A GENERALISATION OF A THEOREM OF NAGATA
ON RULED SURFACES

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MIRAMARE – TRIESTE
January 2000

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1 Introduction

Let $X$ be a smooth projective irreducible curve of genus $g$ over an algebraically closed field $k$, $G$ a connected reductive algebraic group over $k$ and $P$ a parabolic subgroup. For a principal $G$-bundle $E$ over $X$, consider the associated $G/P$-bundle $\pi : E/P \to X$. If $\sigma$ is a section of $\pi$ we denote by $N_\sigma$ the normal bundle of $\sigma(X)$ in $E/P$. The first result proved in this paper is the following.

**Theorem 1.1** There exists section $\sigma$ of $\pi : E/P \to X$ such that

$$\deg(N_\sigma) \leq g \cdot \dim(G/P)$$

where $g$ is the genus of $X$ and $\deg(N_\sigma)$ denotes the degree of the normal bundle $N_\sigma$ considered as a vector bundle on $X$.

The above result was classically known in the case of $G = \text{GL}(2)$ and $P$ a maximal parabolic, in the form of the theorem of M. Nagata [8] and C. Segre, which asserts that a ruled surface on $X$ admits a section whose self intersection number is $\leq g$. It has also been proved for $G = \text{GL}(n)$ and $P$ a maximal parabolic subgroup by Mukai and Sakai [12], and for $G$ a classical group and $P$ a maximal parabolic subgroup by Nitsure [9]. For a general survey of the topic in the case of vector bundles one may refer to Lange [7].

The main idea of our proof of the Theorem 1.1 is a “no-ghosts theorem” for the Hilbert scheme of $E/P$, which asserts that every point of the Hilbert scheme which lies in an irreducible component containing the Hilbert point of a minimal section (i.e. for which $\deg(N_\sigma)$ is minimum), is itself the Hilbert point of a section (Proposition 2.3). We then adapt an argument of Mukai-Sakai to complete the proof of the theorem.

In the second part of the paper we prove the following.

**Theorem 1.2** Let $G$ be a connected reductive algebraic group and $X$ a smooth projective irreducible curve over an algebraically closed field $k$ of arbitrary characteristic. Then the set of isomorphism classes of semi-stable $G$-bundles on the curve $X$ with a given degree is bounded. In particular, if $G$ is semi-simple then the semi-stable $G$-bundles form a bounded family.

For a precise definition of degree see section 3. In characteristic 0, the above theorem is due to Ramanathan [3]. For the classical groups, the result follows in all characteristics (except in characteristic 2 for $G = \text{SO}(n)$) from the observation of Ramanan (see [13], Proposition 4.2) that a $G$-bundle is semi-stable if and only if the underlying vector bundle is so.

2 Minimal sections

Let $X$ be a smooth projective irreducible curve over an algebraically closed field $k$. Let $G$ be a connected reductive algebraic group over $k$ and $P$ a parabolic subgroup of $G$.

Given a principal $G$ bundle $E$ over $X$, denote by $\pi : M \to X$ the associated bundle $E/P$ with $G/P$ as fibres. If $\sigma$ is a section of $\pi : M \to X$, we denote by $N_\sigma$, the vector bundle on $X$ obtained by pulling back by $\sigma$ the normal bundle of $\sigma(X)$ in $M$. Observe that $N_\sigma$ is the pullback $\sigma^*(T_\pi)$ where $T_\pi$ is the tangent bundle along the fibres of $\pi : M \to X$.

In the following lemma we prove that the degree $\deg(N_\sigma)$ of the vector bundle $N_\sigma$ on $X$ is bounded below.

**Lemma 2.1** Given a principal $G$-bundle $E \to X$, there exists a constant $C$ such that $\deg(N_\sigma) \geq C$ for all sections of the associated bundle $\pi : M \to X$. 

