.

APPENDIX A

Here we summarize our notation and the convention. Throughout this paper, we consider the 4-dimensional infinite lattice $\mathbb{Z}^4$. The site of the lattice is denoted by $n$, $m$, etc. The lattice spacing is taken to be unity $a = 1$ unless otherwise stated. The Greek letters $\mu$, $\nu$, etc. denote the Lorentz indices which run from 1 to 4. $\hat{\mu}$ stands for the unit vector in direction $\mu$. For Lorentz indices, the summation of repeated indices is always understood. The Levi-Civita symbol is defined by $\varepsilon_{\mu\nu\rho\sigma} = \varepsilon_{[\mu\nu\rho\sigma]}$ and $\varepsilon_{1234} = 1$.

The forward and the backward difference operators are respectively defined by

$$\Delta_\mu f(n) = f(n + \hat{\mu}) - f(n), \quad \Delta^*_\mu f(n) = f(n) - f(n - \hat{\mu}).$$

(A.1)

The symbol $\partial_\mu$ is reserved for the standard derivative.

$H = R$ or $L$ stands for the chirality of Weyl fermion and we set $\epsilon_R = +1$ and $\epsilon_L = -1$.

$G = \prod_\alpha G_\alpha$ is the (compact) gauge group and here $G_\alpha$ denotes a simple group or a $U(1)$ factor. The Greek indices $\alpha$, $\beta$, etc. are used to label each factor group. $T^a$ stands for the
representation matrix of the Lie algebra, $[T^a, T^b] = i f^{abc} T^c$. The summation of repeated group indices $a, b, c$ etc. is always understood.

$U(n, \mu)$ is the link variable on the link that connects the lattice sites $n$ and $n + \mu$. For the abelian gauge group $G = U(1)^N$, we parameterize the link variable by the gauge potential as $U^a(n, \mu) = \exp A^a_\mu(n)$. In this case, the superscript $a$ distinguishes each $U(1)$ factor. The abelian field strength is defined by

$$F^a_{\mu\nu}(n) = \Delta_\mu A^a_\nu(n) - \Delta_\nu A^a_\mu(n).$$  \hfill (A.2)

We never use this symbol $F^a_{\mu\nu}$ to indicate the nonabelian field strength. The plaquette variable is defined by

$$P(n, \mu, \nu) = U(n, \mu)U(n + \mu, \nu)U(n + \nu, \mu)^{-1}U(n, \nu)^{-1}. \hfill (A.3)$$

## APPENDIX B

In this appendix, we show the calculation of the “Wilson line” $W'$ in eq. (2.17). From eqs. (2.11) and (2.17), $W'$ is given by

$$W' = \exp \left( \epsilon_H \int_0^1 dt \int_0^1 ds \ Tr \ P_t(s) \left[ \partial_s P_t(s), \partial_t P_t(s) \right] \right), \hfill (B.1)$$

where $P_t(s) = P_H|_{U \to U_t(s)}$ and we explicitly indicated $s$-dependences defined through eq. (2.13). If we introduce the transporting operator $Q_t(s)$ for each $s$ by $\partial_t Q_t(s) = [\partial_t P_t(s), P_t(s)] Q_t(s)$ and $Q_0(s) = 1$, we have

$$P_t(s) = Q_t(s) P_0(s) Q_t(s)^\dagger, \hfill (B.2)$$

(note that $Q_t(s)$ is unitary). Substituting this into eq. (B.1), after some calculation, we have

$$W' = \exp \left\{ \epsilon_H \int_0^1 dt \int_0^1 ds \ \left[ \partial_s Tr P_0(s) Q_t^\dagger(s) \partial_t Q_t(s) - \partial_t Tr P_0(s) Q_t^\dagger(s) \partial_s Q_t(s) \right] \right\}. \hfill (B.3)$$

We then apply the Stokes theorem to this 2-dimensional integration. This yields

$$W' = \exp \left( \epsilon_H \int_0^1 dt \ Tr P_0(1) Q_t^\dagger(1) \partial_t Q_t(1) - \epsilon_H \int_0^1 ds \ Tr P_0(s) Q_t^\dagger(s) \partial_s Q_t(s) \bigg|_{t=0}^{t=1} \right). \hfill (B.4)$$

However, from eq. (B.2),

$$\text{Tr} \ P_0(1) Q_t^\dagger(1) \partial_t Q_t(1) = \text{Tr} \ P_0(1) Q_t^\dagger(1) [\partial_t P_t(1), P_t(1)] Q_t(1) = 0, \hfill (B.5)$$

and $\partial_s Q_0(s) = 0$ because $Q_0(s) = 1$. Therefore

$$W' = \exp \left( -\epsilon_H \int_0^1 ds \ Tr P_0(s) Q_t^\dagger(s) \partial_s Q_1(s) \right). \hfill (B.6)$$

Noting $P_0(s) \partial_s P_0(s) P_0(s) = 0$ and $P_1(s) = P_0(s)$ (recall that $U_1(s) = U_0(s)$), it can be
confirmed that eq. (B.6) is equal to

\[ W' = \exp \left\{ -\epsilon_H \int_0^1 ds \, \text{Tr} \left[ 1 - P_0(s) + P_0(s)Q_1(s) \right]^{-1} \partial_s \left[ 1 - P_0(s) + P_0(s)Q_1(s) \right] \right\}. \] 

(B.7)

This shows eq. (2.17).

REFERENCES


43. W. Bietenholz, hep-lat/9901005.
44. T. Fujiwara, H. Suzuki and K. Wu, hep-lat/0001029.
45. T. Aoyama and Y. Kikukawa, hep-lat/9905003.
46. H. Neuberger, hep-lat/9912013.
50. O. Bär and I. Campos, hep-lat/9909081; hep-lat/0001025.
61. See, for example, L. O’Raifeartaigh, Group structure of gauge theories (Cambridge University Press, Cambridge, 1986).
65. D. H. Adams, hep-lat/9910006; hep-lat/0001014.