Proof Let $T_\pi$ be the tangent bundle along the fibres of $\pi$. As observed already, $N_\sigma \cong \sigma^*(T_\pi)$. If $\mathfrak{g}$ (resp. $\mathfrak{p}$) are the Lie algebras of $G$ (resp. $P$) we have an exact sequence of $P$-modules

$$0 \rightarrow \mathfrak{p} \rightarrow \mathfrak{g} \rightarrow \mathfrak{g}/\mathfrak{p} \rightarrow 0.$$ 

Note that $\mathfrak{g}/\mathfrak{p}$ is the tangent space of $G/P$ at $e$. On $M$, we have the principal $P$-bundle $E \rightarrow M$, and the above short exact sequence of $P$-modules gives a short exact sequence of vector bundles on $M$. Pulling it back under $\sigma$ gives us a short exact sequence of vector bundles on $X$, whose middle term is the adjoint bundle of $E$ and the last term is $\sigma^*(T_\pi)$. This implies that $\sigma^*(T_\pi)$ is a quotient of a fixed vector bundle (independent of $\sigma$).

It is known that $\pi : M \rightarrow X$ admits sections. This follows from a theorem of Steinberg (see for example Ramanathan [3], 2.11, p.306).

Suppose $\sigma$ is a section of $\pi : M \rightarrow X$. We say $\sigma$ is a minimal section if $\deg(N_{\sigma})$ is minimum. As sections exists, and as their degrees are bounded below by Lemma 2.1, minimal sections exists.

We will now prove a lemma which is a crucial step in the proof of Theorem 1.1.

Let $Y$ be a one dimensional projective scheme over $k$. If $L$ is a line bundle (locally free sheaf of rank one) on $Y$, we define the degree of $L$ by

$$\deg(Y, L) = \chi(Y, L) - \chi(Y, \mathcal{O}_Y).$$

Note that this is consistent with the usual definition of the degree of a line bundle on a non-singular projective curve.

It is well known that if $L$ is ample on $Y$, then $\deg(Y, L) > 0$ (see for example Iitaka [10], 8.4). Observe that $\deg(Y, L)$ is the sum of $\deg(Y_i, L)$ where $Y_i$ are the connected components of $Y$ (regarded as open subschemes), where the zero dimensional components of $Y$ contribute 0 to the degree.

**Lemma 2.2** Let $X$ be a smooth projective irreducible curve over $k$ and $Y$ a projective one dimensional scheme over $k$.

Let $f : Y \rightarrow X$ be a morphism. Assume that

(a) $\chi(X, \mathcal{O}_X) = \chi(Y, \mathcal{O}_Y)$.

(b) For some line bundle $L$ of degree 1 on $X$, we have $\chi(X, L) = \chi(Y, f^*(L))$.

Then we have the following.

(i) There exists a unique irreducible component $D$ of $Y$ which dominates $X$. Let $D_{\text{red}}$ be the reduced subscheme structure on $D$ induced from $Y$. Then $f|_{D_{\text{red}}} : D_{\text{red}} \rightarrow X$ is an isomorphism.

(ii) Suppose that the component $D$ given by (i) above is the only irreducible component of $Y$ of dimension one. Then $f : Y \rightarrow X$ is an isomorphism (in particular, $Y$ has no zero dimensional components).

(iii) Let $\xi$ be a line bundle on $Y$. Suppose that $Y$ has more than one irreducible component of dimension one. Let $D_1, D_2, \ldots, D_k$ be the one dimensional irreducible components other than $D$ and let $D_{i,\text{red}}$ be the corresponding reduced subscheme of $Y$. Suppose $\xi|_{D_i,\text{red}}$ is ample for all $i$. Then we have

$$\deg(D_{\text{red}}, \xi) < \deg(Y, \xi).$$

**Proof** We prove the proposition in several steps:

**Step (1)** $R^1f_*(\mathcal{O}_Y)$ is a torsion sheaf, in particular, $H^1(X, R^1f_*(\mathcal{O}_Y)) = 0$. 


Proof of step (1) Let $S \subset X$ be the set of points of $X$ over which the fibres of $Y \to X$ is positive dimensional. As $Y$ is one dimensional, it follows from the semi-continuity of the dimension of fibres that $S$ is a finite set of points of $X$. If $U = X - S$ then we observe that $f|_{f^{-1}(U)}$ is quasi-finite and proper hence it is a finite map. Therefore $R^1\psi_* \mathcal{O}_{f^{-1}(U)} = 0$. Now it is clear that $R^1f_*\mathcal{O}_Y$ is supported over $S$, hence it is a torsion sheaf.

Step (2) \deg(Y,f^*(L)) = \chi(X,f_*(\mathcal{O}_Y) \otimes L) - \chi(X,f_*(\mathcal{O}_Y))

Proof of step (2) We have $H^0(Y, \mathcal{O}_Y) = H^0(X, f_*(\mathcal{O}_Y))$, and $H^0(Y, f^*(L)) = H^0(X, f_*f^*(L)) = H^0(X, f_*(\mathcal{O}_Y) \otimes L)$ by projection formula. Since $\dim(X) = 1$, the Leray spectral sequence gives us the following exact sequences

\[
0 \longrightarrow H^1(X, f_*(\mathcal{O}_Y)) \longrightarrow H^1(Y, \mathcal{O}_Y) \longrightarrow H^0(X, R^1f_*(\mathcal{O}_Y)) \longrightarrow 0
\]

and

\[
0 \longrightarrow H^1(X, f_*(\mathcal{O}_Y) \otimes L) \longrightarrow H^1(Y, f^*(L)) \longrightarrow H^0(X, R^1f_*(\mathcal{O}_Y) \otimes L) \longrightarrow 0.
\]

Hence

\[
\chi(Y, \mathcal{O}_Y) = \chi(X, f_*(\mathcal{O}_Y)) - h^0(X, R^1f_*(\mathcal{O}_Y))
\]

and

\[
\chi(Y, f^*(L)) = \chi(X, f_*(\mathcal{O}_Y) \otimes L) - h^0(X, R^1f_*(\mathcal{O}_Y) \otimes L).
\]

Note that as $R^1f_*(\mathcal{O}_Y)$ is torsion by step (1), we have

\[
h^0(X, R^1f_*(\mathcal{O}_Y)) = h^0(X, R^1f_*(\mathcal{O}_Y) \otimes L).
\]

Hence

\[
\deg(Y, f^*(L)) = \chi(Y, f^*(L)) - \chi(Y, \mathcal{O}_Y) = \chi(X, f_*(\mathcal{O}_Y) \otimes L) - \chi(X, f_*(\mathcal{O}_Y)).
\]

Step (3) Rank $f_*(\mathcal{O}_Y) = 1$, in particular, $f$ is dominant.

Proof of step (3) If $T$ is the torsion submodule of $f_*(\mathcal{O}_Y)$, we have the short exact sequence

\[
0 \longrightarrow T \longrightarrow f_*(\mathcal{O}_Y) \longrightarrow Q \longrightarrow 0,
\]

$Q$ being locally free. Since we have

\[
\deg(Y, f^*(L)) = \chi(Y, f^*(L)) - \chi(Y, \mathcal{O}_Y) = \chi(X, L) - \chi(X, \mathcal{O}_X) \text{ (by (a) and (b) of the lemma)} = 1,
\]

from step (2) it follows that

\[
1 = \chi(X, f_*(\mathcal{O}_Y) \otimes L) - \chi(X, f_*(\mathcal{O}_Y)).
\]

From the short exact sequence $0 \longrightarrow T \otimes L \longrightarrow f_*(\mathcal{O}_Y) \otimes L \longrightarrow Q \otimes L \longrightarrow 0$ we see that

\[
\chi(X, f_*(\mathcal{O}_Y) \otimes L) - \chi(X, f_*(\mathcal{O}_Y)) = \chi(X, Q \otimes L) - \chi(X, Q),
\]

as $\chi(X, T \otimes L) = \chi(X, T)$ since $T$ is a torsion sheaf.

Thus $\chi(X, Q \otimes L) - \chi(X, Q) = 1$, in particular we have $Q \neq 0$. Note that this implies that $f$ is dominant. Let $r$ be the rank of $Q$. Since $\deg(L) = 1$, Riemann-Roch on $X$ gives

\[
\chi(X, Q \otimes L) - \chi(X, Q) = (r + \deg(Q) + r(1 - g)) - (\deg(Q) + r(1 - g)) = r.
\]
Thus $r = 1$.

**Step (4) Proof of (i):**

Since $(f, f^* : (Y, \mathcal{O}_Y) \to (X, \mathcal{O}_X)$ is dominant (by step (3)) and $X$ is reduced, the homomorphism $f^* : \mathcal{O}_X \to f_* (\mathcal{O}_Y)$ is injective (see EGA [2], Proposition 5.4.3, p. 284). Since $\text{rank}(f_* (\mathcal{O}_Y)) = 1$ (by step (3)), we have a short exact sequence

$$0 \to \mathcal{O}_X \to f_* (\mathcal{O}_Y) \to T' \to 0$$

where $T'$ is torsion. Let $V = X - \text{Supp}(T')$ and $U$ a non-empty open subscheme of $X$ such that $f^{-1}(U) \to U$ is finite (see step (1)). Then $f' : f^{-1}(U \cap V) \to U \cap V$, $(f') = f|_{f^{-1}(U \cap V)}$ is finite (and hence affine) and $f'_* (\mathcal{O}_{U \cap V}) = \mathcal{O}_{U \cap V}$. Hence $f'$ is an isomorphism. Let $Y_0$ be the schematic image (see EGA-I [2], 6.10, pp. 324-325) of the open inclusion $f^{-1}(U \cap V) \to Y$. Since $f^{-1}(U \cap V)$ is reduced, $Y_0$ is the reduced induced structure induced on $f^{-1}(U \cap V)$ by $Y$. Then $Y_0$ is also irreducible and hence by Zariski’s main theorem, $f|_{Y_0} : Y_0 \to X$ is an isomorphism.

Since $f' : f^{-1}(U \cap V) \to U \cap V$ is an isomorphism, we see that $Y_0$ is the only component of $Y$ which dominates $X$. In the notation of the statement (i) of the lemma, we have $D_{\text{red}} = Y_0$.

**Step (5) Proof of (ii):**

Suppose now that $Y$ has only one irreducible component $D$ of dimension 1. Let $D_{\text{red}}$ be the reduced subscheme of $Y$ with support $D$. Then we have a short exact sequence

$$0 \to I \to \mathcal{O}_Y \to \mathcal{O}_{D_{\text{red}}} \to 0.$$

Since $f^{-1}(U \cap V)$ is reduced and the other components, if any, are zero-dimensional, we see that $I$ is supported at finitely many points. Now by hypothesis, $\chi(Y, \mathcal{O}_Y) = \chi(X, \mathcal{O}_X) = \chi(D_{\text{red}}, \mathcal{O}_{D_{\text{red}}})$ as $D_{\text{red}} \to X$ is an isomorphism. Since

$$\chi(Y, \mathcal{O}_Y) = \chi(Y, \mathcal{O}_{D_{\text{red}}}) + h^0(Y, I) = \chi(X, \mathcal{O}_X) + h^0(Y, I),$$

we see that $h^0(Y, I) = 0$, and since $I$ is torsion, $I = 0$. Thus in this case $f : Y \to X$ is an isomorphism.

**Step (6) Proof of (iii):**

Suppose that $Y$ has other one dimensional components apart from $D$.

Let $D_1, \dots, D_k$ ($k \geq 1$) be the other one dimensional components and $P_1, \dots, P_l$ the 0-dimensional components. Let $Y' = Y - \{P_1, \dots, P_l\}$, considered as an open subscheme of $Y$. Let $W = Y' - \{\text{points of intersection of two distinct components}\}$, considered as an open subscheme of $Y$. Let $W^s$ be the schematic closure of $W$ in $Y'$.

Similarly define $D_i^s$ for any component to be the schematic closure in $Y'$ of $D_i - \{\text{points of intersection of $D_i$ with the other components}\}$. Observe that $D_i^s = D_{\text{red}}$ (see step (4)). Note that $D^s$ and $D_i^s$ are closed subschemes of $W^s$. We have a short exact sequence

$$0 \to T_1 \to \mathcal{O}_{Y'} \to \mathcal{O}_{W^s} \to 0$$

and

$$0 \to \mathcal{O}_{W^s} \to \mathcal{O}_{D_{\text{red}}} \oplus \mathcal{O}_{D_1^s} \oplus \cdots \oplus \mathcal{O}_{D_k^s} \to T_2 \to 0,$$

where $T_1$ and $T_2$ are supported at finite number of points.

For the line bundle $\xi$ on $Y$, we have

$$\deg(Y, \xi) = \deg(Y', \xi) = \deg(W^s, \xi) = \deg(D_{\text{red}}^s, \xi) + \sum_{i=1}^k \deg(D_i^s, \xi).$$
Now $(D_i^r)_{red}$ is the same as the reduced scheme structure $D_i,_{red}$ induced on $D_i$ by $Y'$. Since by hypothesis, $\xi|_{D_i,_{red}}$ is ample, $\xi|_{D_i^r}$ is ample too as can be seen. Hence $\deg(D_i^r,\xi) > 0$ for each $i$. Thus, as $D_{red} = D^s$, we get
\[ \deg(D_{red},\xi) = \deg(D^s,\xi) < \deg(Y,\xi). \]
This completes the proof of the Proposition 2.2. □

We now go back to proving the Theorem 1.1. The above lemma is used in the proof of the following proposition.

Proposition 2.3 Let $\sigma$ be a minimal section of $\pi : M \rightarrow X$ as defined earlier. Let $H$ be the Hilbert scheme of closed subschemes of $M$ (we may restrict ourselves to $\text{Hilb}^P(M)$ where $P$ is the Hilbert polynomial of $\sigma$, with respect to an ample line bundle). Let $Y$ be the closed subscheme of $M$, represented by a point of $H$ which lies in an irreducible component containing the Hilbert point of $\sigma_0(X)$. Then $\pi|_Y : Y \rightarrow X$ is an isomorphism.

Proof Let $L$ be a line bundle of degree 1 on $X$. Let $\eta$ be the line bundle $\det(T_\pi)$ on $M$, where $T_\pi$ is the tangent bundle along the fibres of $\pi$. Consider the diagram
\[
\begin{array}{ccc}
X & \xrightarrow{p_X} & Y \\
\end{array}
\]
Proof  The restriction \( s : S \times X \rightarrow M \) of the universal morphism \( \Pi(M/X) \times X \rightarrow M \) makes the following diagram commute.

\[
\begin{array}{c}
M \\
\pi
\end{array}
\]
Now $\det(N_\sigma)$ is the line bundle associated to the character of $B$ defined by $(-\sum_{\alpha > 0} \alpha)$, sum over all positive roots. Now

$$(-\sum_{\alpha > 0} \alpha) = (-\sum m_i \alpha_i),$$

where $\alpha_i$’s are simple roots taken with respect to a fixed maximal torus contained in $B$ and $m_i > 0$ depending only on the group $G$. Hence

$$\deg(N_\sigma) = -\sum m_i \deg(L_{\alpha_i}) \leq (\sum m_i) \cdot 2g.$$

This remark is sufficient for the applications we have in mind.

### 3 Boundedness for semi-stable $G$-bundles

In this section we use the results of the previous section to prove the boundedness of semi-stable $G$-bundles of a fixed degree on a smooth projective curve $X$ over an algebraically closed field $k$ of arbitrary characteristic, where $G$ is a connected reductive algebraic group over $k$.

For any algebraic group $G$, a set $\mathcal{S}$ of principal $G$-bundles on $X$ is called bounded if there exists a scheme $\mathcal{S}$ of finite type over $k$, and a family of principal $G$-bundles parametrised by $\mathcal{S}_1$, such that each element of $\mathcal{S}$ is isomorphic on $X$ to the $G$-bundle on $X$ obtained by restriction of the given family to some closed point of $\mathcal{S}$.

**Proposition 3.1** Let $B$ be a Borel subgroup of the reductive group $G$ and $T = B/B_u$, where $B_u$ the unipotent radical of $B$. Let $\mathcal{B}_T$ be a bounded set of $T$-bundles on $X$, and let $\mathcal{B}$ be a set of $G$-bundles on $X$ such that each member of $\mathcal{B}$ admits a reduction of structure group to $B$ such that the associated $T$-bundle is isomorphic to a member of $\mathcal{B}_T$. Then $\mathcal{B}$ is a bounded set of $G$-bundles.

**Proof** We first prove it in the case of $G = GL(n)$ and $B = \text{upper triangular matrices}$. We may identify principal $GL(n)$-bundles with their associated vector bundles. By hypothesis, each vector bundle $E$ in $\mathcal{B}$ admits a full flag $0 \subset E_1 \subset E_2 \subset \ldots \subset E_n = E$ such that the degrees of the line bundles $E_i/E_{i-1}$ ($i = 1, \ldots, n$) are bounded. Since the line bundles of a given degree form a bounded family and extensions of vector bundles in bounded families form a bounded family (see FGA [1], 4, Proposition 1.2, p.221), the proposition is proved in this case.

In the general case, we embed $G$ as a closed subgroup of $GL(n)$ for some $n$. Let $B_1$ (resp. $B$) be a Borel subgroup of $GL(n)$ (resp. $G$) with $B \subset B_1$. Since $B_u$ is contained in $B_{1,u}$, we have an induced homomorphism of $T$ into $T_1$, where $T = B/B_u$ and $T_1 = B_1/B_{1,u}$.

Let $\mathcal{B}'$ be the set of $GL(n)$ bundles obtained from $B$ by extension of structure group via $G \hookrightarrow GL(n)$. From the commutative diagram

$$G$$
we see that each bundle in \( B' \) has a reduction to \( B \), such that the corresponding \( T \) bundle is obtained by extension of structure group from an element of the set \( B_T \). Since by hypothesis \( B_T \) is a bounded set, by what has been proved above for \( GL(n) \)-bundles, \( B' \) is a bounded set.

Let \( \mathcal{P} \rightarrow X \times W \) be a family of principal \( GL(n) \)-bundles on \( X \) parametrised by a scheme \( W \) of finite type over \( k \), such that up to isomorphism all the bundles in \( B' \) occur in this family. Using \( \mathcal{P} \) we shall now construct a family of \( G \)-bundles on \( X \) parametrised by a scheme \( S \) of finite type over \( k \), such that every bundle in \( B \) occurs in this family.

By the results of Grothendieck (see FGA, 221, 4.c), there exists a \( \mathcal{V}/S \)-scheme
\[
S = \Pi_{W \times X/W}((\mathcal{P}/G)/W \times X)
\]
which has the following universal property: for any \( W \)-scheme \( U \rightarrow W \), the set of sections of \( (\mathcal{P}/G)_U \rightarrow X \times_W U \) is in bijective correspondence with the set of sections of \( S_U \) over \( U \). In particular, for \( w \in W \), the fibres of \( S \rightarrow W \) consists of the sections of the fibre bundle \( \mathcal{P}_w/G \rightarrow X \), where \( \mathcal{P}_w = \mathcal{P}|_{w \times X} \), and these are exactly the reductions of the \( GL(n) \) bundle \( \mathcal{P}_w \) to \( G \).

Therefore, the universal section of \( (\mathcal{P}/G)_S \rightarrow X \times_W S \) gives a family of \( G \)-bundles parametrised by \( S \), in which each bundle from the set \( B \) occurs. Finally, as \( G \) and \( GL(n) \) are reductive groups, \( GL(n)/G \) is affine, and there is a representation of \( GL(n) \) on a vector space \( V \) which gives a \( GL(n) \)-equivariant closed embedding of \( GL(n)/G \rightarrow V \). Now it is clear that the scheme \( S \) is a closed subscheme of the scheme \( S' = \Pi_{W \times X/W}(V/W \times X) \), where \( V \) is the vector bundle associated to \( \mathcal{P} \) by the representation of \( GL(n) \) on \( V \). Hence \( S \) is of finite type over \( k \) (see Ramanathan [4], Remark 4.8.2, page 425). This completes the proof of the Proposition 3.1.

Let \( G \) be a connected reductive group. Let \( X^*(G) = \text{Hom}(G,k^*) \). Let \( Z \) be the center of \( G \) and \( Z^0 \) its connected component of identity. Then \( G = Z^0 \cdot [G,G] \) and \( Z^0 \cap [G,G] \) is finite. Thus \( X^*(G) \) is a subgroup of \( X^*(Z^0) \) of maximal rank.

If \( E \) is a \( G \)-bundle on \( X \), we have a homomorphism \( d_E : X^*(G) \rightarrow \mathbb{Z} \) given by \( \chi \mapsto \deg(E_\chi) \), where \( E_\chi \) is the line bundle associated to \( E \) by \( \chi \).

**Definition 3.2** We shall call the element \( d_E \in \text{Hom}(X^*(G),\mathbb{Z}) \) the degree of \( E \).

When \( G = GL(n) \), the above definition reduces to the usual definition of the degree of the associated rank \( n \) vector bundle, as \( X^*(GL(n)) = \mathbb{Z} \). Also note that if \( G \) is semi-simple then \( d_E \) is zero as \( \text{Hom}(X^*(G),\mathbb{Z}) = 0 \).

We have the following:

**Lemma 3.3** Let \( T = GL(1)^l \) be a torus and \( L \subset X^*(T) \) be a subgroup of maximal rank. For a \( T \)-bundle \( F \) on \( X \), let \( d_F : X^*(T) \rightarrow \mathbb{Z} \) be the homomorphism as above, and \( d'_F : L \rightarrow \mathbb{Z} \) be the restriction of \( d_F \) to \( L \). If \( S \) is a set of \( T \)-bundles on \( X \) such that the set \( \{d'_F : F \in S\} \) is a finite set, then \( S \) is a bounded set of \( T \)-bundles.

**Proof** We reduce the proof to the case where \( L = X^*(T) \) as follows. If \( L \subset X^*(T) \) is an arbitrary subgroup of maximal rank, then there exists a basis \( \{\chi_1, \ldots, \chi_l\} \) of \( X^*(T) \) such that \( \{\lambda_1\chi_1, \ldots, \lambda_l\chi_l\} \) forms a basis for \( L \), with \( \lambda_i \in \mathbb{Z} \), \( \lambda_i \neq 0 \) for each \( i \). Since \( d_F(\chi_i) = \lambda_i^{-1}d_F(\lambda_i\chi_i) \), the result is true for \( L \) if it is true for \( X^*(T) \).

Hence we can assume that \( L = X^*(T) \). Let \( \{\chi_1, \ldots, \chi_l\} \) be a basis of \( X^*(T) \). Then by our hypothesis the set \( N_0 = \{d'_F(\chi_i) \mid F \in S, 1 \leq i \leq l\} \) is a finite set of integers. Thus the set \( S \) can be considered as a subset of the set of all \( l \)-tuples \( \{L_1, \ldots, L_l \mid L_i \in \text{Pic}(X) \text{ with } \deg(L) \in N_0\} \).
Hence \( S \) is a bounded set. \( \square \)

**Proposition 3.4** Let \( G \) be a connected semi-simple group. Then the family of semi-stable \( G \)-bundles on \( X \) is bounded.

**Proof** Let \( S \) be the set of semi-stable \( G \)-bundles. We shall show that each member \( E \) of \( S \) admits a reduction of structure group to \( B \) such that the associated \( T \)-bundles \( E_T \) (as \( E \) varies) form a bounded family. We then apply the Proposition 3.1 to complete the proof. For any principal \( G \)-bundle \( E \), by Remark 2.6, we can choose a reduction \( \sigma \) of the structure group to \( B \) such that

\[
\text{deg}(N_\sigma) \leq C,
\]

where \( C \) is a constant independent of \( E \). To show that the set of associated \( T \)-bundles \( \{E_T\} \) is bounded, we will show that there is a subgroup \( L \) of \( \mathcal{X}^*(T) \) of maximal rank with the property that \( \{(d_{E_T}|_L) \mid E \in S\} \) is a finite set and then use Lemma 3.3.

Let \( \Lambda_1, \ldots, \Lambda_l \) be the set of fundamental weights with respect to a maximal torus contained in \( B \) and the positive roots being contained in the Lie algebra of \( B \). Let \( m \) be a positive integer with the property that \( m\Lambda_i \) is a character of \( T \) for every \( i \). Let \( L \) be the subgroup of \( \mathcal{X}^*(T) \) generated by \( \{m\Lambda_i \mid 1 \leq i \leq l\} \). Then we observe that \( L \) is of maximal rank. Now the line bundle \( \det(N_\sigma)^{\otimes m} \) is associated to the character

\[
-2 \sum_{i=1}^{l} (m\Lambda_i) = - \sum_{\alpha \text{ root}, \alpha > 0} m\alpha.
\]

Hence for each \( E_T \) as above we have the condition

\[
\sum_{i=1}^{l} d_{E_T}(m\Lambda_i) = -\text{deg}(\det(N_\sigma)^{\otimes m})/2 \geq -mC/2,
\]

where \( d_{E_T}(m\Lambda_i) \) is the degree of the line bundle associated to \( E_T \) by the character \( m\Lambda_i \). On the other hand, if \( E \) is semi-stable then for any reduction of structure group to \( B \) the degree of the line bundle associated to a dominant character if \( B \) is \( \leq 0 \) (see Ramanathan [3]). Thus we have

\[
d_{E_T}(m\Lambda_i) \leq 0.
\]

This together with the above inequality implies that \( -mC/2 \leq d_{E_T}(m\Lambda_i) \leq 0 \) for each \( i \). Hence \( \{(d_{E_T}|_L) \mid E \in S\} \) is a finite set. This completes the proof. \( \square \)

**Proof of the Theorem 1.2**

Let \( S' \) be the set of semi-stable \( G \)-bundles with a fixed degree. For each element \( E \) of \( S' \) we choose a reduction \( \sigma \) of structure group to \( B \) with \( \text{deg}(N_\sigma) \leq C \), \( C \) independent of \( E \). We shall show that the associated \( T \)-bundles form a bounded family and apply Proposition 3.1. This will be shown by proving that there is a subgroup \( L \) of maximal rank in \( \mathcal{X}^*(T) \) such that \( \{(d_{E_T}|_L) \mid E \in S'\} \) is a finite set and then using the Lemma 3.3.

Note that \( T' = T/Z^0 \) is a maximal torus of \( G' = G/Z^0 \), contained in its Borel subgroup \( B' = B/Z^0 \). As we have the isomorphism \( G/B \cong G'/B' \), it follows that the \( G' \)-bundle \( E' \) obtained from \( E \) by extension of structure group is semi-stable, and \( \sigma \) gives rise to a reduction \( \sigma' \) of structure group of \( E \) to \( B' \). We also observe that any dominant character vanishes on \( Z^0 \). Consider the following diagram:

\[
\begin{array}{ccc}
E & \xrightarrow{\sigma} & E' \\
\downarrow & & \downarrow \\
B & \xrightarrow{\text{extension}} & B'
\end{array}
\]
where the row is exact. As already remarked, $X^*(G)$ is a subgroup of maximal rank in $X^*(Z^0)$. Hence the subgroup $L$ generated by $X^*(G)$ and $X^*(G/Z^0)$ is of maximal rank in $X^*(T)$. The set \( \{ d_{E_T} \mid E \in \mathcal{S}' \} \) is finite because $d_{E_T}[\lim (X^*(G))]$ is fixed while \( \{ d_{E_T} \mid \lim (T/Z^0) \mid E \in \mathcal{S}' \} \) is a finite set as shown in Lemma 3.4, since $G'$ is semi-simple and $E'$ is semi-stable). Now the theorem follows from the Lemma 3.3 and Proposition 3.1.

**Acknowledgement.** The first named author would like to thank the Abdus Salam International Centre for Theoretical Physics, Trieste, where this work was done.

**References**


